A Phase Equation of Third-Order in Spatial Derivatives

Yuji MASUTOMI and Kazuhiro NOZAKI

Department of Physics, Nagoya University, Nagoya 464-8602, Japan

(Received October 12, 2001)

We derive a phase equation containing terms of third-order in spatial derivatives. In this equation, a nonlinear dissipative term with a third-order derivative suppresses the lowest-order diffusive instability, in cooperation with the other terms of third-order in spatial derivatives. We find an exact shock solution of the phase equation, whose upstream and downstream states are stable, and a periodic solution stable with respect to modulation that is realized through a Hopf bifurcation.

§1. Introduction

Various phase equations are known to describe the slow evolution of large-scale modulations of periodic patterns. A phase diffusion equation was first derived in the context of convection by Pomeau and Manneville.\(^1\) Since then, many phase equations have been derived and studied in various contexts.\(^2\)

If the coefficient of the diffusive term in a diffusive phase equation passes through zero into the unstable regime, it becomes necessary to add higher-order terms to control the dynamics. Kuramoto presented a classification of such higher-order equations based on symmetry and scaling arguments.\(^3\) As asymptotic equations, various higher-order phase equations can be derived by means of the renormalization group (RG) method.\(^4\) The celebrated higher-order phase equation called the Kuramoto–Sivashinsky equation\(^5\),\(^6\) consists of an unstable diffusive term, a nonlinear phase gradient term of lowest order, and a linear term of fourth-order in the spatial derivative in the transverse direction to the underlying periodic pattern, which stabilizes the lowest-order diffusive instability. In Benney-type equations, a linear fourth-order spatial derivative along the underlying periodic pattern appears as a stabilizing higher-order term.\(^7\),\(^8\),\(^4\) In some cases, nonlinear diffusion terms should be introduced as the other stabilizing terms. Terms that may play such a role are nonlinear diffusive terms with third-order derivatives. Such terms were first introduced in the case of stationary patterns like convective cells.\(^3\) There, the quadratic phase gradient term exactly vanishes, and a nonlinear diffusive term of third-order in the spatial derivative appears as a dominant nonlinear term, while a linear fourth-order derivative term acts as a stabilizing higher-order term. This equation is called the “modified Kuramoto equation” here.

In this paper, we study the case in which the nonlinear diffusive term of third-order in the spatial derivative stabilizes the diffusive instability without the aid of the linear fourth-order derivative. When the frequency of the underlying periodic pattern does not vanish identically, it is possible that the coefficient of the quadratic phase gradient term becomes sufficiently small but finite that the quadratic phase
gradient term is as small as the nonlinear diffusive term with a third-order derivative. In this case, a linear dispersive term of third-order in the spatial derivative is usually dominant over the linear dissipative term of fourth order. Thus, we reach a new type of phase equation that consists of the lowest-order unstable diffusive term, the linear dispersive term with a third-order derivative, the nonlinear diffusive term with a third-order derivative, the quadratic phase gradient term and/or a cubic phase gradient term. This phase equation is called a “phase equation with third-order derivatives” in this paper, since all spatial derivatives up to third-order are included, with no derivatives of higher order. It is mentioned that there are some numerical studies of phase equations with linear dissipative terms of fourth order or other higher-order nonlinear diffusion terms, in addition to all third-order terms in the equation considered presently.\textsuperscript{9,10} However, such equations including various-order terms are questionable, and it is not clear which terms are responsible for controlling the lowest order negative diffusion term. Our phase equation is self-consistent in the sense of asymptotic perturbation theory, and the only possible stabilizing term is the nonlinear diffusive term with a third-order derivative. This nonlinear diffusive term is shown to stabilize the lowest-order diffusive instability in cooperation with the linear dispersive term with a third-order derivative, and the quadratic nonlinear phase gradient term (or the cubic nonlinear phase gradient term). In fact, we obtain an exact shock solution of the phase equation with third-order derivatives, whose upstream and downstream states are stabilized by combined effects of various terms with third-order derivatives. The existence of a periodic solution near a uniform state is proved using the theory of normal form. This periodic solution is shown to be realized as a result of a Hopf bifurcation, which occurs simultaneously with the diffusive instability of the underlying pattern, and we obtain a stability criterion for the periodic solution with respect to a modulational perturbation.

