Phase Transition of Color Superconductivity and Cooling Behavior of Quark Stars

Keiji Yamaguchi,¹ Masaharu Iwasaki² and Osamu Miyamura¹,*

¹Department of Physics, Hiroshima University, Higashi-Hiroshima 739-3526, Japan
²Department of Physics, Kochi University, Kochi 780-8520, Japan

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We discuss color superconductivity and its effect on the cooling behavior of strange quark stars. The neutrino emissivity and specific heat of quark matter are calculated within the BCS theory. In the superconducting phase, the emissivity decreases and causes suppression of the cooling rate. It is shown that the phase transition leads to a sudden discontinuous suppression of the cooling rate in cooperation with the specific heat.

§1. Introduction

Recently, many authors have investigated color superconductivity in quark matter.¹ Although there is no evidence of its existence, it has been pointed out that it might exist in the core of neutron stars and in the central region of high-energy heavy-ion collisions.

What physical quantities must we observe in order to demonstrate the existence of color superconductors? In this paper, we discuss physical phenomena that are characteristic of color superconductivity. To this end, we focus our attention on the cooling behavior of hot strange quark stars.² Recent works have shown that quark matter is transformed from a Fermi gas into a superconducting fluid with the decrease of the temperature; there occurs a phase transition of second order at some critical temperature.³ Therefore it is expected that some physical quantities display singularities when the hot quark matter is cooled down. Such a singularity may be a signal for the existence of color-superconductors.

The cooling phenomenon in quark stars has been studied for many years by several authors.⁴ According to their results, only the weak interaction can release energy from such high-density quark matter. The simplest possible process is neutrino emission from quark matter, and, in particular, the pair of d quark β decay reactions are dominant (URCA):

\[
\begin{align*}
    d &\rightarrow u + e^- + \bar{\nu}_e, \\
    u + e^- &\rightarrow d + \nu_e.
\end{align*}
\]

The neutrino energy loss during an interval \(dt\) through this process is denoted \(C_m dt\). The constant \(C_m\) is called the neutrino emissivity. Noting that this quantity can also be expressed as \(C_V dT\), with the specific heat \(C_V\) of quark matter, we obtain a
differential equation determining the temperature as a function of time:
\[
\frac{dT}{dt} = -\frac{C_m(T)}{C_V(T)}.
\] (2)

This equation describes the cooling behavior of quark matter.

Recently much effort has been made in studying the influence of color superconductivity on cooling behavior. Studies to this time, however, have been restricted to low temperature regions and deal mainly with the specific heat. It is the purpose of this paper to investigate the effect of neutrino emissivity on the cooling behavior around the critical temperature.

In the next section, the mean-field (BCS) theory for color superconductivity is developed in such a way that it is suitable for the finite temperature mean field theory we consider presently. Then we calculate the neutrino emissivity and specific heat of quark matter in §3. Finally, in §4 numerical calculations are carried out for typical quark matter and some comments are given.

§2. BCS theory at finite temperature

First we develop the BCS theory in a three-flavor quark system at finite temperature. The Lagrangian is assumed to be given by
\[
\mathcal{L} = \bar{\Psi} (i \gamma \cdot \partial + \mu \gamma^0) \Psi + g \sum_a (\bar{\Psi} \gamma^\mu \lambda^a \Psi)(\bar{\Psi} \gamma^\mu \lambda^a \Psi),
\] (3)

where $\mu$ is the chemical potential and $\lambda^a$ denotes the color $SU(3)$ matrix. This effective Lagrangian comes from the one-gluon exchange interaction with infinite gluon mass due to the many-body effect in the medium. To study pair correlation, it is convenient to use the Fierz transformation,\(^5\)
\[
\mathcal{L} = \bar{\Psi} (i \gamma \cdot \partial + \mu \gamma^0) \Psi + \frac{2}{3} g \sum_{a,b} (\bar{\Psi} \bar{\gamma}^5 \lambda^a A^b \Psi)(\bar{\Psi} \gamma^{-1} \lambda^a A^b \Psi),
\] (4)

where $A^b$ denotes the flavor $SU(3)$ matrix. Here we have included only the most attractive terms: spin-singlet, color-antisymmetric and flavor-antisymmetric ($a, b = 2, 5, 7$) terms.

