Three-Cluster Equation Using the Two-Cluster RGM Kernel

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We propose a new type of three-cluster equation which uses two-cluster resonating-group-method (RGM) kernels. In this equation, the orthogonality of the total wave function to two-cluster Pauli-forbidden states is essential to eliminate redundant components admixed in the three-cluster systems. The explicit energy dependence inherent in the exchange RGM kernel is self-consistently determined. For bound-state problems, this equation is straightforwardly transformed into the Faddeev equation, which uses a modified singularity-free $T$-matrix constructed from the two-cluster RGM kernel. The approximation of the present three-cluster formalism can be examined with a more complete calculation using the three-cluster RGM. As a simple example, we discuss three di-neutron ($3d'$) and $3\alpha$ systems in the harmonic-oscillator variational calculation. The result of the Faddeev calculation is also presented for the $3d'$ system.

§1. Introduction

All the present-day quark-model descriptions of nucleon-nucleon ($NN$) and hyperon-nucleon ($YN$) interactions incorporate important roles of the quark-gluon degrees of freedom in the short-range region and the meson-exchange processes dominating in the medium- and long-range parts of the interaction.1) For example, we have introduced a one-gluon exchange Fermi-Breit interaction and effective meson-exchange potentials acting between quarks, and have obtained very accurate descriptions of the $NN$ and $YN$ interactions with a limited number of parameters.2)–5) We hope that the interaction derived in these models can be used for realistic calculations in few-baryon systems, like the hypertriton, and various types of baryonic matter. This project, however, involves a non-trivial problem of determining how to extract the effective two-baryon interaction from the microscopic quark-exchange kernel. The basic baryon-baryon interaction is formulated as a composite-particle interaction in the framework of the resonating-group method (RGM). If we rewrite the RGM equation in the form of a Schrödinger-type equation, the interaction term becomes non-local and energy dependent. Furthermore, the RGM equation sometimes involves redundant components, due to the effect of the antisymmetrization, which is related to the existence of the Pauli-forbidden states. In such a case, the full off-shell $T$-matrix is not well defined in the standard procedure, which usually
assumes simple energy-independent local potentials. Since these features are related to the characteristic description of the short-range part in the quark model, it would be desirable if the quark-exchange kernel could be used directly in application to many-baryon systems.

In this paper, we propose a new type of three-cluster equation that employs a two-cluster RGM kernel for the inter-cluster interaction. We assume, for simplicity, three identical clusters having only one Pauli-forbidden state for the inter-cluster relative motion, but the extension to general systems is rather straightforward. We first consider a two-cluster RGM equation and the structure of the $T$-matrix constructed from the two-cluster RGM kernel (RGM $T$-matrix). Next, we formulate a three-cluster equation that employs the two-cluster RGM kernel and a projection operator onto the pairwise Pauli-allowed space. This equation is converted into the Faddeev equation, which uses a non-singular part of the RGM $T$-matrix. Finally, we give some examples of the present formulation with respect to the $0^+$ ground states of three di-neutron ($3d'$) and $3\alpha$ systems. The calculation is performed with a variational method, using the translationally invariant harmonic oscillator basis. For the $3d'$ system, the result of the Faddeev calculation is also presented. Detailed comparison is made with the more complete three-cluster RGM calculation and with some other approximations including the "renormalized RGM" and the well-known orthogonality condition model (OCM).

§2. $T$-matrix of the two-cluster RGM kernel

We use the same notation as used in Ref. 6) and write a two-cluster RGM equation as

$$\left[ \varepsilon - H_0 - V^{\text{RGM}}(\varepsilon) \right] \chi = 0 ,$$

(2.1)

where $\varepsilon$ is the total energy in the center-of-mass system, measured from the two-cluster threshold, $\varepsilon = E - E^\text{int}$, $H_0$ is the relative kinetic-energy operator, and

$$V^{\text{RGM}}(\varepsilon) = V_D + G + \varepsilon K ,$$

(2.2)

