Hierarchical Organization in Smooth Dynamical Systems

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Abstract This article is concerned with defining and characterizing hierarchical structures in smooth dynamical systems. We define transitions between levels in a dynamical hierarchy by smooth projective maps from a phase space on a lower level, with high dimensionality, to a phase space on a higher level, with lower dimensionality. It is required that each level describe a self-contained deterministic dynamical system. We show that a necessary and sufficient condition for a projective map to be a transition between levels in the hierarchy is that the kernel of the differential of the map is tangent to an invariant manifold with respect to the flow. The implications of this condition are discussed in detail. We demonstrate two different causal dependences between degrees of freedom, and how these relations are revealed when the dynamical system is transformed into global Jordan form. Finally these results are used to define functional components on different levels, interaction networks, and dynamical hierarchies.

1 Introduction

Hierarchical organization is a frequently occurring phenomenon in natural processes. It is a tradition in the natural sciences to use this fact, at least implicitly, when constructing models. We refer to an approximate description of the system, typically valid on a specific length and time scale. In physics we even divide the different research fields by the physical scale where they apply, for example, elementary particle physics, nuclear physics, mesoscopic physics, condensed matter physics, and astronomy. Similar domain boundaries, also based on separation of physical scales, exist in biology. A coarse-grained hierarchy may be expressed, for example, as molecules, organelles, cells, tissues, organs, organisms, and ecosystems [17].

However, many aspects of natural phenomena cannot be addressed efficiently if strict domain boundaries, based on length and time scales, are assumed. On the contrary, most complex systems are characterized by having highly nontrivial interactions between dynamics on different scales. In many cases, hierarchical organization still exists, but the separations of different levels are defined by functionality rather than by traditional physical scales. From this perspective, satisfactory descriptions of dynamical hierarchies should use functionality (rather than, e.g., length scale) to define levels. The conceptual framework of this approach to dynamical hierarchies has been outlined in a previous article by Rasmussen et al. [17].

In general, defining separation between levels in a hierarchical structure is nontrivial. In statistical data analysis, for example, detecting structures such as hierarchies of clusters in a data set is an area that still attracts much attention from the research community. If no additional a priori information is provided to guide the definition of clusters and differentiation between levels, the task is not well defined. The same is true for hierarchical structures in dynamical systems. Traditional definitions of separation are based on physical scales defined with respect to physical constants (e.g., Planck's constant), to some natural system-dependent dimensional unit such as lattice distance, or, in statistical physics, to converging averages from a large number of degrees of freedom.

In this article we are concerned with deterministic dynamical systems, described by a phase space with an attached flow. We are interested in hierarchical organization, defined through smooth projective maps from the original phase space to lower-dimensional phase spaces, that is, from a system with many degrees of freedom to systems with fewer degrees of freedom. A projective map defined on the phase space also defines an induced map on the flow, that is, on the dynamics. It is in general not the case that the induced map defines a new deterministic dynamics on the lower-dimensional phase space. Intuitively this is clear: a subset of the degrees of freedom in a dynamical system do not in themselves have a deterministic dynamics (their time evolution depends on the other degrees of freedom). In this article we use this observation to define levels in a hierarchical structure. We denote the original phase space manifold by M and the higher level phase space by N [$\dim(N) < \dim(M)$]; the corresponding flows are denoted by $\phi_M(t)$ and $\phi_N(t)$. The following diagram describes two levels in the hierarchy:

$$\begin{array}{ccc} N & \xrightarrow{\phi_N(t)} & N \\ \pi\Big\uparrow & & \pi\Big\uparrow \\ M & \xrightarrow{\phi_M(t)} & M \end{array} \tag{1}$$

where $\pi : M \rightarrow N$ is the projective map. The projective map π describes a new level of description if the diagram commutes, that is, if there exists a flow $\phi_N(t)$ such that

$$\pi \circ \phi_M(t) \;=\; \phi_N(t) \circ \pi. \tag{2}$$

In this case we refer to π as a projective fiber map. As defined here, the problem of identifying a dynamical hierarchy can be expressed as finding projective fiber maps π such that there exist a nontrivial flow $\phi_N(t)$ fulfilling Equation 2. The essence of this article is to explore the consequences of this equation.

1.1 Outline of the Article

The article starts with a overview of more traditional approaches to defining transitions between levels of descriptions in physics and dynamical systems. In Section 3 the technical framework relating to the definition of levels in a dynamical hierarchy is described. The succeeding section shows how the definition relates directly to invariant subspaces of the Jacobian of the flow. In connection with this we also discuss how a global projection may be constructed by smooth "gluing" of local projections. Dimensional entanglement and subjugation between the degrees of freedom, two generic phenomena that may prevent the existence of global projective fiber maps, are defined. In Section 5, we use dimensional entanglement and subjugation to construct higher order functional components and interaction networks. Finally, we define a dynamical hierarchy as a pair consisting of a set of self-contained dynamical subsystems and a set of invariant manifolds. The invariant manifolds define projections between the dynamical subsystems through a quotient manifold construction.

2 Background

Since hierarchies can be viewed as a generic property, shared by a multitude of natural phenomena, various aspects have been studied in different scientific fields. A short overview of these approaches is given in this section.

2.1 Statistical Physics and Thermodynamics

Perhaps the most interesting and extensively studied phenomenon in statistical mechanics is criticality. Systems near critical points exhibit an infinite hierarchical structure, showing correlations on all length scales simultaneously. The systems are said to be scale invariant, or self-similar. As a result of the great scientific success in this field, highly sophisticated techniques for studying critical phenomena now exist. Most important is perhaps the renormalization group. Many aspects of critical phenomena can be derived from iteration of the renormalization transformation, which almost always converges to a fixed point in the parameter space.[1]

Conceptual models inspired by critical phenomena have been successful in addressing many aspects of hierarchical structures in natural systems. Highly complex energy landscapes produced by frustration of spin glass systems have for example been extensively used for comparing physical relaxation processes to Darwinian evolution, as well as more general complex optimization problems [12]. The more recent theory of self-organized criticality (pioneered by Bak, Sneppen, and others [2, 1]), argues for natural systems to spontaneously tune their model parameters toward critical states, that is, states displaying (in theory infinite) hierarchical structure. Self-organizing criticality was put forward as an attempt to explain the abundant presence of power scaling laws in nature.

The theory of critical phenomena is based on equilibrium statistical mechanics. Many natural systems must however be considered as open systems. Their behavior is often dominated by external driving "forces" and dissipation. For such systems fundamentally different techniques must be developed. Nonequilibrium, or irreversible, thermodynamics was pioneered by Onsager, and later established in definite form by the Brussels school in the 1960s. One aspect of this approach is Prigogine's theory of self-organization in dissipative structures (for which he received a Nobel Prize in 1977) [14]. The fundamental idea in Prigogine's framework is to view the nonequilibrium state of the system as a source of order; the driving of the system constantly counteracts the internal entropy production. Dissipative systems that, by external driving, maintain a nonequilibrium state[2] are referred to as *self-organizing*.