In the next section, we give a phenomenological derivation of the phase equation with third-order spatial derivatives. Its derivation from a general reaction-diffusion system is presented in the Appendix using an asymptotic perturbation theory. In §§3–5, we derive a shock solution and a periodic solution and find some stability conditions for them. In §6, suitable boundary conditions for general phase equations are investigated, making use of a covariant property of phase equations.

§2. Derivation of phase equation

Let us consider a general reaction-diffusion system of equations:

\[
\partial_t U = F(U) + D \nabla^2 U. \tag{2.1}
\]

Here \(U\) is an \(n\)-dimensional vector, \(D\) is an \(n \times n\) constant matrix, and \(F\) is some nonlinear function of \(U\). Suppose (2.1) has a spatially and temporally periodic solution \(U_0 = U_0(k, kx - \omega(k)t + \phi)\) satisfying

\[
-(\omega U_{0,\theta} + F(U_0) + k^2 D U_{0,\theta\theta}) = 0, \tag{2.2}
\]

where \(k\) and \(\phi\) are arbitrary constants, \(\omega(k)\) is a definite function of \(k\), \(\theta = kx - \omega t + \phi\) (where the suffix \(\theta\) denotes differentiation with respect to \(\theta\)), and \(U_0\) is a \(2\pi\) periodic
function of $\theta$.

Here, we derive phase equations using simple scaling arguments. The detailed derivation based on an asymptotic theory, the RG method, is given in the Appendix. Let us assume, without loss of generality, that the slow evolution of the large-scale modulation of the phase $\phi$ is described by the phase equation

$$\phi_t = f(\phi_{xx}, \nabla^2\phi, \nabla \phi_x, (\phi_x)^2, |\nabla \phi|^2, \phi_{xxx}, \cdots),$$

(2.3)

where $\phi(x, y, z, t)$ is a slowly varying phase, $\phi_x = \partial_x \phi$, $\phi_{xx} = \partial^2_x \phi$, and $\nabla \equiv (0, \partial_y, \partial_z)$. This phase equation should retain reflection symmetry in the $yz$ transverse direction, which the original reaction-diffusion system preserves, so that it is invariant under $\nabla \rightarrow -\nabla$. For this reason, we remove such terms as $\nabla \phi_x$ from $f$ in Eq. (2.3). There is no symmetry in the longitudinal $x$ direction, due to the emergence of a symmetry-breaking pattern. Introducing suitable scales of space, time and phase, we derive various phase equations. Balancing the lowest nonlinear terms and the diffusive terms in the longitudinal and transverse directions, we obtain the anisotropic Burgers equation,

$$\phi_t = D_{||} \phi_{xx} + D_\perp \nabla^2 \phi + N_{||}(\phi_x)^2 + N_{\perp}|\nabla \phi|^2,$$

(2.4)

where the scalings are $\partial_x \sim \nabla_\perp \sim O(\epsilon)$ and $\phi \sim O(\epsilon^0)$. The explicit expressions of the coefficients of Eq. (2.4) are given in the Appendix and in Ref. 4). Suppose that both $D_{||}$ and $N_{||}$ are $O(\epsilon)$, while $\partial_k D_{||}$ and $\partial_k N_{||}$ are $O(\epsilon^0)$. Then we obtain a phase equation with third-order spatial derivatives,

$$\phi_t = D_{||} \phi_{xx} + A \phi_{xxx} + N_{||}(\phi_x)^2 + N_{||}(\phi_x)^3 + H \phi_x \phi_{xx},$$

(2.5)

where $\partial_x \sim O(\epsilon)$, $\phi \sim O(\epsilon^0)$, while $\nabla_{\perp} \sim O(\epsilon^n)$, with $n > 3/2$, is assumed so that $(y, z)$-dependence of $\phi$ is ignored for simplicity. In the Appendix, we give the derivation of Eq. (2.5) by means of the RG method and the explicit expressions of its coefficients. As shown in the following sections, terms with third-order spatial derivatives, such as $\phi_x \phi_{xx}, \phi_{xxx}$ and $(\phi_x)^3$ [or $(\phi_x)^2$], stabilize a system with a negative diffusive coefficient $D_{||}$, even in the absence of the fourth-order spatial derivative $\phi_{xxxx}$.