The partition function of our system is given by the functional integral
\[
Z = \text{Tr} \exp(-\beta \hat{H}) = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp(-\int d^4x \mathcal{L}),
\] (5)

where $\beta \equiv 1/T$ and $d^4x = d\tau d^3x$. Next we introduce the auxiliary field $\varphi(x)$ in terms of the identity:
\[
1 = \int \mathcal{D}\varphi^* \mathcal{D}\varphi \exp(-\kappa^2 \int d^4x |\varphi(x)|^2).
\] (6)

Using this equation, we have chosen the constant $\kappa$ in the above function in such a way that the four-Fermi interaction term is eliminated, and we obtain
\[
Z = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \mathcal{D}\varphi^* \mathcal{D}\varphi \exp\left[-\int d^4x \left(\bar{\Psi} (i \gamma \cdot \partial + \mu \gamma^0) \Psi + (\bar{\Psi} \gamma^5 \lambda^a A^b \Psi)(\bar{\Psi} \gamma^{-1} \lambda^a A^b \Psi) + \kappa^2 |\varphi(x)|^2\right)\right],
\] (7)
where the color-flavor matrix $T^\rho$ is defined from $\lambda^a \otimes \Lambda^b$. From the above equation it is clear that the auxiliary field is equivalent to a wave function of Cooper pairs.

Here we use a mean field approximation (BCS theory), assuming that the field $\varphi(x)$ is a constant:

$$\varphi_\rho(x) = \delta_{\rho,2} \delta_{\rho,2} \varphi_0.$$  \hspace{1cm} (8)

This expression means that our Cooper pairs are a $u-d$ spin-singlet with momenta $\pm k$. The reason that we have taken a $u-d$ pairing is that the mass of the $s$ quark is greater than those of $u$ and $d$ quarks.

To carry out the integrations, we expand the Fermi field $\Psi$ in a Fourier series as

$$\Psi(x) = \sum_n \sum_p \sum_s b_{p,s} u(p, s) \exp\{i(p \cdot x - \omega_n \tau)\},$$  \hspace{1cm} (9)

where the Matsubara frequency $\omega_n$ is defined as $(2n+1)\beta^{-1}$ and $s (\pm 1)$ is the eigenvalue of the quark helicity. Substituting this equation into the partition function, it can be represented in the matrix form

$$-S = \beta \sum_p \left[ \begin{array}{cc} b_{p,1}^\dagger & b_{-p,1}^\dagger \\ b_{p,-1} & b_{-p,-1}^\dagger \end{array} \right] \left( \begin{array}{cc} i\omega_n - \zeta_p & \Delta_0 \\ \Delta^*_0 & i\omega_n + \zeta_p \end{array} \right) \left( \begin{array}{c} b_{p,1} \\ b_{p,-1}^\dagger \end{array} \right) + \kappa^2 |\varphi_0|^2,$$

where the usual gap parameter is defined by $\Delta_0 \equiv -2\varphi_0 T^2 (T^2 \equiv \lambda^2 \otimes \Lambda^2)$. This Cooper pair is antisymmetric in spin, color and flavor spaces independently.

Carrying out the functional integrals of the quark fields, we obtain

$$Z = N \exp \left[ \sum_p \text{Tr} \log \beta^2 (-\omega_n^2 - \hat{E}_p^2) \right] - \beta \kappa^2 \Delta_0^2,$$  \hspace{1cm} (10)

where $\hat{E}_p$ is the energy matrix of the quasi-particle and given by

$$\hat{E}_p^2 = \zeta_p^2 + \Delta_0^2 (\lambda^2 \otimes \Lambda^2)^2.$$  \hspace{1cm} (12)

The quasi-particle has the following characteristics: All the $s$ quarks are gapless and have no pair correlation, the $u$ and $d$ quarks with color indices $1$ and $2$ have a finite gap, and the remaining particle is gapless. \hspace{1cm} (6)