is the RGM kernel composed of the direct potential $V_D$, the sum of the exchange kinetic-energy and interaction kernels, $G = G^K + G^V$, and the exchange normalization kernel $K$. We assume that there exists only one Pauli-forbidden state $|u\rangle$, which satisfies the eigenvalue equation $K|u\rangle = \gamma |u\rangle$ with the eigenvalue $\gamma = 1$. The projection operator onto the Pauli-allowed space for the relative motion is denoted by $A = 1 - |u\rangle \langle u|$. Using the basic property of the Pauli-forbidden state $|u\rangle$, $(H_0 + V_D + G)|u\rangle = \langle u|(H_0 + V_D + G) = 0$, we can separate $V^{\text{RGM}}(\varepsilon)$ into two distinct parts as

$$V^{\text{RGM}}(\varepsilon) = V(\varepsilon) + v(\varepsilon) ,$$

(2.3)

where

$$V(\varepsilon) = (\varepsilon - H_0) - A(\varepsilon - H_0)A = \varepsilon |u\rangle \langle u| + \Lambda H_0 A - H_0 ,$$

$$v(\varepsilon) = A V^{\text{RGM}}(\varepsilon) A = A (V_D + G + \varepsilon K) A .$$

(2.4)
Note that $\Lambda V(\varepsilon)A = 0$ and $\Lambda v(\varepsilon)A = v(\varepsilon)$; namely, $V(\varepsilon)$ may be considered an operator acting in the Pauli-forbidden space and $v(\varepsilon)$ an operator acting in the Pauli-allowed space. Using these properties, we can express Eq. (2.1) as

$$\Lambda [\varepsilon - H_0 - v(\varepsilon)] A \chi = 0 \ .$$

(2.5)

The separation of $V_{RGM}(\varepsilon)$ in Eq. (2.3) enables us to deal with the energy dependence of the exchange RGM kernel in the Pauli-forbidden space and that in the allowed space separately. Let us generalize Eq. (2.3) to

$$V(\omega, \varepsilon) = V(\omega) + v(\varepsilon) \ ,$$

(2.6)

which we use in the following three-cluster formulation. We will see that the energy dependence involved with $V(\omega)$ can be eliminated by the condition of orthogonality to the Pauli-forbidden state.

Since the direct application of the $T$-matrix formalism to Eq. (2.1) leads to a singular off-shell $T$-matrix, we first consider the subsidiary equation

$$\left[ \omega - H_0 - V_{RGM}(\varepsilon) \right] \chi = 0 \ ,$$

(2.7)

with $\omega \neq \varepsilon$, and extract a singularity-free off-shell $T$-matrix starting from the standard $T$-matrix formulation for the "potential" $V_{RGM}(\varepsilon)$. A formal solution of the $T$-matrix equation

$$T(\omega, \varepsilon) = V_{RGM}(\varepsilon) + V_{RGM}(\varepsilon) G_0^{(+)}(\omega) T(\omega, \varepsilon) \ ,$$

(2.8)

with $G_0^{(+)}(\omega) = 1/(\omega - H_0 + i0)$, is given by

$$T(\omega, \varepsilon) = \tilde{T}(\omega, \varepsilon) + (\omega - H_0)|u\rangle \frac{1}{\omega - \varepsilon} \langle u| (\omega - H_0) \ ,$$

$$\tilde{T}(\omega, \varepsilon) = T_v(\omega, \varepsilon) - \left( 1 + T_v(\omega, \varepsilon) G_0^{(+)}(\omega) \right) |u\rangle \frac{1}{\langle u| G_0^{(+)}(\omega, \varepsilon)|u\rangle}$$

$$\times \langle u| \left( 1 + G_0^{(+)}(\omega) T_v(\omega, \varepsilon) \right) \ ,$$

(2.9)

where $T_v(\omega, \varepsilon)$ is defined by

$$T_v(\omega, \varepsilon) = v(\varepsilon) + v(\varepsilon) G_0^{(+)}(\omega) T_v(\omega, \varepsilon) \ .$$

(2.10)

This result is obtained through the expression for the full Green function given by

$$G^{(+)}(\omega, \varepsilon) = \frac{1}{\omega - H_0 - V_{RGM}(\varepsilon) + i0} = G_A^{(+)}(\omega, \varepsilon) + |u\rangle \frac{1}{\omega - \varepsilon} \langle u| \ ,$$