Applying a suitable scale transformation, Eq. (2.5) can be rewritten as

$$\phi_t = a \phi_{xx} + b \phi_{xxx} + c (\phi_x)^2 + d (\phi_x)^3 + e \phi_x \phi_{xx},$$

(2.6)

where all of the coefficients are $O(\epsilon^0)$, given by

$$a = D_{||}/\epsilon, \quad c = -\omega/(2\epsilon), \quad b = A, \quad d = -\partial_k \omega/6, \quad e = \partial_k D_{||}.$$

(2.7)

§3. **Shock solution**

In terms of a new dependent variable $u(x, t) = \phi_x(x, t)$, Eq. (2.6) can be rewritten as

$$u_t = au_{xx} + bu_{xxx} + c(u^2)_x + d(u^3)_x + (e/2)(u^2)_{xx},$$

(3.1)
The finite truncation of the WTC singular manifold expansion\(^\dagger\) yields the Cole-Hopf transformation to the variable \(\psi\) defined by

\[
u = h(\ln\psi)_x,
\]

where

\[
dh^2 - eh + 2b = 0 \quad \text{and} \quad e^2 - 8bd \geq 0.
\]

(3.3)

The Cole-Hopf transformation (3.2) transforms Eq. (3.1) or Eq. (2.6) into the bilinear form

\[
\psi(\psi_t - a\psi_{xx} - b\psi_{xxx}) + \psi_x(\tilde{a}\psi_x + \tilde{b}\psi_{xx}) = 0,
\]

(3.4)

where

\[
\tilde{a} = a - ch, \quad \tilde{b} = 3b - eh.
\]

(3.5)

It is easy to see that the bilinear equation (3.4) has exact solutions forming a one-parameter family

\[
\psi = e^{Kx + \Omega(K)t}(1 + e^{kx + \omega(k)t + \delta}),
\]

(3.6)

\[
K = -\frac{\tilde{a} + \tilde{b}k}{2b}, \quad \Omega(K) = (a - \tilde{a})K^2 + (b - \tilde{b})K^3,
\]

(3.7)

\[
\omega(k) = \dot{\Omega}k + \frac{\ddot{\Omega}}{2}k^2 + (b - \tilde{b})k^3,
\]

(3.8)

\[
\dot{\Omega} = \frac{d\Omega}{dK}, \quad \ddot{\Omega} = \frac{d^2\Omega}{dK^2},
\]

(3.9)

where \(k\) is an arbitrary constant. From Eqs. (3.2) and (3.6), we obtain a shock solution for \(\nu\):

\[
u = \frac{h}{2}\left\{ -\frac{\tilde{a}}{b} + k\tanh\left(\frac{kx + \omega(k)t + \delta}{2}\right) \right\}.
\]

(3.10)

For \(x \to \pm \infty\), we have upstream and downstream values \(U_{\pm}\) of the shock,

\[
U_{\pm} = -\frac{h}{2}\left( \frac{\tilde{a}}{b} \pm k \right).
\]

(3.11)

Let us examine the linear stability of the upstream and downstream states \(\nu = U_{\pm}\). Substituting \(\nu = U_{\pm} + \tilde{v}(x, t)\) into Eq. (2.6), we have the linearized equation for \(\tilde{v}(x, t)\)

\[
\tilde{v}_t = (2cU_{\pm} + 3dU_{\pm}^2)\tilde{v}_x + (a + eU_{\pm})\tilde{v}_{xx} + b\tilde{v}_{xxx}.
\]

(3.12)

The asymptotic stability conditions of the constant states \(\nu = U_{\pm}\) are

\[
a + eU_{\pm} = a - \frac{eh}{2}\left( \frac{\tilde{a}}{b} \pm k \right) > 0,
\]

(3.13)

which restrict the values of \(k\). For example, when \(d = 0\), Eq. (3.13) reads

\[
-\frac{2bc}{e} < bk < \frac{2bc}{e},
\]

(3.14)
where the following condition is necessary:

$$0 < \frac{bc}{e}. \quad (3.15)$$

Note that the bilinear equation (3.4) reduces to the linear equation

$$\psi_t = a\psi_{xx} + b\psi_{xxx} \quad (3.16)$$

if the coefficients of Eq. (2.6) satisfy the special relations $\tilde{a} = 0$ and $\tilde{b} = 0$.