From the partition function (7), we can easily calculate the thermodynamic potential,

$$\Omega = -\beta^{-1} \log Z = -4 \sum_p (E_p + 2\beta^{-1} \log(1 + \exp(-\beta E_p)))$$

$$-5 \sum_p (\xi_p + 2\beta^{-1} \log(1 + \exp(-\beta \xi_p))) + \frac{1}{4} \kappa^2 \Delta_0^2.$$  \hspace{1cm} (13)

The gap parameter can be determined by applying the stationary condition to this thermodynamic potential:

$$\frac{\partial \Omega}{\partial \Delta_0} = -4 \sum_p \frac{\Delta_0}{E_p} (1 - 2n_p) + \frac{\kappa^2}{2} \Delta_0 = 0,$$  \hspace{1cm} (14)

with $n_p \equiv (1 + e^{\beta E_p})^{-1}$, the distribution function of the quasi-particle.
§3. The neutrino emissivity and the specific heat

Now we are in a position to study the cooling behavior by using the above results. First we calculate the specific heat of the quark matter. To this end, we must derive the entropy of quark matter, which is written as

\[ S = -\left( \frac{\partial \Omega}{\partial T} \right)_{V,\mu} = -8 \sum_p [n_p \log n_p + (1 - n_p) \log(1 - n_p)] . \] (15)

Here we have omitted the contribution of anti-quarks for the sake of brevity. They will be taken into account in the numerical calculations discussed in the next section. From the entropy, we obtain the specific heat in the usual manner as

\[ C_V = T \left( \frac{\partial S}{\partial T} \right)_{V,\mu} . \] (16)

Note that the other specific heat \( C_p \) is nearly equal to \( C_V \) in the Fermi gas, which is a well-known result in statistical physics. The entropy does not contain the derivative of the gap variable, by virtue of the gap equation (14). Hence it is a continuous function of \( T \) across the critical temperature. The specific heat by contrast does contain the derivative of the gap variable, so that it is discontinuous as a function of the temperature. Such behavior is known to occur in electron superconductors. The sudden enhancement of the specific heat results from the formation of Cooper pairs, because an amount of heat supplied to the superconductor is used in the destruction of Cooper pairs instead of in causing thermal motion.

Next let us consider neutrino emissivity. It was calculated in quark matter in the normal phase many years ago. The result is

\[ C_m^0 = (\text{const})T^6 . \] (17)

The temperature dependence of the emissivity is easily understood. Let us investigate the quark URCA process (1). There are \( u \) and \( d \) quarks and electrons that are approximately degenerate with Fermi momentum \( p_F(i) \). Each Fermion gives one power of \( T \) from the phase space integral \( (d^3p_i \to P_F(i)^2dE_i \propto T) \). Thus we have \( T^3 \) from the \( u \) and \( d \) quarks and electrons. In addition, the phase space integral for the neutrino gives \( d^3 p_\nu \propto E_\nu^2dE_\nu \propto T^3 \). One power of \( T \) from the emitted neutrino energy, \( E_\nu \), cancels a factor \( T^{-1} \) from the energy-conserving \( \delta \) function. Since the momenta of the degenerate particles are restricted to lie close to their respective Fermi surfaces, the angular integrals give no temperature dependence. Altogether, we thus have \( C_m \propto T^6 \). The constant on the right-hand side of the above equation can only be obtained in a more detailed calculation. Since its value is not important for subsequent analysis, we do not consider it further.

This derivation of Eq. (17) is based on the assumption that each Fermi energy is much larger than the temperature: \( E_F \gg T \). Noting that \( p_F \propto \rho^{1/3} \), this condition should be approximately satisfied for high density quark matter near the critical temperature considered in this paper.