The systems studied by Onsager, Prigogine and others were assumed to be close to an equilibrium state. One may argue that the theory is essentially a linear perturbation theory. For systems far from equilibrium, the traditional approach is to use statistical thermodynamics of nonequilibrium systems. Stochastic processes and stochastic differential equations (e.g., the Fokker-Planck, the Langevin, and the master equations) are used to construct high level models of the relevant variables (e.g., fluctuation-dissipation theorems and transport equations). Another approach is to develop a statistical mechanics theory for systems far from equilibrium, that is, techniques where high level thermodynamic relations can be derived directly from the microscopic dynamics. Lately there has also been remarkable success in this area. A beautiful general picture based on results from chaos theory, ergodicity, and strange attractors has emerged [19]. This will be discussed further in the next section. We also mention a connection between nonequilibrium statistical mechanics and dynamical hierarchies. Perhaps surprisingly, this connection is based on a fundamental result in nonlinear time series analysis.

2.2 Chaos, Strange Attractors, and Time Series Analysis

The time evolution of a dissipative driven dynamical system can often be characterized geometrically by simultaneous expansion (from the driving forces) and folding (from dissipation) of the configuration space. The time evolution of such a system typically displays irregular chaotic behavior. Chaos arguably implies practical nondeterminism. At first such systems may therefore seem uninteresting for predictive science. Surprisingly, however, attentive analysis has shown that many different systems in (e.g.) physics and biology, despite diverse origins and chaotic dynamics, share

1 In two dimensions this explains universality of scaling exponents through topological classifications of the fixed points in the parameter space.

2 A state where the entropy is not maximized.

many qualitative properties. When applied to nonequilibrium statistical mechanics, it may actually be easier to argue for (e.g.) ergodicity and probabilistic behavior in systems with chaotic dynamics than for nonchaotic dynamical systems. Nonlinear dynamics and chaos has become a cornerstone in nonequilibrium statistical mechanics.

In a nonlinear dissipative driven dynamical system, as time approaches infinity the trajectory of the system often spans a highly nontrivial geometrical object in the configuration space, usually referred to as a *strange attractor* [18].[3] Some features of strange attractors can be understood intuitively from a general principle of expanding and folding the configuration space often associated with chaotic dynamics; one example is frequently occurring fractal structure. Other attributes are more convoluted, such as the various entropy measures and characteristic exponents (Lyapunov exponents). These observables, combined with mathematical objects (e.g., invariant probability measures on the attractor), provide the fundamental entities studied in chaos theory. A single relevant trajectory in the configuration space is replaced by probability measures over a strange attractor. These concepts are similar in spirit to ensemble averaging and show a deep connection with (nonequilibrium) statistical mechanics [19, 3].

A result that first[4] appeared in the turbulence community is the attractor reconstruction technique for nonlinear time series analysis. The idea is the following. When observing a system, we usually only capture a low-dimensional representation of the full system. However, due to a somewhat surprising mathematical result, *Takens' embedding theorem* [21], we know that under very general circumstances the projected information is sufficient for the topology of the original attractor to be reconstructed. By topology we mean the geometric shape of the attractor up to a diffeomorphic transformation. Since a diffeomorphism corresponds to a smooth change of descriptive variables, this reconstruction can in fact be argued to capture all interesting features of the original dynamical system. Observables such as dimensionality, entropy, and characteristic exponents on the attractor are invariant under diffeomorphisms. Takens' theorem also includes a constructive technique, the delay coordinate map, which provides an explicit formula for the reconstruction. Virtually all techniques in nonlinear time series analysis are based on this result [10].

The concept of dynamical hierarchies is closely related to Takens' embedding theorem. The original attractor can *almost always*[5] be reconstructed from a generic projection map of the degrees of freedom of the system (our observed data). As we will discuss in detail later, we define a higher order description of the system as a (singular) situation when the original attractor *cannot* be reconstructed from the coarse-grained time evolution. There is therefore a direct, if somewhat convoluted, connection between attractor embedding and the perspective on dynamical hierarchies presented in this article.

2.3 Invariant Manifolds and Inertial Manifolds

Analysis of nonlinear dynamical systems is a challenging task, and it undoubtedly remains an area of active research effort. Among the most notable research objectives in nonlinear dynamic analysis is to characterize the existence of invariant manifolds and to solve the associated problem of finding (computing) them [23, 8]. The problem under consideration is of great importance, and it has been traditionally motivated by efforts to develop systematic methods for simplifying the analysis of nonlinear systems by effectively reducing their dimensionality. In particular, the study of invariant manifolds has been conducted in connection with the existence problem of stable, unstable, and center manifolds for nonlinear dynamical systems, their long-term asymptotic behavior, and the associated stability characteristics, as well as bifurcation analysis. Later in this article we will see that invariant manifolds also play a central role for dynamical hierarchies.

3 Strange attractors may be viewed as generalizations of fixed point and limit cycles in more "well-behaved" dynamical systems.

4 To be more precise, the idea of attractor embedding was actually first put forward in the complex systems community by Packard et al. [15], but the first mathematical proof was presented almost simultaneously, and independently, by Takens [21].

5 *Almost* is meant in a measure theoretical sense, that is, "except on a set with zero measure." It is in fact the degenerate set of projections not allowing reconstruction of the attractor that is of interest for dynamical hierarchies.

Strange attractors, as well as many of the generic properties of such objects, can be generalized to infinite-dimensional systems. In the same spirit as the standard stable and center manifold theory for finite-dimensional dynamical systems, conceptual and technical extensions have been introduced in the case of singularly perturbed systems [20]. The classification of the corresponding invariant manifolds as slow and fast is a natural consequence of a time scale separation property, a phenomenon of great importance in our construction of dynamical hierarchies. Moreover, a similar geometric notion of a positively invariant finite-dimensional manifold that attracts (almost) every trajectory was introduced in the study of the dynamic model reduction problem of parabolic partial differential equations, and is referred to as an inertial manifold [6]. These results have been much appreciated in the turbulence community through the Navier-Stokes equations, which constitute perhaps the least understood of all fundamental models in classical mechanics. In fact, many of the results on strange attractors have their origin in the turbulence community.

3 Hierarchies in Smooth Dynamical Systems

This article is concerned with hierarchical structures in dynamical systems, that is, with dynamical systems that allow (or do not allow) multiple simultaneous levels of description. The first technical issue is therefore how to mathematically define levels within such hierarchies, and thereafter to find tools that can be used to identify these levels. In the introduction we discussed an intuitive definition of description levels, based on the assumption that the dynamics should be self-contained and deterministic at each level within the hierarchy. In this section we start by elaborating this definition and make it more precise; thereafter we present a technique that may be used to identify higher levels of description.

The objects we study are smooth finite-dimensional dynamical systems with continuous time evolution.[6] A higher level of description is defined through a projective map of the degrees of freedom on a lower level, as well as the induced map of the dynamics. Whether or not a projective map constitutes a new level of description can be determined by studying the *induced dynamics on the higher level*, that is, by determining if the map induces a well-defined flow describing a new deterministic dynamics on a higher level.