(2.11)

where

$$G_A^{(+)}(\omega, \varepsilon) = G_v^{(+)}(\omega, \varepsilon) - G_v^{(+)}(\omega, \varepsilon)|u\rangle \frac{1}{\langle u| G_v^{(+)}(\omega, \varepsilon)|u\rangle}$$

$$\times \langle u| G_v^{(+)}(\omega, \varepsilon) \ ,$$

(2.12)
and \( G_v^{(+)}(\omega, \varepsilon) = 1/(\omega - H_0 - v(\varepsilon) + i0) \) is the solution of
\[
G_v^{(+)}(\omega, \varepsilon) = G_0^{(+)}(\omega) + G_0^{(+)}(\omega) v(\varepsilon) G_v^{(+)}(\omega, \varepsilon) .
\]
(2.13)

In fact, the simple relationship
\[
V(\varepsilon) = V(\omega) - (\omega - \varepsilon)|u\rangle\langle u|
\]
yields
\[
\omega - H_0 - V_{\text{RGM}}(\varepsilon) = \Lambda [\omega - H_0 - v(\varepsilon)] \Lambda + (\omega - \varepsilon)|u\rangle\langle u| .
\]
(2.15)

Then, using the property
\[
\Lambda [\omega - H_0 - v(\varepsilon)] \Lambda \cdot G_A^{(+)}(\omega, \varepsilon)
\]
\[
= G_A^{(+)}(\omega, \varepsilon) \cdot \Lambda [\omega - H_0 - v(\varepsilon)] \Lambda = \Lambda ,
\]
(2.16)
it is easily found that
\[
\begin{aligned}
\left[\omega - H_0 - V_{\text{RGM}}(\varepsilon)\right] G^{(+)}(\omega, \varepsilon) \\
= \{ \Lambda [\omega - H_0 - v(\varepsilon)] \Lambda + (\omega - \varepsilon)|u\rangle\langle u| \} \left\{ G_A^{(+)}(\omega, \varepsilon) + |u\rangle \frac{1}{\omega - \varepsilon} \langle u| \right\}
\end{aligned}
\]
\[
= \Lambda + |u\rangle\langle u| = 1 .
\]  
(2.17)

The expression of \( T(\omega, \varepsilon) \) in Eq. (2.9) is most easily obtained from
\[
G^{(+)}(\omega, \varepsilon) = G_0^{(+)}(\omega) + G_0^{(+)}(\omega) T(\omega, \varepsilon) G_0^{(+)}(\omega)
\]
or
\[
T(\omega, \varepsilon) = (\omega - H_0) G^{(+)}(\omega, \varepsilon) (\omega - H_0) - (\omega - H_0) .
\]
(2.19)

The basic relationship that will be used in the following is
\[
G_0^{(+)}(\omega) T(\omega, \varepsilon) = G^{(+)}(\omega, \varepsilon) V_{\text{RGM}}(\varepsilon)
\]
\[
= G_A^{(+)}(\omega, \varepsilon) V_{\text{RGM}}(\varepsilon) + |u\rangle \frac{1}{\omega - \varepsilon} \langle u| V_{\text{RGM}}(\varepsilon)
\]
\[
= G_A^{(+)}(\omega, \varepsilon) V(\omega, \varepsilon) - |u\rangle \langle u| + |u\rangle \frac{1}{\omega - \varepsilon} \langle u|(\omega - H_0)
\]
\[
= G_0^{(+)}(\omega) \tilde{T}(\omega, \varepsilon) + |u\rangle \frac{1}{\omega - \varepsilon} \langle u|(\omega - H_0) ,
\]
(2.20)

where \( \tilde{T}(\omega, \varepsilon) \) satisfies
\[
G_0^{(+)}(\omega) \tilde{T}(\omega, \varepsilon) = G_A^{(+)}(\omega, \varepsilon) V(\omega, \varepsilon) - |u\rangle \langle u| .
\]
(2.21)

These can be shown by using Eqs. (2.11) and (2.14). The full \( T \)-matrix, \( T(\omega, \varepsilon) \), in Eq. (2.9) is singular at \( \varepsilon = \omega \), while \( \tilde{T}(\omega, \varepsilon) \) does not have such a singularity. For \( \varepsilon \neq \omega \), \( T(\omega, \varepsilon) \) satisfies the relationship
\[
\langle u| G_0^{(+)}(\omega) T(\omega, \varepsilon) |\phi\rangle = \langle \phi| T(\omega, \varepsilon) G_0^{(+)}(\omega) |u\rangle = 0
\]
(2.22)
for the plane wave solution $|\phi\rangle$ with energy $\varepsilon$, i.e., $(\varepsilon - H_0)|\phi\rangle = 0$. This relationship is a direct result of the more general relationship
\[
\langle u | \left[ 1 + G_0^{(+)}(\omega)\tilde{T}(\omega, \varepsilon) \right] |u\rangle = 0 ,
\]
which follows simply from Eq. (2.21).