Our interest is now in the modification of the emissivity due to color superconductivity and its influence on the cooling behavior. Again we consider the weak
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There are two kinds of $d$ quarks, (Cooper) paired particles and unpaired particles. For the former type, the $d$ quark might decay in the following mode:

$$(ud) \rightarrow u + u + e^- + \bar{\nu}_e.$$  \hspace{1cm} (18)

However, noting that $\Delta_0 \sim 100 \text{ MeV} < T_c$, this process is prohibited by the energy conservation law. Therefore, the $\beta$-decay is suppressed by the formation of Cooper pairs. The rate is estimated to be

$$r \simeq \frac{\text{number of unpaired } d \text{ quarks}}{\text{total number of } d \text{ quarks}}.$$  \hspace{1cm} (19)

The second process of (1),

$$(ud) + e^- \rightarrow d + d + \nu_e,$$  \hspace{1cm} (20)

is suppressed in the same manner as the first one. Consequently, the neutrino emissivity in the super phase is given by

$$C_{\text{em}} = r^2 C_{\text{em}}^{(0)}.$$  \hspace{1cm} (21)

According to the usual BCS theory,\(^9\) the rate of the unpaired $d$ quark is given by the following equation:

$$r = \frac{1}{3} - \frac{2}{Tk_F^4} \int_0^\infty k^4 \frac{\partial n_k}{\partial E_k} dk.$$  \hspace{1cm} (22)

The first term here comes from the contribution of the gapless $d$ quarks, and the second comes from those of the unpaired $u$ and $d$ quarks. As a result, the neutrino emissivity is suppressed by the pairing correlation.

§4. Numerical results

A numerical calculation has been carried out with a chemical potential $\mu = 500 \text{ MeV}$. If the coupling constant $g$ is taken as $1 \text{ GeV}$, the resulting gap energy becomes about $100 \text{ MeV}$, which is a typical value used in other works. It is assumed that the emissivity (21) derived in the previous section is valid under these conditions.

First, the behavior of the specific heat is plotted in Fig. 1. The large-dotted curve represents the result in the super phase and the small-dotted curve that in the normal phase. We can see a discontinuity at the critical temperature, indicating a phase transition of second order. Below the critical temperature, the specific heat decreases with an exponential curve. Contrastingly, it is linear in the normal phase, so that the cooling rate in the super phase becomes larger than that in normal phase at lower temperature.

The rate of unpaired particles defined by Eq. (22) is shown in Fig. 2. Since Cooper pairs are formed in the super phase, the number of the unpaired quarks decreases below the critical temperature. Substituting the specific heat and emissivity considered above into Eq. (2), the cooling rate can be calculated, and the result is displayed in Fig. 3 as a function of $T$. We can understand the pairing effects on the
cooling rate as follows. First, a sudden suppression of the cooling rate occurs across the critical temperature. This suppression is strengthened by the emissivity. Below the critical temperature, the decrease of the emissivity causes overall suppression of the cooling rate, in particular at lower temperature.

We end by giving some comments. The sudden suppression of the cooling rate may provide a manifest signal of color superconductivity in quark matter if we could determine the temperature in the core of the stars in some way. However, this is a very difficult task at present. Another possible method is the direct measurement of the decrease of the neutrino emissivity, such as a measurement of the neutrino energy from a supernova. But this would also be very difficult. In any case, the observation of such a signal is a future problem.

The second comment regards the other cooling process in which gluons excited

![Fig. 1. The specific heat of quark matter as a function of temperature. The large-dotted (small-dotted) curve represents the result in the super (normal) phase.](image1)

![Fig. 2. The rate of unpaired $d$-quarks squared as a function of temperature. The rate is defined by Eq. (22).](image2)

![Fig. 3. The cooling rate $-dT/dt = C_m(T)/C_V(T)$ as a function of temperature.](image3)
in hot quark matter decay into neutrino-antineutrino pairs:  

\[ \text{gluon} \rightarrow q + \bar{q} \rightarrow Z_0 \rightarrow \nu + \bar{\nu}. \]  

(23)

Here, \( q + \bar{q} \) represents a particle-hole state in the quark matter. This process seems to give large contribution to the cooling. In the super phase, however, the particle-hole state should be replaced by the two-quasi-particle state whose excitation energy is larger than \( 2\Delta_0 \). Noting that the gluon energy is smaller than \( T_c < 2\Delta_0 \), this process is prohibited by the energy conservation law. Therefore, this cooling process is suppressed by the formation of Cooper pairs and may be ignored in investigations of the cooling behavior.

In conclusion, the neutrino emissivity decreases on account of the color superconductivity and causes suppression of the cooling rate. This phase transition leads to a sudden discontinuous suppression of the cooling rate in cooperation with the specific heat.

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