This can be formulated in the language of differential geometry. Assume that, on the lowest level, the dynamics of the system is described by a trajectory in configuration space. The trajectories are confined to some m-dimensional manifold M. The trajectory is an integral curve of a vector field, the infinitesimal generator of the flow. The vector field belongs to the tangent bundle TM of the manifold, that is, the union of all tangent spaces of the manifold.

An example of a dynamical system, one that we use again in Section 4.5, is the famous Lorenz system:

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= -xz + rx - y, \\ \dot{z} &= xy - bz. \end{aligned} \tag{3}$$

The time evolution of a typical trajectory on the strange attractor is shown in Figure 2 in Section 4.5 below. The phase space manifold is simply $\mathbb{R}^3$, and the corresponding tangent space at each point is $TM|_x = \mathbb{R}^3$.[7] The vector field generating the dynamics is given by $\mathbf{v} = \sigma(y - x)\partial_x + (-xz + rx - y)\partial_y + (xy - bz)\partial_z$.

6 The continuous time assumption is not critical, and most results presented hold in the discrete time case as well. For clarity we restrict this presentation to continuous time.

7 For a Euclidean space, the tangent space at each point is always isomorphic to the base space. This is not true for general manifolds.

We assume that a transition to a higher level can be described by a smooth map $\pi : M \rightarrow N$, where N is another manifold, $\text{rank}(\pi) = \dim(N) = n < m$, and π is onto. The rank deficiency guarantees a decrease of the degrees of freedom. Since π is a smooth map from M to N, it induces a differential map between the respective tangent spaces $\pi_* : TM|_x \rightarrow TN|_{\pi(x)}$ at each point (see, e.g., [4]). Since π is not a diffeomorphism, π_* does typically not define a new vector field on N. This means that the induced dynamics on N is usually not well defined, that is, not deterministic. We are interested in the cases when π actually does define a new deterministic dynamical system on a higher level. We make the following definition:

DEFINITION 1: *Let $\pi : M \rightarrow N$ be a smooth map from a manifold M, with dimension m, to a manifold N, with dimension n, where $n < m$. Let $\mathbf{v}$ be a vector field in the tangent bundle TM. If the differential $\pi_* : TM|_x \rightarrow TN|_{\pi(x)}$ maps $\mathbf{v}$ onto a well-defined vector field $\mathbf{w}$ in TN for all x in M, that is, $\mathbf{w}|_{\pi(x)} = \pi_*(\mathbf{v}|_x)$ for all x in M, then we call π a projective fiber map with respect to $\mathbf{v}$.*

We conclude that:

LEMMA 1: *A system allows a higher level of descriptions if and only if there exists a projective fiber map with respect to the flow (or the vector field generating the flow).*

If π is a projective fiber map, then the infinitesimal generators of the dynamics $\mathbf{v}$ and $\mathbf{w}$ are said to be π-related. The following lemma provides necessary and sufficient conditions for a map π to be a projective fiber map:

LEMMA 2: *Let $\pi : M \rightarrow N$ be a smooth map, and $\mathbf{v}$ be a vector field on M. Assume further that π is onto. If $\pi_*(\mathbf{v}|_x) = \pi_*(\mathbf{v}|_y)$ whenever $\pi(x) = \pi(y)$, for all x and y in M, then π is a projective fiber map with respect to $\mathbf{v}$; otherwise not. Furthermore, if π is a projective fiber map, then integral curves of $\mathbf{v}$ map on integral curves of $\pi_*(\mathbf{v})$, that is,*

$$\pi \circ \exp(\tau \cdot \mathbf{v}) = \exp(\tau \cdot \pi_*(\mathbf{v})) \circ \pi. \tag{4}$$

Proof. For a proof of this lemma we refer to any text on differential geometry, such as [4]. Note that Equation 4 agrees with the intuitive argument in Equations 1 and 2. □

4 Invariant Manifolds and Fiber Projections

To gain intuition for the results in the previous section, we start with the simplest nontrivial case, namely, linear dynamics and linear projective maps:

$$\begin{aligned} \dot{x} &= Ax, \\ \pi(x) &= Px, \end{aligned} \tag{5}$$

where A is an $m \times m$ matrix and P is an $n \times m$ matrix. The phase spaces are Euclidean: $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$.

PROPOSITION 1: *Consider a linear dynamical system and a linear projective map, as defined in Equation 5. Then P describes a projective fiber map if and only if ker(P) is A-invariant.*[8]

8 By invariance under matrix multiplication we will always mean left multiplication, that is, $x \in \ker(P)$ implies that $Ax \in \ker(P)$.

Proof. According to Lemma 2, we need to show that $Px = Py$ implies $\pi_*(\mathbf{v}|x) = \pi_*(\mathbf{v}|y)$. We start by noting that linearity implies

$$P_x = P_y \Longleftrightarrow x - y \in \ker(P).$$

In the linear case, there is a direct equivalence between the base manifold M and the tangent space $TM|_x$, which simplifies the differential map[9] $\pi_*(\mathbf{v}|_x) = PAx$; therefore

$$\pi_*(\mathbf{v}|_x) = \pi_*(\mathbf{v}|_y) \iff A(x-y) \in \ker(P).$$

It follows that ker(P) is A-invariant.

On the other hand, assume that ker(P) is not A-invariant, that is, there exists a $z \in \ker(P)$ such that $AZ \notin \ker(P)$. Further, there exist z' and z'' such that $z = z' - z''$, and since $z \in \ker(P)$, we have $Pz' = Pz''$, but since $PA(z' - z'') \neq 0$, we have $PAz' \neq PAz''$, and P is not a projective fiber map according to Lemma 2. □

In the general case both the dynamics and the projective map are nonlinear. In an infinitesimal neighborhood of a point, we may linearize both the vector field and the projective map. It is therefore clear that Proposition 1 implies the following result for the general case:

PROPOSITION 2: *Let $\dot{x} = F(x)$ be a dynamical system, and $J|_x$ denote the Jacobian of F at x, which we assume to be nonzero (x is not a quadratic singularity). If π is a projective fiber map, then* $\ker(\pi_*|_x)$ *is $J|_x$-invariant.*[10]

Proof. This follows immediately from local linearization and Lemma 1 (constants added to the field and the projective map do not change the result in the proposition). □

It is important to note that Proposition 2 only provides a necessary condition for a projective fiber map. An additional condition on π is continuity. In Section 4.5 we show that this requirement is nontrivial.

4.1 Invariant Manifolds and Projective Maps

DEFINITION 2: *A (global) invariant manifold of an autonomous dynamical system $\dot{x} = F(x)$ is defined as a set $\mathcal{S}$ such that if $x(0) \in \mathcal{S}$, then $x(t) \in \mathcal{S}$ for all $t \geq 0$.*

The tangent vectors of an invariant manifold lie in the invariant subspace of the Jacobian. Combining this observation with Proposition 2, we get the following important result.

PROPOSITION 3: *If $\pi : M \rightarrow N$ is a fiber projective map, then the kernel* $\ker(\pi_*) \in TM$ *spans the tangent space at each point $x \in M$ of an invariant manifold with respect to the flow.*

As a result, calculating fiber projective maps is to a large extent equivalent to calculating global invariant manifolds. For further discussion of invariant manifolds and of stable, unstable, and center

9 Remember that the differential is locally defined by the Jacobian, and in the linear case the local and the global differential are therefore equivalent.

