Random Wandering around Homoclinic-Like Manifolds in a Symplectic Map Chain

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We present a method to construct a symplecticity preserving renormalization group map of a chain of weakly nonlinear symplectic maps and obtain a general reduced symplectic map describing its long-time behavior. It is found that the modulational instability in the reduced map triggers random wandering of orbits around some homoclinic-like manifolds. This behavior is understood as Bernoulli shifts.

§1. Introduction

Hamiltonian systems with more than one degree of freedom can exhibit trajectories with complex behavior. Discrete symplectic maps are of interest, because the Poincaré surface of section leads naturally to a mapping of the dynamical trajectory onto a subspace of the phase space of the Hamiltonian flow. Symplectic maps allow for much easier numerical calculations of the motion than Hamiltonian flows. These maps are particularly convenient for studying systems of many degrees of freedom. In a two-dimensional symplectic mapping, which corresponds to a Poincaré mapping of a continuous-time Hamiltonian system with two degrees of freedom, the main ingredients of this chaotic motion are hyperbolic-type orbits and their invariant manifolds. The existence of a transversal intersection between stable and unstable manifolds leads to non-integrability of Hamiltonian systems. The area of a lobe enclosed by stable and unstable manifolds represents the flux from inside (resp. outside) the separatrix to outside (resp. inside). Manifolds of codimension one separate the space into disjoint regions.

For symplectic mappings of four or more dimensions, there are some studies concerning homoclinic bifurcations near fully hyperbolic fixed points. Although the dimensions of unstable and stable manifolds of a fully hyperbolic fixed point take maximum possible values, transversal intersection occurs on a nearly one-dimensional homoclinic submanifold, and an exponentially small splitting of the homoclinic manifold leads to very weak chaos, which is almost indiscernible in naive numerical experiments.

A chain of weakly coupled nonlinear oscillators can be used to model of dynamics near a fully elliptic fixed point. The equilibrium state of almost independent oscillators corresponds to a fully elliptic fixed point in the phase space that is encircled by periodic orbits or tori. Resonant tori may be disrupted by the well-known resonance mechanism. If a torus is hyperbolic, unstable and stable manifolds are attached to the torus as a “whisker”. Some dynamics of such whiskers attached to tori can be studied near the resonance junction by means of a naive averaging method.
In the case of a chain of weakly coupled nonlinear oscillators, for which both the coupling and nonlinearity are weak, tori may be destabilized by another mechanism called the modulational instability. Here, we study such modulationally unstable tori and their unstable manifolds. We start with a chain of symplectic maps, each of which represents a weakly nonlinear oscillator. In order to avoid a special dependence of the results on the specific form of the nonlinearity, our investigation is focused on the long-time asymptotic dynamics of the system. However, conventional asymptotic methods, such as the averaging method and the method of the multiple-time scales, may not be directly applicable to discrete systems. Instead, we employ the perturbative renormalization group (RG) method, which can be applied to a discrete system and leads to a reduced map as the RG map. However, a naive reduced map does not preserve symplectic symmetry and fails to describe the long time behavior of the original symplectic map. For two-dimensional maps or special four-dimensional maps, the symplecticity is recovered through a simple “regularization” procedure. For higher-dimensional symplectic maps, we do not have a general procedure of “regularization”. The first purpose of this paper is to develop a general procedure of “regularization” of the RG map for a higher-dimensional symplectic map. Our regularization procedure is based on the symplectic integration method for a continuous Hamiltonian flow and includes the previously developed regularization procedure as a special case. By means of this regularization procedure, we can derive a reduced symplectic map as the RG map. This symplectic map asymptotically approximates a chain of weakly nonlinear symplectic maps.

The reduced map obtained here is general in a sense similar to that in which the nonlinear Schrödinger equation asymptotically describes a general weakly nonlinear dispersive wave. In fact, the reduced map is found to be a space-time discrete version of a nonlinear Schrödinger equation. Since the reduced map is not integrable, while its continuous limit is integrable, it is worth studying chaotic solutions of the reduced map. This is the second purpose of this paper. The reduced map has a periodic solution, which is destabilized by a modulational perturbation under certain conditions (the modulational instability). In this case, the periodic solution becomes hyperbolic and is associated with unstable and stable manifolds, whose transversal intersection is the origin of chaos (homoclinic chaos). It should be mentioned that various discrete versions of the nonlinear Schrödinger equation have been studied in connection with numerically induced chaos of the nonlinear Schrödinger equation. In those studies, systems with many large degrees of freedom were investigated in connection with the numerical integration of the nonlinear Schrödinger equation, while little attention has been directed to the behavior of chaotic solutions themselves.

Here, we are mainly interested in chaotic solutions near the onset of symplectic chaos, where the phase space of the map has a dimension higher than two. Near the onset of homoclinic chaos, an unstable manifold is found to be very close to a stable manifold and the homoclinic-like structure is approximately preserved. In a two-dimensional symplectic map, homoclinic chaos close to the onset is difficult to observe, because the distance between the unstable and stable manifolds is small, beyond all orders of the perturbation parameter. However, in the higher-dimensional map considered in this paper, weak chaos is found to be manifested as a random
rotation of some homoclinic-like orbits, which is characterized by the Bernoulli shift.

In §2, some reduced maps are derived from a chain of symplectic maps as the regularized RG map. The conditions of the modulational instability in the reduced maps are presented, and the unstable manifolds of unstable periodic orbits are depicted in the cases of two, three and four sites in §3. In §4, a random rotation of homoclinic-like orbits is described for three- and four-site maps. This is interpreted as the Bernoulli shift of three or four numbers.