We note that $\tilde{T}(\omega, \varepsilon)$ satisfies
\[
\tilde{T}(\omega, \varepsilon) = V(\omega, \varepsilon) - |u\rangle \langle u| (\omega - H_0) + \tilde{T}(\omega, \varepsilon) G_0^{(+)}(\omega) \tilde{T}(\omega, \varepsilon) .
\]
However, these asymmetric forms of the $T$-matrix equations do not determine the solution $\tilde{T}(\omega, \varepsilon)$ uniquely, since the resolvent kernel $\left[ 1 - V(\omega, \varepsilon) G_0^{(+)}(\omega) \right]^{-1}$ has a singularity related to the existence of the trivial solution $|u\rangle$:
\[
\langle u | \left[ 1 - V(\omega, \varepsilon) G_0^{(+)}(\omega) \right] |u\rangle = 0 .
\]
The leading terms, $V(\omega, \varepsilon) - |u\rangle \langle u| (\omega - H_0)$, etc. in Eq. (2.24), guarantee the existence of the solution, except for an arbitrary admixture of the $(\omega - H_0)|u\rangle$ component. In order to eliminate this ambiguity and make $\tilde{T}(\omega, \varepsilon)$ symmetric, one has to impose some orthogonality conditions. This will be discussed in a separate paper.

§3. Three-cluster equation

Let us consider a system composed of three identical spinless particles, interacting via the two-cluster RGM kernel $V_{RGM}(\varepsilon)$. The energy dependence involved with $V_{RGM}(\varepsilon)$ should be treated properly by calculating the expectation value of the two-cluster subsystem, at least for $\nu(\varepsilon)$. On the other hand, the energy dependence of $V(\varepsilon)$ is of kinematical origin, and could be modified so as to be best suited to the three-cluster equation. The three-body equation we propose is expressed as
\[
P \left[ E - H_0 - V_{RGM}^{\alpha}(\varepsilon_{\alpha}) - V_{RGM}^{\beta}(\varepsilon_{\beta}) - V_{RGM}^{\gamma}(\varepsilon_{\gamma}) \right] P\Psi = 0 ,
\]
where $H_0$ is the free three-body kinetic energy operator and $V_{RGM}^{\alpha}(\varepsilon_{\alpha})$ represents the RGM kernel Eq. (2.2) for the $\alpha$ pair, etc. The two-cluster relative energy $\varepsilon_{\alpha}$ in the three-cluster system is self-consistently determined through
\[
\varepsilon_{\alpha} = \langle P\Psi| h_{\alpha} + V_{RGM}^{\alpha}(\varepsilon_{\alpha}) |P\Psi\rangle ,
\]
using the normalized three-cluster wave function $P\Psi$ with $\langle P\Psi|P\Psi\rangle = 1$. Here, $h_{\alpha}$ is the free kinetic energy operator for the $\alpha$ pair. Also, $P$ is the projection operator onto the $[3]$ symmetric Pauli-allowed space, as defined below. $^9$ We solve the eigenvalue problem
\[
\sum_{\alpha} |u_{\alpha}\rangle \langle u_{\alpha}| \Psi_{\lambda} \rangle = \lambda |\Psi_{\lambda}\rangle
\]
in the $[3]$ symmetric model space ($|\Psi_\lambda\rangle \in [3]$) and define $P$ as a projection onto the space spanned by eigenvectors with eigenvalue $\lambda = 0$:

$$P = \sum_{\lambda=0} |\Psi_\lambda\rangle \langle \Psi_\lambda| .$$

(3-4)

It is easy to prove that $P$ has the following properties:

(i) $\Lambda_\alpha P = P \Lambda_\alpha = P$ for $\forall \alpha$.

(ii) when $\Psi \in [3]$, $\forall \langle u_\alpha | \Psi \rangle = 0 \longleftrightarrow P \Psi = \Psi$.