10 Note that $J|_x : TM|_x \rightarrow TM|_x$, so $\pi_*|_x$ must be a map $TM|_x \rightarrow TM|_x$, not $TM|_x \rightarrow TN|_x$. This can however be solved by interpreting N as a submanifold of M, which is allowed as long as π has constant rank.

manifolds, see, for example, [23, 8]. The remainder of this section is a discussion of spectral techniques for calculating invariant manifolds.

4.2 Invariant Subspaces and the Spectrum of the Jacobian

It is instructive to connect the results in Proposition 3 with the intuitive picture presented in Equation 1.

To this end we use the following results from linear algebra concerning invariant subspaces of a matrix:

PROPOSITION 4: *Let $A \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{m\times n}$, where $n \leq m$. We assume that rank$(P) = n$. Then the range of P is A-invariant if and only if there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that*

$$AP = PB.$$

Furthermore, the spectrum of B, denoted by $\lambda(B)$, is contained in the spectrum of A:

$$\lambda(B) \subseteq \lambda(A),$$

where equality if $n = m$.

The proposition can be understood as a condition for the diagram

$$\begin{array}{ccc} N & \xrightarrow{\quad B \quad} & N \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P} \\ \mathcal{R}(P) \subseteq M & \xrightarrow{A} & \mathcal{R}(P) \subseteq M \end{array}$$

to commute. There is also a dual version of this proposition:

PROPOSITION 5: *Let $A \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{m\times n}$, where $n \leq m$. We assume that rank$(P) = n$. Then the kernel of P is A-invariant if and only if there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that*

$$PA = BP.$$

Furthermore, the spectrum of B is contained in the spectrum of A:

$$\lambda(B) \subseteq \lambda(A),$$

where equality holds if $n = m$.

The corresponding commuting diagram is

$$\begin{array}{ccc} N & \xrightarrow{\quad B \quad} & N \\ {\scriptstyle P}\uparrow & & {\scriptstyle P}\uparrow \\ \ker(P) \subseteq M & \xrightarrow{A} & \ker(P) \subseteq M \end{array}$$

Proof. Proof of Proposition 4 can be found, for example, in [7]. Proposition 5 follows from observing that the linear operator, formally written as $B = PAP^{-1}$, can be well defined on the quotient space $\mathbb{R}^m$ / $\ker(P) \approx \mathbb{R}^n$ if and only if the subspace $\ker(P) \subset \mathbb{R}^m$ is A-invariant. □

From Proposition 5 we learn that $\ker(\pi_*)$ being J-invariant implies that there exists a J' such that

$$PJ = J'P, \tag{6}$$

where P is a matrix representation of π_*, that is, the Jacobian of π. Since the differential of composite maps between manifolds satisfies $(f \circ g)_* = f_* \circ g_*$, Equation 6 may be regarded as an infinitesimal version of Equation 2 and Equation 4. This clarifies the relation between invariant manifolds and the commuting diagrams representing projective fiber maps.

Another direct consequence of Proposition 5 and Equation 6 is that the characteristic exponents of the projected dynamical system are already present in the original system, that is,

COROLLARY 1: *A projective fiber map can eliminate, but not alter, characteristic exponents, that is,*

$$\lambda(J') \subseteq \lambda(J),$$

where J and J' are the Jacobians of the original and the projected vector field, respectively.

4.3 Projective Maps and Projections

In this article, when we refer to a projective map from one manifold to another, we mean a map π: $M \rightarrow N$ where $\dim(N) < \dim(M)$. This is not equivalent to the standard definition of a *projection*, which usually is defined as an idempotent map, that is, a map π: $M \rightarrow M$ where $\pi \circ \pi = \pi$. The set of projections is a subset of the family of projective maps, where N can be viewed as a submanifold immersed in M, and $P \circ P = P$. In the rest of this section, we examine the relation between projective maps and projections.

If projections are considered, instead of projective maps, then Proposition 4 and Proposition 5 can be combined into a simpler result:

PROPOSITION 6: *Let P be a bounded smooth idempotent map M into M, then*

$$M = \ker(P) \oplus \mathcal{R}(P),$$

where $\mathcal{R}(P)$ denotes the range of P. Suppose further that A is a bounded linear map from M into M. Then both kernel and range of P are invariant under A if and only if

$$AP = PA.$$

The corresponding commuting diagram is

$$\begin{array}{ccc}
\mathcal{R}(P) \subseteq M & \xrightarrow{A} & \mathcal{R}(P) \subseteq M \\
{\scriptstyle P}\uparrow & & {\scriptstyle P}\uparrow \\
\ker(P) \subseteq M & \xrightarrow{A} & \ker(P) \subseteq M
\end{array}$$

Proof. See, for example, [11]. For completeness we sketch the proof. For the decomposition $M = \ker(P) \oplus \mathcal{R}(P)$ to be valid we need to show that $\mathcal{R}(P) \cap \ker(P) = \{0\}$. Let $x \in \mathcal{R}(P) \cap \ker(P)$; then there exists a y such that $x = Py$. Multiplied by P, this relation gives $Px = P^2 y = Py$, but $Px = 0$ then implies $Py = 0$, so $x = 0$. This ends the first part of the proof.

Note that combining Proposition 4 and Proposition 5 gives the second half of the proposition directly. Alternatively we use $M = \ker(P) \oplus \mathcal{R}(P)$ to decompose an arbitrary vector $x = y + z$, where $y \in \mathcal{R}(P)$ and $z \in \ker(P)$. Note that $y \in \mathcal{R}(P) \Rightarrow y = Py$, so A invariance of $\mathcal{R}(P)$ gives $PAy = Ay = APy$. Furthermore, A invariance of $\ker(P)$ gives $PAz = 0 = APz$. This shows that invariance of $\mathcal{R}(P)$ and $\ker(P)$ gives $PA = AP$. The argument also works in reverse, giving the equivalence. □

Proposition 6 shows a connection between linear projective maps and projections on a Banach space. Furthermore, any linear projective map can be transformed into a projection through the following construction.

PROPOSITION 7: *Let P be a linear projective map from M onto N. Then there exists a corresponding projection $\hat{P}$ from M into M, which is related to P through a smooth change of variables on the manifold N.*

Proof. Start by noting that $\mathcal{R}(P)$ is a subspace of M, so extending the domain of P from $N = \mathcal{R}(P)$ to M is straightforward. The manifold N can therefore be immersed in M.

Consider the following construction:

$$\hat{P} = P^{-}P,$$

where P^{-} denotes the pseudo-inverse of the matrix P, that is, $PP^{-}P = P$. A pseudo-inverse of a linear transformation always exists, and we will in fact give an explicit construction shortly, but it may not be unique. Since rank $(P^{-}) = \dim(\mathcal{R}(P))$, it follows that $P \to \hat{P} = P^{-}P$ can be interpreted as a change of coordinates on N. Further, $\hat{P}$ is idempotent, since $\hat{P}\hat{P} = P^{-}PP^{-}P = P^{-}P = \hat{P}$.