§2. Symplectic map chain and reduced map

In this section, we consider a symplectic map chain and present a method to construct a symplecticity preserving renormalization group map (a reduced symplectic map) from the symplectic map chain.

2.1. Symplecticity preserving renormalization group method

Here we present a method of deriving a symplecticity preserving reduced map from a nonlinear symplectic map of three or more sites that is non-integrable even in the time-continuous limit. This map is given by

\[ x_{j+1}^{n} = x_j^n + \tau p_j^{n+1}, \]
\[ p_{j+1}^{n+1} = p_j^n + \tau \left( -\Omega^2 x_j^n + \varepsilon \left\{ \nu (x_{j+1}^n - 2x_j^n + x_{j-1}^n) - \alpha (x_j^n)^3 \right\} \right), \]

with \( N (\geq 3) \)-periodic boundary conditions: \( (x_j^n, p_j^n) = (x_{j+N}^n, p_{j+N}^n) \). Here \( x_j^n \) and \( p_j^n \) are real canonical variables defined on the site \( j \) at time \( n \), while \( \Omega, \alpha, \nu, \) and \( \tau \) are real parameters and \( \varepsilon \) is a small perturbation parameter satisfying \( 0 < \varepsilon \ll 1 \). We set \( \varepsilon = 0.01, \tau = 1, \Omega = 1, \nu = -0.25 \) and \( \alpha = 1/3 \) for numerical calculations. Eliminating the variables \( p_j^n \) from Eqs. (2.1) and (2.2), we obtain a map of the variables \( x_j^n \) alone,

\[ x_{j+1}^{n} - 2 \cos(\theta) x_j^n - x_{j-1}^{n-1} = \varepsilon \tau^2 \left\{ \nu (x_{j+1}^n - 2x_j^n + x_{j-1}^n) - \alpha (x_j^n)^3 \right\}, \]

where \( \cos(\theta) = (1 - \Omega^2 \tau^2/2) \).

We now construct a reduced map from (2.3) for \( 0 < \varepsilon \ll 1 \) by means of the RG method. Expanding a solution of Eq. (2.3) as \( x_j^n = x_j^{n(0)} + \varepsilon x_j^{n(1)} + \mathcal{O}(\varepsilon^2) \) \((j = 1, 2, \cdots)\), we have

\[ x_j^{n(0)} = A_j \exp(-i\theta n) + \text{c.c.}, \]
\[ \mathcal{L} x_j^{n(1)} = \tau^2 \left\{ \nu \triangle_j^2 A_j - 3\alpha |A_j|^2 A_j \right\} \exp(-i\theta n) + \text{c.c.}, \]

where

\[ \mathcal{L} x_j^{n(1)} \equiv x_{j+1}^{n+1} - 2 \cos \theta \cdot x_j^{n(1)} + x_{j-1}^{n(1)}, \]
\[ \triangle_j^2 A_j \equiv A_{j+1} - 2A_j + A_{j-1}, \]
c.c. denotes the complex conjugate of the preceding terms, and $A_j$ is an integration constant associated with the site $j$. The first order solution has a term that is secular with respect to $n$,

$$x_j^{n(1)} = n \frac{i \tau^2}{2 \sin \theta} \left\{ \nu \Delta_j^2 A_j - 3\alpha |A_j|^2 A_j \right\} \exp(-i \theta n) + \text{c.c.} + \text{n.r.},$$

where n.r. denotes non-resonant terms. In order to remove this secular term, we define a renormalization transformation $A_j \rightarrow A^n_j$ by

$$A^n_j = A_j + \varepsilon \left\{ n \frac{i \tau^2}{2 \sin \theta} \left( \nu \Delta_j^2 A_j - 3\alpha |A_j|^2 A_j \right) \right\} + \mathcal{O}(\varepsilon^2). \quad (2.6)$$

A discrete version of the RG equation is constructed by taking the difference of consecutive values of $A^n_j$:

$$A^{n+1}_j - A^n_j = \varepsilon \left\{ n \frac{i \tau^2}{2 \sin \theta} \left( \nu \Delta_j^2 A^n_j - 3\alpha |A^n_j|^2 A^n_j \right) \right\}. \quad (2.7)$$

Substituting the expression for $A_j$ in terms of $A^n_j$ given in Eq. (2.6) into Eq. (2.7), we can eliminate the secular term up to $\mathcal{O}(\varepsilon)$ and obtain the naive RG map

$$A^{n+1}_j = A^n_j + \varepsilon \left\{ n \frac{i \tau^2}{2 \sin \theta} \left( \nu \Delta_j^2 A^n_j - 3\alpha |A^n_j|^2 A^n_j \right) \right\}, \quad (2.8)$$

or

$$\frac{A^{n+1}_j - A^n_j}{\tau} = \varepsilon \left\{ n \frac{i \tau^2}{2 \sin \theta} \left( \nu \Delta_j^2 A^n_j - 3\alpha |A^n_j|^2 A^n_j \right) \right\}. \quad (2.9)$$

This naive RG map does not preserve symplectic symmetry. To recover the symplecticity of Eq. (2.8), we apply the symplectic integration method to the continuous-time limit ($\tau \rightarrow 0$) of the naive RG map (2.9). The continuous-time limit takes the following symplectic form (which is also derived from the continuous-time limit of Eqs. (2.1) and (2.2) in Appendix A):

$$\frac{dA_j}{dt} = i\varepsilon \frac{1}{2\Omega} \left( \nu \Delta_j^2 A_j - 3\alpha |A_j|^2 A_j \right) = \frac{\partial H^{\text{RG}}}{\partial A_j^*}, \quad \frac{dA_j^*}{dt} = -\frac{\partial H^{\text{RG}}}{\partial A_j}. \quad (2.10)$$