(iii) when $\Psi \in [3]$, $P \psi = 0 \longleftrightarrow \gamma \langle f \rangle$,

such that $\Psi = |u_\alpha\rangle |f_\alpha\rangle + |u_\beta\rangle |f_\beta\rangle + |u_\gamma\rangle |f_\gamma\rangle$.

(3-5)

Note that all these relations are considered in the $[3]$ symmetric model space, and $P$ and $Q \equiv 1 - P$ are both $[3]$ symmetric three-body operators. Using the property (i), we can simplify Eq. (3-1) as

$$P \left[ E - H_0 - v_\alpha (\varepsilon_\alpha) - v_\beta (\varepsilon_\beta) - v_\gamma (\varepsilon_\gamma) \right] P \Psi = 0 .$$

(3-6)

In order to derive the Faddeev equation corresponding to Eq. (3-1), it is convenient to rewrite Eq. (3-1) or (3-6) as

$$P \left[ E - H_0 - V^{(3)}_\alpha (E, \varepsilon_\alpha) - V^{(3)}_\beta (E, \varepsilon_\beta) - V^{(3)}_\gamma (E, \varepsilon_\gamma) \right] P \Psi = 0 ,$$

(3-7)

where $V^{(3)}_\alpha (E, \varepsilon_\alpha)$ is the three-body operator defined by $V(\omega, \varepsilon)$ in Eq. (2-6) through

$$V^{(3)}_\alpha (E, \varepsilon_\alpha) = V_\alpha (E - h_\alpha, \varepsilon_\alpha) = (E - H_0) - \Lambda_\alpha (E - H_0) \Lambda_\alpha + v_\alpha (\varepsilon_\alpha) .$$

(3-8)

Here $h_\alpha$ is the kinetic energy operator between the $\alpha$ pair and the third particle. The last equation of Eq. (3-8) is derived by using $h_\alpha + h_\alpha = H_0$. The validity of Eq. (3-7) is easily seen from, for example, $P V^{(3)}_\beta (E, \varepsilon_\beta) P = P \Lambda_\beta V^{(3)}_\beta (E, \varepsilon_\beta) \Lambda_\beta P = P v_\beta (\varepsilon_\beta) P$, which uses property (i) of Eq. (3-5). The expression following the first $P$ on the left-hand side of Eq. (3-7) is symmetric with respect to the exchange of the three particles.\(^\dagger\)

Thus, by applying property (iii) in Eq. (3-5), we find

$$\left[ E - H_0 - V^{(3)}_\alpha (E, \varepsilon_\alpha) - V^{(3)}_\beta (E, \varepsilon_\beta) - V^{(3)}_\gamma (E, \varepsilon_\gamma) \right] P \Psi = |u_\alpha\rangle |f_\alpha\rangle + |u_\beta\rangle |f_\beta\rangle + |u_\gamma\rangle |f_\gamma\rangle .$$

(3-9)

where $|f\rangle$ is an unknown function and $|f_\beta\rangle$ and $|f_\gamma\rangle$ are simply obtained from $|f_\alpha\rangle$ by cyclic permutations. Here we invoke the standard ansatz, setting

$$P \Psi = \psi_\alpha + \psi_\beta + \psi_\gamma ,$$

(3-10)

and define $\psi_\alpha$ as the solution of

$$(E - H_0) \psi_\alpha = V^{(3)}_\alpha (E, \varepsilon_\alpha) P \Psi + |u_\alpha\rangle |f_\alpha\rangle .$$

(3-11)

\(^\dagger\) The two-cluster relative energies, $\varepsilon_\alpha$, $\varepsilon_\beta$ and $\varepsilon_\gamma$, are actually all equal, since we are considering three identical particles.
This equation can be written as

\[
E - H_0 - \mathcal{V}_\alpha(E, \varepsilon_\alpha) \psi_\alpha = \mathcal{V}_\alpha(E, \varepsilon_\alpha)(\psi_\beta + \psi_\gamma) + |u_\alpha| f_\alpha, \tag{3.12}
\]

or by using Eq. (3.8), as

\[
A_\alpha \left[ E - H_0 - v_\alpha(\varepsilon_\alpha) \right] A_\alpha \psi_\alpha = \mathcal{V}_\alpha(E, \varepsilon_\alpha)(\psi_\beta + \psi_\gamma) - |u_\alpha\rangle \langle u_\alpha| E - H_0 \psi_\beta + \psi_\gamma. \tag{3.13}
\]