§ 4. Periodic solution

In this section, we prove the existence of a periodic solution of Eq. (2.6) by means of the normal form theory. Introducing new coordinates as $x \rightarrow x - \lambda t$, $t \rightarrow t$ and the transformation

$$\phi \rightarrow \phi - \Lambda x - \Omega(K)t, \quad (4.1)$$

we rewrite Eq. (2.6) as

$$\phi_t = \lambda' \phi_x + a'\phi_{xx} + b\phi_{xxx} + c' (\phi_x)^2 + d(\phi_x)^3 + e\phi_x \phi_{xx}, \quad (4.2)$$

where

$$a' = a + eK, \quad c' = c + 3dK, \quad (4.3)$$

and $\Omega(K) = \lambda K + cK^2 + dK^3$, $\lambda' = \lambda + 2cK + 3dK^2$ and $\lambda$ is an arbitrary constant.

In the steady state (\partial_t = 0) and for $b \neq 0$, Eq. (4.2) is transformed into the following ordinary differential equation in $(u = \phi_x, v = u_x)$:

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\lambda'/b & -a'/b \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -(c'/b)u^2 - (d/b)u^3 - (e/b)uv \end{pmatrix}. \quad (4.4)$$

We seek a periodic solution of Eq. (4.4) around the origin $(u, v) = (0, 0)$. In order to transform Eq. (4.4) into Jordan normal form, we introduce the linear coordinate transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \quad (4.5)$$

where $\alpha + \beta i$ is an eigenvalue of the constant matrix in Eq. (4.4) given by

$$\alpha = -\frac{a'}{2b}, \quad \beta = \sqrt{\frac{\lambda'}{b} - \alpha^2} \quad (\frac{\lambda'}{b} > \alpha^2). \quad (4.6)$$

Then, Eq. (4.4) reduces to the normal form

$$\begin{pmatrix} q_x \\ p_x \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} -(e/b)qp - d/(b\beta)p^3 - c'/(b\beta)p^2 - e\alpha/(b\beta)p^2 \\ 0 \end{pmatrix}. \quad (4.7)$$
From the normal form theory,\textsuperscript{12) we reach the conditions that (4.7) has a periodic solution near \((q, p) = (0, 0)\),

\[
d'c'e/b > 0, \quad \lambda'/b > \alpha^2.
\] (4.8)

The last inequality in (4.8) restricts the range of \(\lambda\). This type of periodic solution will be shown to emerge as a result of a Hopf bifurcation of a uniform state of \(u = \phi_x\) in the next section.

§5. Hopf bifurcation

We now show that a Hopf bifurcation produces the periodic solution of Eq. (4.2) whose existence is proved in the preceding section. We also obtain a stability criterion of the periodic solution with respect to modulational perturbations. Our main task in this section is to derive an equation for the slowly varying amplitude of a periodic solution of Eq. (4.2) near the bifurcation point \(a' \sim O(\delta^2)\), where \(\epsilon \ll \delta < 1\). The method of the derivation is based on the perturbative RG reduction.\textsuperscript{13) Expanding \(\phi\) in terms of the small parameter \(\delta\) as

\[
\phi = \delta \phi_1 + \delta^2 \phi_2 + \cdots, \tag{5.1}
\]

at first order, we have a periodic solution,

\[
\phi_1 = A e^{i\Theta} + \text{c.c.}, \tag{5.2}
\]

where \(A\) is a complex constant and

\[
\Theta = \kappa x, \quad \kappa^2 = \lambda'/b(> 0). \tag{5.3}
\]

The second-order equation is

\[
\{\partial_t - b\kappa^3(\partial_\Theta^2 + 1)\partial_\Theta\} \phi_2 = \mathcal{L} \phi_2 = c' \kappa^2 (\phi_{1,\Theta})^2 + \frac{e\kappa^3}{2} (\phi_{1,\Theta})^2_\Theta, \tag{5.4}
\]

from which we have

\[
\phi_{2,\Theta} = -\frac{i e\kappa + c'}{3b\kappa} A^2 e^{2i\Theta} + \text{c.c.} - \frac{2c'}{b\kappa} |A|^2. \tag{5.5}
\]