Here, $\Omega = \frac{\sin(\theta)}{\tau}$ as $\tau \rightarrow 0$, and $H^{\text{RG}}$ is given by

$$H^{\text{RG}} = H^\nu + H^\alpha,$$

$$H^\nu = -\frac{i\varepsilon \nu}{2\Omega} \sum_j |A_{j+1} - A_j|^2,$$

$$H^\alpha = \frac{3i\varepsilon \alpha}{4\Omega} \sum_j |A_j|^2.$$

The Hamiltonian flow defined by $H^\alpha$ can be solved as

$$A_j(t + \tau') = A_j(t) \exp \left( -\frac{3i\varepsilon \alpha}{2\Omega} |A_j(t)|^2 \tau' \right),$$
where $\tau'$ is an arbitrary constant, and we have

$$A_{j}^{n+1} = A_{j}^{n} \exp \left( -\frac{3i\varepsilon\alpha}{2\Omega} |A_{j}^{n}|^{2} \tau \right). \tag{2.11}$$

The flow defined by $H^{\nu}$ can be discretized by the symplectic implicit midpoint rule as follows. \(^1\) A general Hamiltonian flow

$$\frac{dz}{dt} = f(z),$$

is discretized as

$$z^{n+1} = z^{n} + \tau f\left( \frac{1}{2}(z^{n+1} + z^{n}) \right).$$

In our case, the scheme gives

$$A_{j}^{n+1} = A_{j}^{n} + \tau \frac{i\varepsilon\nu}{2\Omega} \left( \Delta_{j}^{2} A_{j}^{n+1} + \Delta_{j}^{2} A_{j}^{n} \right), \tag{2.12}$$

or

$$\left( 1 - i\tau \frac{i\varepsilon\nu}{2\Omega} \Delta_{j}^{2} \right) A_{j}^{n+1} = \left( 1 + i\tau \frac{i\varepsilon\nu}{2\Omega} \Delta_{j}^{2} \right) A_{j}^{n}. \tag{2.13}$$

Combining the two symplectic transformations (2.11) and (2.13), we obtain a symplectic scheme for $H^{RG} = H^{\nu} + H^{\alpha}$ as

$$\left( 1 - iT \Delta_{j}^{2} \right) A_{j}^{n+1} = \left( 1 + iT \Delta_{j}^{2} \right) \exp \left( -\frac{3i\varepsilon\alpha}{2\Omega} |A_{j}^{n}|^{2} \tau \right) A_{j}^{n}. \tag{2.14}$$

For brevity, we rewrite this symplectic RG map as

$$\left( 1 - iT \Delta_{j}^{2} \right) A_{j}^{n+1} = \left( 1 + iT \Delta_{j}^{2} \right) \exp \left( iQ |A_{j}^{n}|^{2} \right) A_{j}^{n},$$

where

$$T \equiv \frac{\nu \tau^{2}}{4 \sin \theta} \in \mathbb{R}, \quad Q \equiv \frac{-3\alpha \tau^{2}}{2 \sin \theta} \in \mathbb{R}.$$

The quantity $\sum_{j} |A_{j}^{n}|^{2}$ is conserved for this map (see Ref. 13)).

It is possible to give a matrix form of Eq. (2.14). For instance, the three-cite map is written as

$$\begin{pmatrix}
A_{1}^{n+1} \\
A_{2}^{n+1} \\
A_{3}^{n+1}
\end{pmatrix}
= \frac{1 - 3iT}{1 + 9T^{2}}
\begin{pmatrix}
1 - iT & 2iT & 2iT \\
2iT & 1 - iT & 2iT \\
2iT & 2iT & 1 - iT
\end{pmatrix}
\begin{pmatrix}
B_{1}^{n} \\
B_{2}^{n} \\
B_{3}^{n}
\end{pmatrix},$$

where

$$B_{j}^{n} \equiv \exp(iQ |A_{j}^{n}|^{2}) A_{j}^{n}.$$
exponentiation procedure introduced in Ref. 9). Note that Eq. (2.14) with \( N \gg 1 \) is identical to a difference scheme for the nonlinear Schrödinger equation.\(^{13}\) In fact, the time-continuous limit of the RG map (2.14) is just a spatially discretized nonlinear Schrödinger equation (see Appendix A). The map (2.14) also admits some oscillating solutions, including a spatially uniform one.

2.2. Nonlinear symplectic map with two sites

Let us consider a nonlinear symplectic map with two sites. This system is treated separately because the continuous time limit \((\tau \to 0)\) of its RG map is integrable. This map is given by

\[
x_1^{n+1} = x_1^n + \tau p_1^{n+1}, \quad x_2^{n+1} = x_2^n + \tau p_2^{n+1},
\]

\[
p_1^{n+1} = p_1^n + \tau \left\{ -\Omega^2 x_1^n + \varepsilon \left( \nu (x_2^n - x_1^n) - \alpha x_1^n \right) \right\},
\]

\[
p_2^{n+1} = p_2^n + \tau \left\{ -\Omega^2 x_2^n + \varepsilon \left( \nu (x_1^n - x_2^n) - \alpha x_2^n \right) \right\}.
\]