To derive an explicit form for the pseudo-inverse we need to use a singular value decomposition: $P = U\Sigma V^T$, where U and V are orthonormal matrices ($UU^T = 1$, $VV^T = 1$), and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0)$ where k = rank (P). Define the pseudo-inverse of the diagonal matrix Σ as

$$\sum^{-} = \mathrm{diag}\left(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_k}, 0, \ldots, 0\right).$$

A pseudo-inverse of P is then given by

$$P^{-} = V\Sigma^{-}U^T.$$

The condition $PP^{-}P = P$ follows from the orthonormality of U and V. We can go one step further and write an explicit expression for the projection

$$\hat{P} = V\hat{\Sigma}V^T,$$

where $\hat{\Sigma} = \Sigma\Sigma^{-} = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$ with k ones. □

We conclude this section with a definition of fiber map projections:

DEFINITION 3: *Let $\pi : M \rightarrow N$ be a projective fiber map, and N a submanifold of M. If π is a projection (i.e., idempotent: $\pi \circ \pi = \pi$), then we call π a fiber projection.*

4.4 Linear Projections

The dynamics of a linear system can be decomposed into eigenmodes corresponding either to one real eigenvalue or to a pair of complex conjugate eigenvalues. Degenerate eigenvalues cause additional complications and will be discussed later in this section. We start by ignoring this complication and try to identify candidates for projective maps and projections corresponding to a linear transformation.

From Propositions 5 and 6, it is clear that

$$P_{\{\lambda_i\}} = \prod_{\{\lambda_i\} \subseteq \lambda(A)} (A - \lambda_i \cdot \mathbb{1}), \tag{7}$$

where $\{\lambda_i\}$ is a subset of the eigenvalues of A (either neither or both members of a complex conjugate pair are included), is a projective fiber map. The corresponding fiber projection is given by

$$\begin{aligned} \hat{P}_{\{\lambda_i\}} &= \frac{1}{2\pi \mathrm{i}} \oint_\Gamma d\zeta \, (A - \zeta \mathbb{1})^{-1} \\ &= P^{-}_{\{\lambda i\}} P_{\{\lambda i\}}, \end{aligned} \tag{8}$$

where Γ is a closed curve in the complex plane and the set of eigenvalues $\{\lambda_i\}$ is in the exterior of Γ; see Figure 1. The contour integral formulation is mainly used in the infinite-dimensional case, when the spectrum may be continuous.

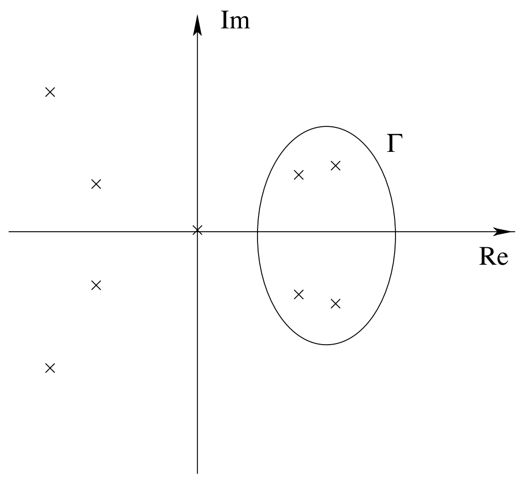


Figure 1. The closed curve Γ encloses a subset of the spectrum. The eigenvalues (or resolvent) in the exterior of Γ constitute the kernel of the corresponding projection.

Projective maps and projections on the generic form given in Equations 7 and 8 are exhaustive if the spectrum of A is nondegenerate, that is, if A is diagonalizable. In the degenerate case, Equations 7 and 8 still define projective maps, but all possibilities are not covered by these generic cases. To reveal the structure of all invariant subspaces with respect to a general matrix A, we need to transform the matrix into Jordan canonical form. Before discussing the Jordan form it is instructive to take an intermediate step, the Schur decomposition.[11] Schur's decomposition theorem states that any matrix A can be written as

$$A = UTU^*$$

where U is unitary (i.e., $UU^*=1$) and T is upper triangular with the eigenvalues of A on the diagonal (in any order we prefer). If A is a real matrix, the eigenvalues come in complex conjugate pairs, an observation that can be used to derive a real Schur decomposition, $A = QRQ^T$, where R is quasi-triangular with a possibility of having 2×2 blocks on the diagonal corresponding to complex conjugate pairs of eigenvalues.

As a direct consequence of R being quasi-triangular, the upper left $k \times k$ corner of R corresponds to an invariant subspace of dimension k. Naturally k must be such that no 2×2 block is cut in two. The Schur decomposition therefore reveals at least $m/2$ real invariant subspaces of an $m \times m$ matrix, even in the case of total eigenvalue degeneracy. The first k columns of Q form an orthogonal basis for the invariant subspace. The last $m-k$ columns define a projective map.

PROPOSITION 8: *The last m columns q_i, $i = 1, \ldots, m$, of a transformation matrix Q from a Schur decomposition define a projective map in the form*

$$P = \begin{pmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_m^T & - \end{pmatrix},$$

where the vectors q_i are mutually orthogonal.

As an example consider the matrix

$$Q^T AQ = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (9)$$

A projective map of the form in Equation 7 eliminates the invariant subspaces spanned by $\{Q \cdot (1, 0, 0)^T, Q \cdot (0, 0, 1)^T\}$. The triangular form reveals an additional two-dimensional invariant subspace, spanned by $\{Q \cdot (1, 0, 0)^T, Q \cdot (0, 1, 0)^T\}$.

11 While the Jordan decomposition is important for theoretical understanding of invariant subspaces, it is useless in numerical calculations, due to its computational complexity and its instability (the B^{-1} causes the instability). Schur factorization is a unitary transformation, and is therefore stable. In addition it can be computed efficiently. Numerical software often uses Schur decomposition to calculate the eigenvalues of a matrix. For further discussions on efficient implementations of Schur decompositions, see, for example, [7].

There is in fact yet another invariant subspace, which is not obvious from the Schur decomposition. To discover this, we use the Jordan decomposition theorem, stating that for every matrix A there is a nonsingular matrix B such that $A = BIB^{-1}$ and

$$I = \begin{pmatrix} I_1 & 0 & \cdots & 0 \\ 0 & I_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_M \end{pmatrix}, \tag{10}$$

where

$$I_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}$$

and λ_i are the eigenvalues of A. The submatrices I_i are referred to as Jordan blocks, and they describe independent generalized eigenmodes of A. Similarly to the Schur decomposition, if A is real, there is a real Jordan decomposition where the elements in I_i are allowed to be 2×2 blocks corresponding to pairs of conjugate eigenvalues.

The Jordan form reveals all invariant subspaces of the matrix. Each Jordan block is a triangular matrix decoupled from the rest. Therefore, by the same argument as for the Schur decomposition, there are between m_i and $m_i/2$ real invariant subspaces corresponding to the Jordan block I_i of size $m_i \times m_i$. For the matrix in Equation 9, the Jordan form is

$$B^{-1}AB = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which reveals two one-dimensional, linearly independent, invariant subspaces spanned by $B \cdot (1, 0, 0)^T$ and $B \cdot (0, 0, 1)^T$, as well as the two-dimensional invariant subspace spanned by $\{B \cdot (1, 0, 0)^T, B \cdot (0, 1, 0)^T\}$.