The third-order perturbed field \(\phi_3\) is governed by

\[
\mathcal{L} \phi_3 = \left(\frac{a'}{\delta^2}\right) \kappa^2 \phi_{1,\Theta\Theta} + 2c' \kappa^2 \phi_{1,\Theta}\phi_{2,\Theta} + d\kappa^3 (\phi_{1,\Theta})^3 + e\kappa^3 (\phi_{1,\Theta}\phi_{2,\Theta})_\Theta
\]

\[
= -\kappa^2 \left(\frac{a'}{\delta^2} + \beta |A|^2\right) A e^{i\Theta} + \text{c.c.} + \text{non-secular terms}, \tag{5.6}
\]

where \(\beta = \beta_r + i\beta_i, \quad \beta_r = -c'e/b, \quad \beta_i = (10c'^2 + e^2\kappa^2 - 9b\kappa^2)/3b\kappa\). The secular solution of Eq. (5.6) is expressed as

\[
\phi_3 = P(\Theta, t) e^{i\Theta} + \text{c.c.}, \tag{5.7}
\]
where \( P(\Theta, t) \) is a secular solution of
\[
\{ \partial_t + 2b\kappa^3 \partial_\Theta - 3ib\kappa^3 \partial^2_\Theta - b\kappa^3 \partial^3_\Theta \} P \equiv \dot{L}P = -\kappa^2 \{(a'/\delta^2) + \beta|A|^2\}A. \tag{5.8}
\]
Here, \( \dot{L} \) is called a proto-RG operator. In order to remove the secular term \( P(\Theta, t) \) in \( \phi_3 \) by renormalizing the complex amplitude \( A \), we introduce the renormalization transformation
\[
\tilde{A}(\Theta, t) = A + \delta^2 P(\Theta, t). \tag{5.9}
\]
Following the proto-RG approach,\(^{14}\) we apply the proto-RG operator \( \dot{L} \) to Eq.\,(5.9). Then, taking into account Eq.\,(5.8) and \( A = \tilde{A} + O(\delta^2) \), we obtain the following proto-RG equation for the renormalized amplitude \( \tilde{A} \):
\[
(\partial_t + 2b\kappa^3 \partial_\Theta - 3ib\kappa^3 \partial^2_\Theta - b\kappa^3 \partial^3_\Theta) \tilde{A} = \kappa^2 (-a' - \delta^2 \beta|\tilde{A}|^2)\tilde{A}. \tag{5.10}
\]
Introducing new coordinates as \( \Theta \rightarrow \Theta - 2b\kappa^3 t, \ t \rightarrow t \) and supposing that \( \partial^2_\Theta \gg \partial^3_\Theta \), due to large-scale modulation, we obtain the complex Ginzburg-Landau (cGL) equation:
\[
(\partial_t - 3ib\kappa^3 \partial^2_\Theta) \tilde{A} = \kappa^2 (-a' - \beta \delta^2 |\tilde{A}|^2)\tilde{A}. \tag{5.11}
\]
The cGL equation (5.11) has a simple sinusoidal solution under the condition
\[
-a'\beta_r = \frac{a'c'e}{b} > 0, \tag{5.12}
\]
which agrees with Eq. (4.8). A supercritical Hopf bifurcation occurs when
\[
a' < 0, \quad \beta_r = -c'e/b > 0. \tag{5.13}
\]
Thus, we find a periodic solution that is stable with respect to uniform perturbation. In the case of the modified Kuramoto equation, the bifurcation is always subcritical, and a stable periodic solution does not exist near the bifurcation point.\(^{15}\) The uniformly oscillating solution of the cGL equation (5.11) is modationally stable if the Newell criterion\(^{16}\) \( 3b\kappa^3 \beta_i > 0 \) is satisfied, i.e. if
\[
(e^2 - 9bd)\kappa^2 + 10c'^2 > 0, \tag{5.14}
\]
which gives the condition that the periodic solution with an arbitrary wavenumber \( \kappa \) is modationally stable as
\[
a' < 0, \quad c'e/b < 0, \quad (e^2 - 9bd) \geq 0. \tag{5.15}
\]
Therefore, any periodic solution near the Hopf bifurcation point is stable if this condition is met. Note that the condition (5.15) is satisfied even if either of the coefficients \( c \) or \( d \) of the nonlinear phase gradient terms vanishes. The assumption that \( |a'| = |a + cK| \sim O(\delta^2) < 1 \) may impose a considerable restriction on \( K \), the uniform component of \( u = \phi_x \). However, as we see in the following and in the next section, the value of \( K \) can be adjusted so that \( |a'| = |a + cK| < 1 \) by slightly resetting the value of \( k \) (the wave number of the underlying pattern).
It is found that the bifurcation parameter of the Hopf bifurcation in Eq. (5.11) is the same as that of the diffusive instability of the original phase equation (4.2). From Eqs. (2.7) and (4.3), we have