We rewrite Eqs. (2.15)–(2.17) as

\[
x_1^{n+1} - 2 \cos \theta \cdot x_1^n + x_1^{n-1} = \varepsilon \tau^2 \left\{ \nu (x_2^n - x_1^n) - \alpha (x_1^n)^3 \right\},
\]

\[
x_2^{n+1} - 2 \cos \theta \cdot x_2^n + x_2^{n-1} = \varepsilon \tau^2 \left\{ \nu (x_1^n - x_2^n) - \alpha (x_2^n)^3 \right\},
\]

where \( \theta \) is defined in Eq. (2.3). Setting \( x_j^n \approx A_j^n \exp(-i\theta n) + \text{c.c.} \) and following the same procedure as in the case of a model with three or more sites, we obtain the regularized RG map (up to \( \mathcal{O} (\varepsilon) \))

\[
(1 - iTL_j) A_j^{n+1} = (1 + iTL_j) \exp(iQ|A_j^n|) A_j^n, \quad (j = 1, 2)
\]

where the operator \( L_j \) is defined by

\[
L_1 A_1^n \equiv -A_1^n + A_2^n, \quad L_2 A_2^n \equiv A_1^n - A_2^n.
\]

A detailed derivation of this regularized RG map is given in Appendix B. It is worth noting that the map (2.18) has not only symplectic symmetry but also the conserved quantity

\[
\sum_{j=1,2} |A_j^n|^2 = \sum_{j=1,2} |A_j^0|^2.
\]

The map (2.18) has the oscillating solutions

\[
A_1^n = A_2^n = A_0^n \exp \left( iQ |A_0^n|^2 n \right),
\]

or

\[
A_1^n = -A_2^n = A_0^n \exp \left( iQ |A_0^n|^2 n \right),
\]
where $A^0$ is a constant. The family of such solutions lies on a one-dimensional torus in the phase space spanned by $A_j^m$ and $A_j^m \ast$. The (modulational) stability of solutions (2.21) and (2.22) is of interest, because this stability determines whether the torus is hyperbolic or elliptic. If the torus is hyperbolic, unstable and stable manifolds attach to the torus like whiskers, and for this reason we call such unstable and stable manifolds “whiskers” in this paper. The behavior of such whiskers is the main object of investigation in §§3 and 4.

We now demonstrate the effectiveness of the regularized RG map with the results of numerical calculations. In Fig. 1, the trajectories obtained from the naive RG map (2.8) and the regularized RG map (2.14) are compared to an exact trajectory of the original map [Eqs. (2.1)–(2.2)]. All initial points are set very close to a hyperbolic torus. The trajectory given by the regularized RG map is consistent with the exact solution.

Owing to the conserved quantity (2.20), the continuous time limit ($\tau \to 0$) of the RG map (2.18) is integrable. An analytical expression of whiskers, which are exactly homoclinic in this case, is given in Appendix C.

Fig. 1. Time sequences of the three-site model: (a) a trajectory constructed from the original map, (b) the naive RG map, and (c) the regularized RG map.

§3. Whiskered tori

In this section, we analyze the modulational instability of spatially uniform oscillating solutions (tori) of the reduced maps derived in the previous section and introduce whiskered tori, that is, the hyperbolic structure near periodic manifolds. It is found that whiskers retain approximate homoclinic structure near the onset of the modulational instability, where the homoclinic chaos is weak.

3.1. Two-site case

Let us analyze the reduced symplectic map with two sites, whose continuous-time limit is integrable.

One of the tori of Eq. (2.18) is the spatially uniform solution

$$A_j^n = A^0 \exp(iQ|A^0|^2n). \quad (j = 1, 2)$$

To study the phase space structure around this solution, we consider a perturbed solution of the form

$$A_j^n = A^0 \exp(iQ|A^0|^2n) \left(1 + \mu_j^n\right), \quad (3.1)$$
where $|\mu_j^n|^2 \ll 1$. Substituting this expression into (2.18) and retaining terms first order in $\mu_j^n$, we obtain the expression

$$
\left( \begin{array}{cc}
1 + iT & -iT \\
-iT & 1 + iT
\end{array} \right)
\left( \begin{array}{c}
\mu_1^{n+1} \\
\mu_2^{n+1}
\end{array} \right)
= (1 + iQ|A_0^0|^2)
\left( \begin{array}{cc}
1 - iT & iT \\
 iT & 1 - iT
\end{array} \right)
\left( \begin{array}{c}
\mu_1^n \\
\mu_2^n
\end{array} \right)
+ iQ|A_0^0|^2
\left( \begin{array}{cc}
1 - iT & iT \\
 iT & 1 - iT
\end{array} \right)
\left( \begin{array}{c}
\mu_1^n \ast \\
\mu_2^n \ast
\end{array} \right),
$$

(3.2)

along with its complex conjugate. The eigenvalues of Eq. (3.2) are

$$
1, 1, \lambda_\pm = \beta \pm \sqrt{\beta^2 - 1},
$$

(3.3)

where $\beta \equiv (1 - 4T^2 + 4Q|A_0^0|^2 T)/(1 + 4T^2)$. We thus obtain the conditions for instability

$$
|A_0^0|^2 > \frac{2T}{Q},
$$

(3.4)

and

$$
|A_0^0|^2 < -\frac{1}{2QT},
$$

(3.5)

which are derived from the relation $\beta^2 > 1$.

Hereafter, we concentrate on the condition (3.4), under which the torus becomes hyperbolic for an amplitude $|A_0^0|$ larger than the critical value $\sqrt{2T/Q}$. From the spectrum of eigenvalues (3.3), we find that whiskers are of two types, one-dimensional unstable manifolds (departing whiskers) and one-dimensional stable manifolds (arriving whiskers). To construct whiskers numerically, we transform $A_j^n$ into $a_j^n$ ($j = 1, 2$) according to

$$
A_j^n \equiv a_j^n \exp(iQ|A_0^0|^2 n).
$$

Fig. 2. (a) The phase space of the two-site model near the hyperbolic torus for $|A_0^0| = |A_c| + 0.005$. Here (0) corresponds to the uniform solution $A_j^n = A_0^0 \exp(iQ|A_0^0|^2 n)$, and (1) and (2) correspond to $a_1^n$ and $a_2^n$, respectively. The initial conditions are as follows: $\text{Re} A_1^n = |A_0^0|/\sqrt{2} + 0.0002$, $\text{Re} A_2^n = |A_0^0|/\sqrt{2} + 0.0001$, $\text{Im} A_1^n = |A_0^0|/\sqrt{2} + 0.0$, $\text{Im} A_2^n = |A_0^0|/\sqrt{2} + 0.0$. (b) The initial conditions here are the same as in (a) for the case $|A_0^0| = |A_c| + 0.5$. 