Using the Jordan form, we can gain detailed understanding of projective maps of the form in Equation 7. In general, $\dim(\ker(A - \lambda\mathbb{1}))$ equals the number of Jordan blocks of A with eigenvalue λ. This is easy to see, since each of these blocks produces a zero column through its upper right corner when the matrix $\lambda\mathbb{1}$ is subtracted. The projective map $P = A - \lambda\mathbb{1}$ therefore has the undesired property that it eliminates a one-dimensional invariant subspace from *every* Jordan block with eigenvalue λ. Similarly, $P' = (A - \lambda\mathbb{1})^k$ eliminates a $\min(k, m_i)$-dimensional invariant subspace for each Jordan block I_i with eigenvalue $\lambda_i = \lambda$. We summarize this as follows.

PROPOSITION 9: *A matrix of the form*

$$P = (A - \lambda\mathbb{1})^k,$$

describes a projective fiber map with respect to A. Furthermore, if λ corresponds to a single Jordan block, then dim (ker(P)) $= k$ *and all projective maps with respect to λ are similar*[12] *to this form.*

Note that a nondegenerate spectrum is a special case where $k = 1$ is exhaustive.

For completeness we also mention the Jordan-Schur decomposition. This is the most complete decomposition available using only unitary (orthogonal) transformations. It is quite complicated, so we only summarize its properties here. Like the Jordan form, it explicitly shows what the sizes of the Jordan blocks are and gives explicit bases for many more invariant subspaces than the Schur form. For more details see [22].

4.5 Dimensional Entangling and Subjugated Degrees of Freedom

After analyzing fiber projective maps and fiber projections for linear systems, we now return to the nonlinear case. In Proposition 2 we give a necessary condition for a projective fiber map. The kernel of the differential is invariant under the Jacobian of the flow at each point in the phase space. From the discussion above it is clear that we can construct at least $m/2$ nonsimilar fiber projective maps at each point on the m-dimensional manifold M. However, when these projective maps are constructed independently at each point, there is no guarantee that we can "glue" them together to define a *smooth* global (nonlinear) fiber projective map.

The eigenvalues (unlike the eigenvectors) are smooth functions of the corresponding matrix. Since the Jacobian varies smoothly over M, we know that the eigenvalues do the same, as well as any function of the Jacobian and the eigenvalues (e.g., P in Eq. 7). While this is true in general, there is still a problem with degenerate eigenvalues. Note that $\ker(P) = \ker(J|_x - \lambda|_x \mathbb{1})$ can be discontinuous when λ becomes degenerate; and, as discussed above, dim(ker(P)) may vary. If there exist regions in phase space (with nonzero measure) where the eigenvalues of the Jacobian are degenerate, a more general definition based on the Jordan decomposition must be used to construct a global smooth projective fiber map. To reveal all invariant manifolds of a dynamical system, we assume it to be transformed into Jordan form[13]:

DEFINITION 4: *Suppose we have a dynamical system $\dot{x} = F(x)$ in a phase space $x \in M$. The Jacobian of the flow at a point x is denoted by $J|_x$. Suppose that we make a diffeomorphic change of variables $y = \psi(x)$. The Jacobian is transformed as a $(1, 1)$ tensor, that is, $\tilde{J}|_{\psi(x)} = T^{-1}|_x \cdot J|_x \cdot T|_x$, where $T|_x$ is the Jacobian of ψ at x ($\cdot$ denotes matrix multiplication). Now, if $\tilde{J}|_{\psi(x)}$ is in Jordan canonical form as in Equation (10) for every $x \in M$, then we say that the new dynamical system $\dot{y} = \tilde{F}(y)$, where $y = \psi(x)$, is in Jordan form.*

With the full structure of the dynamical system revealed, we need to consider the following two phenomena:

First, a degenerate eigenvalue may split into a complex conjugate pair. In this situation, the kernel of a global smooth projective fiber map cannot split the corresponding invariant subspace. We make the following definition:

DEFINITION 5: *Two degrees of freedom contained in the kernel of a projective fiber map corresponding to a pair of eigenvalues that, in some region of the phase space, constitute a complex conjugated pair are said to be locally dimensionally entangled. For tactical reasons (see below) we also define a degree of freedom as dimensionally entangled with itself.*

Global dimensional entanglement is defined through reachability on the graph generated by local entanglement: x_i and x_j are defined as globally entangled if there exists a chain $x_i, \ldots, x_j$ such that each member of the chain is entangled with its neighbors.

Note that dimensional entanglement is an equivalence relation.

12 Two matrices A and B are said to be similar if there exists a nonsingular matrix Q such that $A = Q^{-1}BQ$.

13 This transformation can be computationally very demanding, especially for high-dimensional systems.

Second, a degenerate eigenvalue may result in a nontrivial Jordan block. In this case the degrees of freedom are hierarchically organized as invariant subspaces, starting at the lowest level with a one-dimensional invariant subspace corresponding to the upper left corner of the Jordan block. Each invariant subspace depends on the degrees of freedom in the subspace above in the hierarchy, but not vice versa. We make the following definition:

DEFINITION 6: *Suppose that a dynamical system is in Jordan form. Let two degrees of freedom x_1 and x_2 be such that in some region of the phase space they are contained in the invariant subspace of a single Jordan block of the Jacobian of the flow. If x_1 corresponds to a position higher up along the diagonal of the Jordan block than x_2 (i.e., lower down in the hierarchy described above), then we say that x_1 is locally subjugated*[14] *by x_2.*

Global subjugation is defined through reachability on the directed graph generated by local subjugation: x_1 locally subjugated by x_2, and x_2 locally subjugated by x_3, implies x_1 globally subjugated by x_3.

Note that subjugation is not a symmetric relation.

To demonstrate dimensional entanglement we use the famous Lorenz system. Figure 2 shows how dimensional entanglement prevents the existence of a global projective map for the Lorenz system. This is hardly a surprising result, but it is important in that it shows that the existence of projective fiber maps is nontrivial. On the other hand there are dynamical systems for which fiber projective maps exist—for example, (most) linear systems and systems with a cluster of eigenvalues that is well separated from the rest of the spectrum (over the entire phase space).[15]

5 Networks and Hierarchies

In the previous section we learned that a hierarchical structure of a dynamical system is defined by (global) dimensional entanglement and subjugation between degrees of freedom. Graphically this can be nicely described by a graph with nodes representing the degrees of freedom—undirected edges representing global dimensional entanglement, and directed edges showing subjugation. The result is a graphical representation that clearly displays functional components of the dynamical system and the interactions between them.

It should be mentioned that in some situations it might be enlightening to study a time dependent interaction network, rather than the static one described here. Altering the definition to accommodate this situation is straightforward, but it will not be discussed here.