\[ a' = a(k) + \epsilon K = D_\| (k) / \epsilon + \dot{D}_\| (k) K \approx D_\| (k + \epsilon K) / \epsilon = a(k'), \quad (5.16) \]

\[ c' = c(k) + 3dK = -\ddot{\omega}(k) / (2\epsilon) - \partial_k \ddot{\omega}(k) K / 2 \approx -\ddot{\omega}(k') / (2\epsilon) \approx c(k'), \quad (5.17) \]

where \( k' = k + \epsilon K \) is a slightly modified wavenumber of the underlying pattern. Here, \( K \) or \( k' \) plays the role of a bifurcation parameter for both equations (4.2) and (5.11). This fact can also be understood through the general covariant property of the phase equation, as discussed in the next section.

Therefore, as soon as the diffusive instability of the underlying periodic pattern sets in, a periodically modulated phase pattern emerges, as a result of the supercritical Hopf bifurcation, if the condition (5.15) is satisfied.

§6. Covariant property

As seen in previous sections, the value of a uniform component of \( u = \phi_x \) plays a crucial role in the stability of solutions. Furthermore, the boundedness of general solutions may also depend on this value. Let us impose periodic boundary conditions with period \( L \). Then \( \bar{u} \equiv (1/L) \int_0^L u dx \) is a constant of motion, which is chosen arbitrarily. However, the arbitrariness of \( \bar{u} \) is unified with another arbitrariness inherent in all types of phase equations.

The phase \( \phi \) is defined through the underlying periodic solution \( U_0(k, kx - \omega(k)t + \phi) \). Since the wavenumber \( k \) is arbitrary, at least in some range, the phase equation should be covariant with \( k \):

\[ \phi_t = F(k, \phi_x, \phi_{xx}, \cdots), \quad (6.1) \]

\[ \phi'_t = F(k', \phi'_x, \phi''_x, \cdots), \quad (6.2) \]

\[ kx - \omega(k)t + \phi = k'x - \omega(k')t + \phi'. \quad (6.3) \]

If \( k \) is changed slightly to \( k' = k + K, \ (|K| \ll |k|) \), Eq. (6.3) gives

\[ \phi' = \phi - Kx + \Omega(K)t, \quad \Omega(K) = \omega(k)K + \frac{\ddot{\omega}(k)}{2}K^2 + \cdots, \quad (6.4) \]

or

\[ u = u' + K, \quad \phi_t = \phi'_t - \Omega(K), \quad (6.5) \]

where \( u' = \phi'_x \), and \( K \) is an arbitrary constant \( (|K| \ll |k|) \). Thus, a change in the uniform component \( \bar{u} \) is interpreted as a change in \( k \) in general. If \( F(k', \cdots) \approx F(k, \cdots) \) is satisfied, as is often the case, the difference in the uniform component \( \bar{u} \) does not yield a significant change in the dynamics. However, in the case of the present phase equation (4.2), \( \bar{u} \) (or \( k' \)) is a bifurcation parameter, as shown explicitly in the previous section. Even in this case, by virtue of the arbitrariness of \( K \), we can choose an appropriate phase equation for \( \phi' \) so that the value of the uniform component of \( u' \) is set to a desirable value, e.g. \( \bar{u}' = 0 \).
The value of $\bar{u}$ has a close connection with the boundary conditions for $\phi$ according to

$$\bar{u} = (1/L) \int_0^L \phi x dx = (1/L) (\phi(L) - \phi(0)).$$

(6.6)

If $\phi$ has the same periodic boundary conditions, we have

$$\phi(L) - \phi(0) = 2\pi n = L \bar{u},$$

(6.7)

where $n$ is an integer. Therefore, the boundary condition for $\phi$ is fixed by choosing $\bar{u}$. Demonstrating the boundedness of general solutions of the original phase equation (2.6) under the fixed boundary conditions (6.7) is an interesting future problem.

§7. Summary

When a diffusive instability due to a dissipative term with a second-order derivative occurs in a phase equation, a dissipative term with a fourth-order derivative is usually introduced to stabilize the instability.\(^3\) Here, we derived a phase equation with third-order spatial derivatives instead of a fourth-order spatial derivative and presented another type of stabilizing mechanism, in which a nonlinear dissipative term with a third-order derivative suppresses the lowest-order instability in cooperation with a linear dispersive term and a small quadratic nonlinear phase gradient term or a cubic nonlinear phase gradient term.