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Tori with departing whiskers are depicted in Fig. 2. We find that even if $|A^0|$ is considerably larger than the critical value, whiskers are almost homoclinic, and their behavior appears to be regular. This apparent regularity reflects the integrability of the continuous-time limit of the map with two sites, and is in sharp contrast to the case of three or more sites studied in the subsequent sections. Since the difference between the map (2.18) and its integrable continuous-time limit is very small (beyond all orders of $\varepsilon$), the weak homoclinic chaos here may be almost indiscernible in numerical calculations with the precision used here.

3.2. Three-site case

Let us consider the regularized RG map with three sites (2.14). When the number of sites is odd, a general stability analysis is possible. Such an analysis is given in Appendix D. A linearized analysis around a uniformly oscillating solution similar to that used in the previous subsection leads to the conditions for instability

$$|A^0|^2 > \frac{3T}{Q},$$

and

$$|A^0|^2 < \frac{-1}{3QT}.$$  

As in the case of the two-site model, we consider the condition (3.6) only. The spectrum of the linearized map is given by

$$\lambda_a \equiv \beta_a \pm \sqrt{\beta_a^2 - 1}, \quad (a = 0, 1)$$

$$\beta_0 = 1, \quad \text{(singlet)}$$

$$\beta_1 = \frac{1 + 6Q|A^0|^2T - 9T^2}{1 + 9T^2}.$$  

This spectrum shows that both departing and arriving whiskers are two dimensional for the three-site case. The tori with their unstable manifolds are depicted in Fig. 3.

![Fig. 3](https://example.com/fig3.png)

Fig. 3. (a) The phase space of the three-site model near the hyperbolic torus for $|A^0| = |A_c| + 0.005$. Here (0) corresponds to the uniform solution $A_n^0 = A^0 \exp(iQ|A^0|^2n)$, and (1), (2), (3) correspond to $a_1^n$, $a_2^n$, $a_3^n$, respectively. The initial conditions are as follows: $\Re A_1^n = |A^0|/\sqrt{2} + 0.0003$, $\Re A_2^n = |A^0|/\sqrt{2} + 0.0002$, $\Re A_3^n = |A^0|/\sqrt{2} + 0.0001$, $\Im A_1^n = |A^0|/\sqrt{2} + 0.0$, $\Im A_2^n = |A^0|/\sqrt{2} + 0.0$, $\Im A_3^n = |A^0|/\sqrt{2} + 0.0$. (b) The initial conditions here are the same as in (a) for the case $|A^0| = |A_c| + 0.5$. 


These figures show that the whiskers’ behavior differs from that in the two-site case. The difference may be attributed to a difference in the integrability properties of their continuous time limits. The behavior of the whiskers becomes increasingly complicated as the amplitude of the tori increases. We have found that when the amplitude just exceeds the critical value $\sqrt{3T/Q}$ for instability, the whiskers are still homoclinic-like, as shown in Fig. 3 (a).

3.3. Four-site case

We analyze the modulational instability of a uniformly oscillating solution of the regularized RG map with four sites (2.14) separately, because general analysis is possible only for the case of an odd number of sites (see Appendix D).

The linearized map around a uniformly oscillating solution (a torus) has the following eight eigenvalues:

$$
\lambda_{a\pm} \equiv \beta_a \pm \sqrt{\beta_a^2 - 1}, \quad (a = 0, 1, 2)
$$

$$
\beta_0 = 1, \quad \text{(singlet)}
$$

$$
\beta_1 = \frac{1 + 4Q|A^0|^2T - 4T^2}{1 + 4T^2}, \quad \text{(doublet)}
$$

$$
\beta_2 = \frac{1 + 8Q|A^0|^2T - 16T^2}{1 + 16T^2}, \quad \text{(singlet)}
$$

These eigenvalues imply the following conditions for instability of the torus:

$$
|A^0|^2 > \frac{2T}{Q} \quad \text{(doublet)}, \quad |A^0|^2 > \frac{4T}{Q} \quad \text{(singlet)}, \quad (3.9)
$$

and

$$
|A^0|^2 < -\frac{1}{2QT} \quad \text{(doublet)}, \quad |A^0|^2 < -\frac{1}{4QT} \quad \text{(singlet)}. \quad (3.10)
$$

Here we consider only the conditions (3.9), as before. This spectrum of eigenvalues

Fig. 4. (a) The phase space of the four-site model near the hyperbolic torus for $|A^0| = |A_c| + 0.005$. Here (0) corresponds to the uniform solution $A_{j} = A^0 \exp(iQ|A^0|^2n)$, and (1), · · · , (4) correspond to $a_n^0$, · · · , $a_4^0$, respectively. The initial conditions are as follows: $\text{Re}A_1^0 = |A^0|/\sqrt{2} + 0.0004$, $\text{Re}A_2^0 = |A^0|/\sqrt{2} + 0.0003$, $\text{Re}A_3^0 = |A^0|/\sqrt{2} + 0.0002$, $\text{Re}A_4^0 = |A^0|/\sqrt{2} + 0.0001$, $\text{Im}A_1^0 = |A^0|/\sqrt{2} + 0.0$, $\text{Im}A_2^0 = |A^0|/\sqrt{2} + 0.0$, $\text{Im}A_3^0 = |A^0|/\sqrt{2} + 0.0$, $\text{Im}A_4^0 = |A^0|/\sqrt{2} + 0.0$. (b) The initial conditions here are the same as in (a) for the case $|A^0| = |A_c| + 0.5$. 