5.1 Effective Interaction Networks

Further clarity in the global structure of the dynamical system can be achieved by defining higher level functional components according to the following scheme:

1. An equivalence class of dimensionally entangled degrees of freedom is defined as an *atomic component*, denoted by $\mathcal{D}_i$.
2. A set of atomic components is called a *composite component.*
3. A composite component with no outgoing arrows is called a *subjugated component*, denoted by $\mathcal{S}_\alpha$.
4. A composite component with no incoming or outgoing arrows is called a *decoupled component.*
5. A composite component with no incoming arrows is called a *free component*, denoted by $\mathcal{F}_\mu$.
6. A composite component with cyclic subjective interdependence is called a *cyclic component.*

14 The term is chosen for association with Haken's term "enslaving," but the technical definitions are not equivalent [9].

15 If the separation in the spectrum is along the imaginary axis, this phenomenon is usually referred to as separation of fast and slow degrees of freedom; see the discussion in Section 2.3.

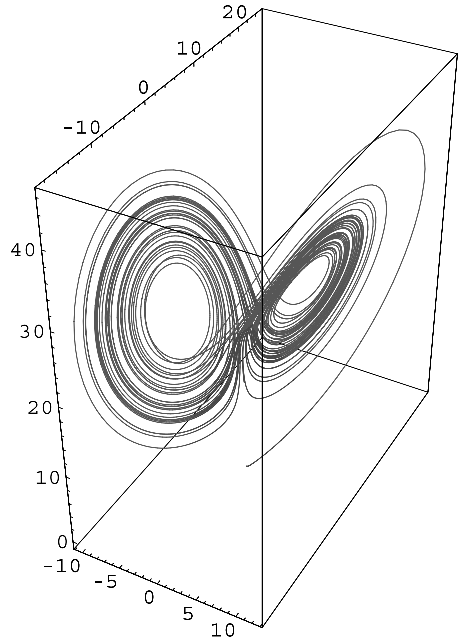


Figure 2. The Lorenz attractor. The trajectory is black when the Jacobian has three real eigenvalues, and light gray when the spectrum consists of one real eigenvalue and a complex conjugate pair of eigenvalues. The dynamical system is defined in Equation 3. Here we use parameter values $\sigma = 3$, $b = 1$, and $r = 26.5$.

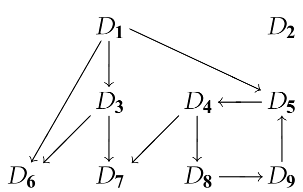
The atomic components can be viewed as the smallest functional units of a dynamical system, except for the degrees of freedom themselves.[16] The atomic components can be combined into higher order structures, which we call composite components. From Proposition 3 it follows that

PROPOSITION 10: *The degrees of freedom in a subjugated component define a global invariant manifold of the dynamical system. Further, if the kernel of a projective map* π *consists of a subjugated component, then* π *is a projective fiber map.*

Also, the complement of a subjugated component is a free, and possibly decoupled, component.

COROLLARY 2: *The range of a fiber projective map is always a free component.*

It makes sense to simplify the interaction network described above by agglomerating dimensionally entangled degrees of freedom. This leaves the atomic components as new nodes in a higher level graph. We call the resulting graph an *effective interaction network* of the dynamical system. Note that the only interaction type in an effective interaction network is subjugation. As a consequence all edges are directed. Here is an example:



16 It is tempting to call the degrees of freedom *elementary components.*

with nine atomic components and their interactions. In this example, $\mathcal{D}_1$ and $\{\mathcal{D}_1, \mathcal{D}_3\}$ are among the free components; $\mathcal{D}_2$ is the only decoupled component; $\{\mathcal{D}_6, \mathcal{D}_7\}$ and $\{\mathcal{D}_3, \mathcal{D}_4, \ldots, \mathcal{D}_9\}$ are among the subjugated components; and $\{\mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_8, \mathcal{D}_9\}$ is a cyclic component.

5.2 Dynamical Hierarchies

To define a dynamical hierarchy it is useful to add algebraic structure to the effective interaction network. We start with some definitions.

Let $\mathcal{S}$ be the set of all subjugated components, $\mathcal{F}$ the set of all free components, and finally $\mathcal{D}$ the set of all atomic components, of a dynamical system. Now, $\mathcal{S}$ defines all invariant manifolds of the dynamical system, and every member of $\mathcal{F}$ defines a self-contained deterministic dynamical system. Fiber projective maps take the form $\Pi : \mathcal{S} \times \mathcal{F} \to \mathcal{F}$, and can be defined as a quotient

$$\Pi(\mathcal{S}_\alpha, \mathcal{F}_\mu) \doteq \mathcal{F}_\mu / \mathcal{S}_\alpha, \tag{11}$$

where the quotient eliminates, from $\mathcal{F}_\mu$, all degrees of freedom contained in $\mathcal{S}_\alpha$. If a degree of freedom in $\mathcal{S}_\alpha$ is not present in $\mathcal{F}_\mu$, the operation is void. So Π is in fact a fiber projection.

To clarify the idea behind Equation 11, we make a short detour into differential geometry. As discussed above, the degrees of freedom in $\mathcal{S}_\alpha$ define an invariant manifold $\tilde{N}$, which is a submanifold in the phase space M. At each point on $\tilde{N}$, the tangent space is spanned by a set of tangent vectors, which globally defines a set of vector fields. From Frobenius' theorem we know that the set of vector fields form an involution at each point on $\tilde{N}$, and therefore also a Lie algebra $\mathfrak{g}$, with a corresponding Lie group G (for details see, e.g., [4]). The orbit of the action of the Lie group G on the manifold M, $\Psi : G \times M \to M$, spans the invariant manifold $\tilde{N}$. The fiber projection, as defined in Equation 11, is really a quotient manifold construction: M/G, which due to the conditions imposed on $\mathcal{S}_\alpha$ commutes with the flow of the dynamical system. From this perspective, a submanifold $\tilde{N}$, with corresponding Lie algebra $\mathfrak{g}$ spanning its tangent space, defines an invariant manifold (or subjugated component) if the Lie bracket is closed with respect to the flow: $[\mathbf{v}, \mathfrak{g}] \subseteq \mathfrak{g}$. Rather than the local transformation into Jordan form, the relation $[\mathbf{v}, \mathfrak{g}] \subseteq \mathfrak{g}$ can be used as a starting point for identifying invariant manifolds and fiber projections. Exploring this avenue is a work in progress by the author that will be published shortly.