Using the Cole-Hopf transformation the phase equation with third-order spatial derivatives can be reduced to a bilinear equation, from which we obtained an exact shock solution of the phase equation. We found a condition under which the upstream and downstream states of the shock are stable. This ‘stable’ shock persists even if the cubic nonlinear phase gradient term vanishes.

We derived existence conditions for a periodic solution near a constant wavenumber using the normal form theory. The RG method can be used to show that a periodic solution is produced as a result of a supercritical Hopf bifurcation, which coincides with the diffusive instability of the underlying periodic pattern. We obtained a stability criterion of the periodic solution with respect to modulational perturbations.

It was shown that an appropriate value of a uniform component of the wavenumber can be chosen by virtue of a covariant property of the phase equation. This choice fixes the boundary conditions for the phase $\phi$ and may allow boundedness conditions to be relaxed for general solutions of the present phase equation.

Acknowledgements

We would like to thank the members of R-lab at Nagoya University for fruitful discussions. The present work is, in part, supported by the Japan Society for Promotion of Science Grant-in-Aid for Scientific Research (C) 13640402.
Appendix

We now derive Eq. (2.5) using the perturbative RG method, following a procedure similar to that used as in Ref. 13. The diffusion coefficient $D_\parallel$ and the nonlinear coefficient $N_\parallel$ are given in terms of the adjoint null-function ($\bar{U}$) of the linearized operator $L$ of Eq. (2.1) as

$$D_\parallel = \frac{\langle \bar{U} \cdot (DU_{0,\theta} + MU_{0,k}) \rangle}{\langle \bar{U} \cdot U_{0,\theta} \rangle},$$  \hspace{1cm} (A.1)

$$N_\parallel = -\frac{\bar{\omega}}{2}.$$  \hspace{1cm} (A.2)

Both $D_\parallel$ and $N_\parallel$ are assumed to be $O(\epsilon)$, so that

$$D_\parallel(k) \propto \langle \bar{U} \cdot (DU_{0,\theta} + MU_{0,k}) \rangle \sim O(\epsilon),$$  \hspace{1cm} (A.3)

$$N_\parallel(k) \propto \bar{\omega} \sim O(\epsilon),$$  \hspace{1cm} (A.4)

while $\partial_k D_\parallel$ and $\partial_k N_\parallel$ are assumed to be $O(\epsilon^0)$. Equation (A.1) implies that there is periodic vectors $V(k, \theta)$ such that

$$LV(k, \theta) = DU_{0,\theta} + MU_{0,k} - D_\parallel U_\theta.$$  \hspace{1cm} (A.5)

Differentiating Eq. (A.5) with respect to $\theta$ and $k$, we have

$$LV_\theta = DU_{0,\theta\theta} + MU_{0,\theta k} + F'U_{0,\theta}V - D_\parallel U_{0,\theta\theta},$$  \hspace{1cm} (A.6)

$$LV_k = 3DU_{0,\theta k} + MU_{0,kk} + F''U_{0,k}V + MV_{\theta} + \bar{\omega}U_{0,k}$$

$$-\partial_k D_\parallel U_{0,\theta} - D_\parallel U_{0,\theta k}.$$  \hspace{1cm} (A.7)

We seek a secular solution close to $U_0(k, \theta)$ in the form

$$U = U_0(k + \kappa, \theta + \delta) + \bar{U}(\theta, x, r_\perp, t),$$  \hspace{1cm} (A.8)

where $\delta = \delta(x, r_\perp, t)$ and $\kappa = \kappa(x, r_\perp, t)$ are small secular deviations from the constant phase $\phi$ and the wavenumber $k$, respectively, and $\bar{U}$ represents a small perturbed fields that modifies the 0-th order field pattern $U_0$ and is not expressed in terms of differentials of $U_0(k, \theta)$. Let us expand $\delta, \kappa$ and $u$ as

$$\delta = \epsilon(P_1 + \epsilon P_2 + \cdots), \quad \kappa = \epsilon(P_1x + \epsilon P_2x + \cdots)$$  \hspace{1cm} (A.9)

$$\bar{U} = \epsilon^2(\bar{U}_{2}(\theta) + \epsilon \bar{U}_{3}(\theta, t, x, r_\perp) + \cdots).$$  \hspace{1cm} (A.10)

where $P_j (j = 1, 2, \cdots)$ are polynomials of $(x, r_\perp, t)$ that have increasing degrees with $j$, so that polynomial secular terms are renomalizable in the sense of the Lie approach of the RG method. \(^{13}\)