Random Wandering around Homoclinic-Like Manifolds

The torus with its unstable manifold is depicted in Fig. 4. As in the three-site case, the behavior of the whiskers becomes increasingly complicated as the amplitude of the torus increases.

§4. Random rotation of homoclinic-like orbits

We study the behavior of chaotic orbits slightly above the critical amplitude in the three- and four-site cases. In such cases, an orbit wanders randomly around some homoclinic-like manifolds (whiskers), and it appears as a random sequence of homoclinic-like orbits, as shown in Fig. 5. Each homoclinic-like orbit differs only with regard to which site takes the largest amplitude in the orbit. In other words, a random rotation of site numbers occurs in each homoclinic-like orbit. This random rotation of site numbers can be understood in terms of the Bernoulli shift.

We represent each homoclinic-like orbit at each time by the site number of the largest amplitude at that time, so that an orbit is represented in time by a sequence of site numbers, for example,

\[ 10210120 \cdots, \quad 112301 \cdots, \]

corresponding to the sequences in Figs. 6 and 7, where 0 corresponds to the site number 3 in the first case and 4 in the second. Such a random sequence of numbers is understood as a Bernoulli sequence generated by a simple map. The above two sequences are respectively generated by the following maps:

\[ w^{n+1} = 3w^n, \quad \text{(mod 1)} \]
\[ w^{n+1} = 4w^n, \quad \text{(mod 1)} \]
S. Goto, K. Nozaki and H. Yamada

Fig. 6. The time sequences of $|a_n^3|^2$ for the three-site model near the hyperbolic torus for $|A_0| = |A_c| + 0.1$. Here (1), (2) and (0) correspond to $|a_n^1|^2$, $|a_n^2|^2$ and $|a_n^3|^2$, respectively. The orbit is characterized by the sequence of site numbers 10210120⋯. The initial conditions are as follows: $\text{Re}A_0^1 = |A_0|/\sqrt{2} + 0.003$, $\text{Re}A_0^2 = |A_0|/\sqrt{2} + 0.002$, $\text{Re}A_0^3 = |A_0|/\sqrt{2} + 0.001$, $\text{Im}A_0^1 = |A_0|/\sqrt{2} + 0.0$, $\text{Im}A_0^2 = |A_0|/\sqrt{2} + 0.0$, $\text{Im}A_0^3 = |A_0|/\sqrt{2} + 0.0$.

A periodic sequence of homoclinic-like orbits corresponds to a periodic solution of the Bernoulli map for a rational initial value on the interval $[0, 1)$, while a random sequence is generated for an irrational initial value.

Some number sequences of homoclinic-like whiskers obtained by numerical iteration in the three-site and four-site cases are presented in Fig. 8. These figures suggest that the sequences of homoclinic-like orbits is not periodic, but random. It should be noted that such a random rotation of homoclinic-like orbits does not appear in
§5. Conclusion

We have presented a regularization procedure to preserve the symplectic structure of the RG map near a fully elliptic fixed point of a chain of weakly nonlinear symplectic maps. The regularization is accomplished by extended exponentiation of the naive RG map, and we derived a general reduced symplectic map as an asymptotic RG map.

Analyzing the modulational instability of a uniformly oscillating solution of the reduced map, we found a hyperbolic torus with “whiskers”. We observed that these whiskers retain a homoclinic-like structure in the case the amplitude of the uniformly oscillating solution slightly exceeds a critical value. For cases in which there are three or more sites, we found that there is a random sequence of the homoclinic-like orbits. Such an orbit can be represented by a random sequence of site numbers generated by Bernoulli shifts. Although the chaos corresponding to this random sequence may be produced by the homoclinic mechanism and is very weak, it is easily observed as a random sequence of homoclinic-like orbits. Such an easily observed irregularity of homoclinic-like orbits may be a general feature of high-dimensional chaos and provide an important observational tool for understanding the onset of chaotic dynamics in high-dimensional spaces.

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Appendix A

Continuous-Time RG Equation with Three or More Sites

Here, we give an RG equation for the continuous-time limit ($\tau \to 0$) of the weakly nonlinear chain (2.1) and (2.2), which is a spatially discretized nonlinear
Schrödinger equation. In the continuous-time limit, Eqs. (2.1) and (2.2) become
\[
\frac{dx_j}{dt} = p_j = \frac{\partial H}{\partial p_j},
\]
\[
\frac{dp_j}{dt} = -\Omega^2 x_j + \varepsilon \left\{ \nu \left( x_{j+1} - 2x_j + x_{j-1} \right) - \alpha x_j^3 \right\} = -\frac{\partial H}{\partial x_j},
\]
\[
H = \sum_l \left[ p_l^2 + \Omega^2 x_l^2 + \varepsilon \alpha x_l^4 \right] + \sum_l \varepsilon \nu \left( \frac{(x_{l+1} - x_l)^2}{2} \right).
\]
Expanding \(x_j\) as \(x_j = x_j^{(0)} + \varepsilon x_j^{(1)} + \mathcal{O}(\varepsilon^2)\), a naive perturbed solution is obtained as
\[
x_j^{(0)} = A_j \exp(-i\Omega t) + \text{c.c.},
\]
\[
x_j^{(1)} = \frac{it}{2\Omega} \left( \nu \Delta_j^2 A_j - 3\alpha |A_j|^2 A_j \right) \exp(-i\Omega t) + \text{c.c.} + \text{n.r.},
\]
where \(A_j\) is a complex valued integration constant for the \(j\)-th site. To remove secular terms proportional to \(t\), we define the renormalization transformation \(A_j \mapsto \tilde{A}_j(t)\) by collecting all terms proportional to the fundamental harmonic \(\exp(-i\Omega t)\) and writing
\[
\tilde{A}_j(t) \equiv A_j + \varepsilon \frac{it}{2\Omega} \left( \nu \Delta_j^2 A_j - 3\alpha |A_j|^2 A_j \right) + \mathcal{O}(\varepsilon^2).
\]
From this transformation, the prescription of the RG method\(^7\) gives the RG equation
\[
\frac{d\tilde{A}_j}{dt} = i\varepsilon \frac{1}{2\Omega} \left( \nu \Delta_j^2 \tilde{A}_j - 3\alpha |\tilde{A}_j|^2 \tilde{A}_j \right).
\] (A.1)
This RG equation automatically possesses the symplectic property and is known as a spatially discretized nonlinear Schrödinger equation, which is also non-integrable, except in the case of two sites.