Let us continue to explore some of the structure of $\mathcal{F}$ and $\mathcal{S}$, starting with the following relations:

PROPOSITION 11: *The following relations hold for the sets $\mathcal{F}$ and $\mathcal{S}$:*

1. *$\mathcal{F} \cap \mathcal{S}$ is the set of all decoupled components, and includes $\emptyset$ and $\mathcal{D}$.*
2. *If $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ are in $\mathcal{S}$, then $\mathcal{S}_\alpha \cup \mathcal{S}_\beta$* is in $\mathcal{S}$.
3. *If $\mathcal{F}_\mu$ and $\mathcal{F}_\nu$ are in $\mathcal{F}$, then $\mathcal{F}_\mu \cup \mathcal{F}_\nu$ is in $\mathcal{F}$.*
4. *The complement of $\mathcal{S}_\alpha$ is in $\mathcal{F}$, and the complement of $\mathcal{F}_\mu$ is in $\mathcal{S}$.*

The symmetry between $\mathcal{F}$ and $\mathcal{S}$ is further emphasized by equal cardinality:

$$|\mathcal{F}| = |\mathcal{S}|.$$

Relation 2 in Proposition 11 implies a commutative property of fiber projections:

$$\begin{aligned}\Pi(\mathcal{S}_\alpha, \Pi(\mathcal{S}_\beta, \mathcal{F}_\mu)) &= \Pi(\mathcal{S}_\alpha \cup \mathcal{S}_\beta, \mathcal{F}_\mu) \\ &= \Pi(\mathcal{S}_\beta, \Pi(\mathcal{S}_\alpha, \mathcal{F}_\mu)),\end{aligned}$$

or, in more compact notation,

$$\Pi_{\mathcal{S}_\alpha} \circ \Pi_{\mathcal{S}_\beta} = \Pi_{\mathcal{S}_\beta} \circ \Pi_{\mathcal{S}_\alpha}, \tag{12}$$

where the compact notation $\Pi_{\mathcal{S}_\alpha}(\mathcal{F}_\mu) \doteq \Pi(\mathcal{S}_\alpha, \mathcal{F}_\mu)$ is used. Moreover,

$$\begin{aligned} \Pi(\mathcal{S}_\alpha, \Pi(\mathcal{S}_\alpha, \mathcal{F}_\mu)) &= \Pi(\mathcal{S}_\alpha \cup \mathcal{S}_\alpha, \mathcal{F}_\mu) \\ &= \Pi(\mathcal{S}_\alpha, \mathcal{F}_\mu), \end{aligned}$$

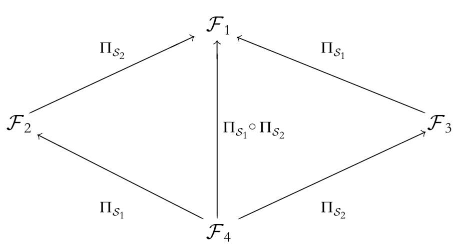
which shows that Π is indeed idempotent, as required for a projection.

DEFINITION 7: *A dynamical hierarchy is a doublet*

$$\{\mathcal{F}, \mathcal{S}\}_\Pi,$$

where $\mathcal{F}$ contains the elements of the hierarchy, that is, a set of self-contained dynamical subsystems, and $\Pi_{\mathcal{S}_\alpha}$, $\mathcal{S}_\alpha \in \mathcal{S}$, define projections between the dynamical subsystems in $\mathcal{F}$ ($\mathcal{S}$ is a set of invariant manifolds $\mathcal{S}_\alpha$, which defines the kernel of the projection $\Pi_{\mathcal{S}_\alpha}$).

A graphical representation of a dynamical hierarchy can be constructed as follows. Each member of $\mathcal{F}$ is represented by a node, and there is a directed edge from node $\mathcal{F}_\mu$ to node $\mathcal{F}_\nu$ if there exists a member $\mathcal{S}_\alpha \in \mathcal{S}$ such that $\Pi(\mathcal{S}_\alpha, \mathcal{F}_\mu) = \mathcal{F}_\nu$. The graph is typically not a treelike structure, since Equation 12 shows that



is a typical component of a dynamical hierarchy ($\mathcal{F}_2 = \mathcal{F}_1$ if $\mathcal{S}_2 \subset \mathcal{S}_1$).

According to relation 4 in Proposition 11, the complement of any member of $\mathcal{F}$ is a member of $\mathcal{S}$, and vice versa. This implies that a dynamical hierarchy has a dual representation, which can be graphically represented by members in $\mathcal{S}$ as nodes and the union operator ($\mathcal{S}_\alpha \cup \mathcal{S}_\beta$) defining the directed graphs between nodes. Sometimes the dual representation might be advantageous, especially since the whole structure of the dynamical hierarchy is defined by a set of group generators, $\hat{\mathcal{S}}_\alpha \in \hat{\mathcal{S}} \subseteq \mathcal{S}$ (any member of $\mathcal{S}$ can be generated through a sequence of applications of the generators on each other), with the following properties:

i. Abelian: $\hat{\mathcal{S}}_\alpha \cdot \hat{\mathcal{S}}_\beta = \hat{\mathcal{S}}_\beta \cdot \hat{\mathcal{S}}_\alpha$.

ii. Idempotent: $\hat{\mathcal{S}}_\alpha \cdot \hat{\mathcal{S}}_\alpha = \hat{\mathcal{S}}_\alpha$.

6 Discussion, Conclusions, and Outlook

In this article we have outlined a rather ambitious program for analyzing interesting properties of dynamical systems. While the original aim was to derive dynamical hierarchies, the technical exploration has shown it to be natural, if not unavoidable, to take an intermediate step and define interaction networks. The analysis is performed according to the following strategy:

dynamical hierarchy
↑
effective interaction network
↑
interaction network
↑ invariant manifolds
dynamical system

In the scheme above, the numerical computation of the interaction network, through the Jordan decomposition, is by far the most time-consuming step. There are two obvious ways to remedy this problem: develop more efficient techniques for calculating invariant manifolds, or through analytic arguments and numerical experiments derive typical statistical properties of interaction networks. The statistics should depend on the properties of the dynamical system, such as dimensionality, dissipation, locality, and symmetries. Of cause, to derive heuristics we are in need of the first approach for the numerical experiments. Once realistic interaction networks can be generated, deriving the corresponding effective interaction network and dynamical hierarchy is relatively straightforward.

Alternatively, we could study dynamical hierarchies in isolation. This approach is preferred when exploring the fundamental mathematical structure of dynamical hierarchies. This is in fact the motivation when defining a dynamical hierarchy as a doublet $\{\mathcal{F}, \mathcal{S}\}_\Pi$, and deriving the algebraic properties of such structures.

Lastly, general interaction networks have received enormous attention, especially from physicists. The properties of these networks were originally assumed to be random graphs [16, 5], and with connectivity as the only (order) parameter. Recently, however, much effort has been devoted to experimental, numerical, and analytical studies of more realistic structured networks, such as the internet, social networks, and gene regulatory networks. For a review see [13]. The statistical properties of the graph are still derived at a very high level. The interaction networks defined in this article are defined directly from an underlying dynamical system. In fact the approach outlined is more general: the nodes of the network are not defined a priori, but have to be identified through a nonlinear transformation of the degrees of freedom, or nodes, into Jordan form.

Even though it has not been stressed explicitly in this article, a dynamical hierarchy is a topological object, a diffeomorphic invariant, that is, its structure does not change under a smooth transformation of the degrees of freedom. While this is obvious from the construction, it could possibly have deep implications. In fact, the term *fiber projection* was chosen to reflect that the projection has to respect the fiber bundle structure of the manifold and the vector field as a section of the tangent bundle. The existence of fiber projections could possibly be rephrased in terms of an integrability condition directly related to the global structure of the phase space and the vector field. These theoretical explorations are currently a work in progress.

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