We obtain the following secular terms up to $O(\epsilon^3)$:

$$P_1 = P_{1,x}x, \quad P_2 = P_{2,xx}x^2 + \nabla_\perp P_2r_\perp,$$  \hspace{1cm} (A.11)

$$P_3 = P_{3,xxx}x^3/3! + x(r_\perp \cdot \nabla_\perp)P_3 + P_{3,t}t,$$  \hspace{1cm} (A.12)

$$\bar{U}_{2}(\theta) = R_2V(k, \theta), \quad \bar{U}_{3} = R_3(x)V(\theta) + \bar{U}_{3}(\theta).$$  \hspace{1cm} (A.13)
Here $R_2 = P_{2,xx}$, $R_3 = P_{3,xxx}$, and all other coefficients of the polynomials are arbitrary constants. Thus, we obtain the renormalized secular solution up to $O(\epsilon^3)$

$$U = U_0(k + \tilde{\phi}_x, kx - \omega t + \tilde{\phi}) + \tilde{R}V(k + \tilde{\phi}_x, kx - \omega t + \tilde{\phi}),$$  \hspace{1cm} (A.14)

where $\tilde{\phi}$ and $\tilde{R}$ are renormalized variables defined by the renormalization transformations

$$\tilde{\phi}(x, r_\perp, t) = \phi + \delta(x, r_\perp, t) = \phi + \epsilon(P_1 + \epsilon P_2 + \cdots), \quad (A.15)$$

$$\tilde{R} = R_2 + \epsilon R_3, \quad (A.16)$$

from which we have

$$\tilde{\phi}_x = \epsilon P_{1,x}, \quad \tilde{\phi}_{xx} = \epsilon^2 P_{2,xx}, \quad \tilde{\phi}_{xxx} = \epsilon^3 P_{3,xxx}, \quad \tilde{\phi}_t = \epsilon^3 P_{3,t}. \quad (A.17)$$

The compatibility condition for a periodic solution $\bar{U}_3(\theta)$ requires

$$-\frac{\ddot{\omega}}{2\epsilon} (P_{1,x})^2 \langle \tilde{U} \cdot U_{0,\theta} \rangle - \frac{\partial_k(\ddot{\omega})}{3!} (P_{1,x})^3 \langle \tilde{U} \cdot U_{0,\theta} \rangle + \frac{D_\parallel}{\epsilon} P_{2,xx} \langle \tilde{U} \cdot U_{0,\theta} \rangle + P_{3,xxx} (\langle \tilde{U} \cdot MV \rangle + \langle \tilde{U} \cdot DU_{0,k} \rangle) + \partial_k D_\parallel P_{1,x} P_{2,xx} \langle \tilde{U} \cdot U_{0,\theta} \rangle - P_{3,t} \langle \tilde{U} \cdot U_{0,\theta} \rangle = 0. \quad (A.18)$$

Substituting (A.17) into (A.18), we arrive at the phase equation with third-order spatial derivatives:

$$\tilde{\phi}_t = D_\parallel \tilde{\phi}_{xx} + A \tilde{\phi}_{xxx} + N_\parallel (\tilde{\phi}_x)^2 + N_\parallel (\tilde{\phi}_x)^3 + H \tilde{\phi}_x \tilde{\phi}_{xx}, \quad (A.19)$$

where

$$A = B + A', \quad B = \frac{\langle \tilde{U} \cdot MV \rangle}{\langle \tilde{U} \cdot U_{0,\theta} \rangle}, \quad A' = \frac{\langle \tilde{U} \cdot DU_{0,k} \rangle}{\langle \tilde{U} \cdot U_{0,\theta} \rangle}, \quad (A.20)$$

$$N_\parallel = -\frac{\partial_k \ddot{\omega}}{6}, \quad (A.21)$$

$$H = \partial_k D_\parallel. \quad (A.22)$$

References

4) Y. Masutomi and K. Nozaki, Physica D 151 (2001), 44.