**Appendix B**

*Derivation of a Reduced Map of a Two-Site Nonlinear Symplectic Map*

In this subsection, we derive a regularized RG map from a nonlinear two-site symplectic map chain.

Expanding \(x_1^n\) and \(x_2^n\) as \(x_j^n = x_j^{n(0)} + \varepsilon x_j^{n(1)} + \mathcal{O}(\varepsilon^2)\) \((j = 1, 2)\), we have the following naive perturbative solution of Eqs. (2.15)–(2.17):
\[
x_j^{n(0)} = A_j \exp(-i\theta n) + \text{c.c.},
\]
\[
x_1^{n(1)} = \tau^2 \left\{ \nu (A_2 - A_1) - 3\alpha |A_1|^2 A_1 \right\} \exp(-i\theta n) + \text{c.c.} + \text{n.r.},
\]
\[
x_2^{n(1)} = \tau^2 \left\{ \nu (A_1 - A_2) - 3\alpha |A_2|^2 A_2 \right\} \exp(-i\theta n) + \text{c.c.} + \text{n.r.}
The renormalization transformation $A_j \mapsto A^*_j$ to remove the secular terms in the coefficient of $\exp(-i\theta n)$ is given by

$$
A^*_1 \equiv A_1 + \varepsilon n \frac{i\tau^2}{2\sin \theta} \left\{ \nu (A_2 - A_1) - 3\alpha |A_1|^2 A_1 \right\} + \mathcal{O} (\varepsilon^2),
$$

$$
A^*_2 \equiv A_2 + \varepsilon n \frac{i\tau^2}{2\sin \theta} \left\{ \nu (A_1 - A_2) - 3\alpha |A_2|^2 A_2 \right\} + \mathcal{O} (\varepsilon^2),
$$

(B.1)

from which we have the naive RG map up to $\mathcal{O} (\varepsilon)$

$$
A^n_j = A^n_j + \varepsilon \frac{i\tau^2}{2\sin \theta} \left( \nu L_j A^n_j - 3\alpha |A^n_j|^2 A^n_j \right), \quad (j = 1, 2)
$$

(B.2)

where $L_j A_j$ is defined by Eq. (2.19). The naive RG map (B.2) should be regularized by means of the symplectic integration method. This yields the regularized RG map

$$
\left( 1 - iTL_j \right) A^{n+1}_j = \left( 1 + iTL_j \right) \exp \left( iQ_j |A^n_j|^2 \right) A^n_j,
$$

(B.3)

which has the following matrix form useful for numerical calculations:

$$
\begin{pmatrix}
1 + iT & -iT \\
-iT & 1 + iT
\end{pmatrix}
\begin{pmatrix}
A^{n+1}_1 \\
A^{n+1}_2
\end{pmatrix} =
\begin{pmatrix}
1 - iT & +iT \\
+iT & 1 - iT
\end{pmatrix}
\begin{pmatrix}
B^n_1 \\
B^n_2
\end{pmatrix}.
$$

(B.4)

By inverting the matrix in Eq. (B.4), we obtain

$$
\begin{pmatrix}
A^{n+1}_1 \\
A^{n+1}_2
\end{pmatrix} = \frac{1 - 2iT}{1 + 4T^2}
\begin{pmatrix}
1 & 2iT \\
2iT & 1
\end{pmatrix}
\begin{pmatrix}
B^n_1 \\
B^n_2
\end{pmatrix}.
$$

**Appendix C**

Continuous-Time RG Equation with Two Sites

In order to obtain a clear picture of a homoclinic manifold resulting from the modulational instability, let us analyze a continuous-time RG equation with two sites, which is integrable. In this case, the continuous-time RG equation has the form (see Eq. (A.1))

$$
\frac{d \tilde{A}_1}{dt} = \frac{i\varepsilon}{2\Omega} \left\{ \nu (\tilde{A}_2 - \tilde{A}_1) - 3\alpha |\tilde{A}_1|^2 \tilde{A}_1 \right\},
$$

$$
\frac{d \tilde{A}_2}{dt} = \frac{i\varepsilon}{2\Omega} \left\{ \nu (\tilde{A}_1 - \tilde{A}_2) - 3\alpha |\tilde{A}_2|^2 \tilde{A}_2 \right\}.
$$

Since the regime in which the modulational instability occurs is of interest, we suppose $\nu = -|\nu| < 0$ and $\alpha > 0$ and introduce two new variables:

$$
t' = |\nu| \frac{\varepsilon}{2\Omega} t, \quad Q_j = \sqrt{\frac{3\alpha}{|\nu|}} \exp \left\{ \frac{i\nu \varepsilon t}{2\Omega} \right\} \tilde{A}_j. \quad (j = 1, 2)
$$
Then, we have
\[
\frac{dQ_1}{dt'} = -i\left( Q_2 + |Q_1|^2Q_1 \right) = \frac{\partial H}{\partial Q_1^*}, \quad \frac{dQ_1^*}{dt'} = -\frac{\partial H}{\partial Q_1},
\]
\[
\frac{dQ_2}{dt'} = -i\left( Q_1 + |Q_2|^2Q_2 \right) = \frac{\partial H}{\partial Q_2^*}, \quad \frac{dQ_2^*}{dt'} = -\frac{\partial H}{\partial Q_2},
\]
\[
H = -i\left\{ Q_1^*Q_2 + Q_1Q_2^* + (|Q_1|^4 + |Q_2|^4)/2 \right\},
\]
which has the uniformly oscillating solution
\[
Q_j(t') = Q_0(0) \exp \left\{ -i(1 + |Q_0(0)|^2) t' \right\}. \quad (j = 1, 2)
\]
To construct a homoclinic manifold of the unstable periodic solution, we transform \( Q_j \) into \( a_j \) as
\[
Q_j(t') \equiv a_j(t') \exp \left\{ -i(1 + |Q_0(0)|^2) t' \right\},
\]
and we have
\[
\frac{da_1}{dt'} = -i \left\{ |a_1|^2 - (1 + |Q_0|^2) \right\} a_1 + a_2 \quad \text{(C.1)}
\]
\[
\frac{da_2}{dt'} = -i \left\{ |a_2|^2 - (1 + |Q_0|^2) \right\} a_2 + a_1 \quad \text{(C.2)}
\]
This system has two conserved quantities, the Hamiltonian and \( |a_1|^2 + |a_2|^2 \), and is integrable. After some manipulations of Eqs. (C.1) and (C.2), we obtain an explicit expression of the homoclinic manifold as
\[
a_1 = b\sqrt{1 + \Gamma} \exp\{i(\Theta + \Delta)/2\},
\]
\[
a_2 = b\sqrt{1 - \Gamma} \exp\{i(\Theta - \Delta)/2\},
\]
with
\[
\Gamma(t') = \frac{2}{b^2} \sqrt{b^2 - 1} \sech(2\sqrt{b^2 - 1}t'),
\]
\[
\Delta(t') = \cos^{-1}\left( \frac{1 - b^2\Gamma^2/2}{\sqrt{1 - \Gamma^2}} \right),
\]
\[
\Theta(t') = \tan^{-1}\left\{ \frac{B}{\sqrt{1 - B^2}} \tanh\left( 2\sqrt{b^2 - 1}t' \right) \right\} + \text{const},
\]
where \( B \equiv 2\sqrt{b^2 - 1}/b^2 \) and \( b \equiv |Q_0(0)| > 1 \).

Appendix D

Modulational Instability for the Case of an Odd Number of Sites

In this section, we derive instability conditions for a uniformly oscillating solution of the reduced map (2.14) with an odd number sites by means of Fourier analysis.
First, we substitute Eq. (3.1) into Eq. (2.14), where the perturbation is chosen as

\[ \mu_j^n = \sum_a \hat{\mu}_j^a \exp \left( \frac{2\pi i}{N} aj \right), \]  

(D.1)

\[ a \in \left\{ -\frac{N-1}{2}, -\frac{N-1}{2} + 1, \ldots, 0, \ldots, \frac{N-1}{2} - 1, \frac{N-1}{2} \right\}, \]

and \( N \) is an odd number (\( N \geq 3 \)). Assuming the periodic boundary conditions \( \mu_{j+N} = \mu_j \), we have the linearized equation for \( \mu_j \)

\[
\begin{pmatrix}
\hat{\mu}_{j+1}^n \\
\hat{\mu}_j^{n+1}
\end{pmatrix} = \begin{pmatrix}
d_a(1 + iQ|A_0|^2) & d_aiQ|A_0|^2 \\
d_a^{-1}(-iQ|A_0|^2) & d_a^{-1}(1 - iQ|A_0|^2)
\end{pmatrix} \begin{pmatrix}
\hat{\mu}_a^n \\
\hat{\mu}_a^n *
\end{pmatrix},
\]  

(D.2)

where \( a \neq 0 \) and

\[ d_a \equiv \frac{1 - 4iT \sin^2(\pi a/N)}{1 + 4iT \sin^2(\pi a/N)}. \]

The eigenvalues of Eq. (D.2) are

\[ \lambda_{a\pm} = \beta_a \pm \sqrt{\beta_a^2 - 1}, \]  

(D.3)

where

\[ \beta_a \equiv \frac{1 + 8QT|A_0|^2 \sin^2(\pi a/N) - \left( 4T \sin^2(\pi a/N) \right)^2}{1 + \left( 4T \sin^2(\pi a/N) \right)^2} \in \mathbb{R}. \]

Note that \( \lambda_{a+} + \lambda_{a-} = 1 \). The uniformly oscillating solution is destabilized when

\[ \beta_a > 1 \quad \text{and} \quad \beta_a < 1, \]

that is, when

\[ |A_0|^2 > \frac{4T \sin^2(\pi a/N)}{Q}, \quad \text{and} \quad |A_0|^2 < -\frac{1}{4QT \sin^2(\pi a/N)}. \]  

(D.4)

The instability conditions (D.4) and the spectrum of eigenvalues (D.3) give information about the dimensions of invariant manifolds around the uniformly oscillating solution (a torus) characterized by \( |A_0| \).

References

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