The unknown function \( |f_\alpha\rangle \) can be determined if we multiply this equation by \( \langle u_\alpha| \) from the left and use \( \langle u_\alpha| \mathcal{V}_\alpha(E, \varepsilon_\alpha) = \langle u_\alpha| (E - H_0): \)

\[
|f_\alpha\rangle = -\langle u_\alpha| E - H_0 |\psi_\beta + \psi_\gamma. \tag{3.14}
\]

Thus we obtain

\[
A_\alpha \left[ E - H_0 - v_\alpha(\varepsilon_\alpha) \right] A_\alpha \psi_\alpha = \mathcal{V}_\alpha(E, \varepsilon_\alpha)(\psi_\beta + \psi_\gamma) - |u_\alpha\rangle \langle u_\alpha| E - H_0 |\psi_\beta + \psi_\gamma. \tag{3.15}
\]

By employing the two-cluster relation Eq. (2.21) in the three-cluster model space,

\[
G^{(+)}_\alpha(E, \varepsilon_\alpha) \psi_\alpha = G^{(+)}_0(E) \tilde{T}^{(3)}_\alpha(E, \varepsilon_\alpha) + |u_\alpha\rangle \langle u_\alpha|, \tag{3.16}
\]

and the relationship

\[
G^{(+)}_\alpha(E, \varepsilon_\alpha) A_\alpha \left[ E - H_0 - v_\alpha(\varepsilon_\alpha) \right] A_\alpha = G^{(+)}_\alpha(E, \varepsilon_\alpha) = \Lambda_\alpha, \tag{3.17}
\]

Eq. (3.15) yields

\[
A_\alpha \psi_\alpha = G^{(+)}_0(E) \tilde{T}^{(3)}_\alpha(E, \varepsilon_\alpha)(\psi_\beta + \psi_\gamma) + |u_\alpha\rangle \langle u_\alpha| \psi_\beta + \psi_\gamma. \tag{3.18}
\]

Since \( \langle u_\alpha| \psi_\beta + \psi_\gamma = -\langle u_\alpha| \psi_\alpha \) from Eq. (3.10), we finally obtain

\[
\psi_\alpha = G^{(+)}_0(E) \tilde{T}^{(3)}_\alpha(E, \varepsilon_\alpha)(\psi_\beta + \psi_\gamma). \tag{3.19}
\]

Note that \( \tilde{T}^{(3)}_\alpha(E, \varepsilon_\alpha) \) is essentially the non-singular part of the two-cluster RGM \( T \)-matrix Eq. (2.9),

\[
\tilde{T}^{(3)}_\alpha(E, \varepsilon_\alpha) = \tilde{T}_\alpha(E - h_\alpha, \varepsilon_\alpha), \tag{3.20}
\]

and that the solution of Eq. (3.19) automatically satisfies \( \langle u_\alpha| \psi_\alpha + \psi_\beta + \psi_\gamma = 0 \) due to Eq. (2.23). Since \( \psi_\alpha + \psi_\beta + \psi_\gamma \in [3] \), the property (ii) of Eq. (3.5) yields \( \Psi = P\Psi \) if we set \( \Psi = \psi_\alpha + \psi_\beta + \psi_\gamma \). We can also start from Eq. (3.19) and derive Eq. (3.1) by using the properties (i) and (ii) of Eq. (3.5) and thus establish the equivalence between Eqs. (3.1) and (3.19).
§4. Three di-neutron system

As a simplest non-trivial example, we first consider a three di-neutron \((d')\) system, in which the internal wave function of the \((d')\) is assumed to be the \((0s)\) harmonic oscillator (h.o.) wave function. The normalization kernel \(K\) for the \(2d'\) system is given by \(K = \Lambda = 1 - |u\rangle\langle u|\), and the \(A(\varepsilon K)A\) term in \(v(\varepsilon)\) disappears. Here \(|u\rangle\) is the \((0s)\) h.o. wave function given by \(u(r) = r|u\rangle = (2\nu/\pi)^{3/4}e^{-\nu r^2}\). We assume a very simple two-nucleon interaction of the Serber type,

\[
v_{ij} = -v_0 e^{-\kappa r^2} \frac{1}{2} (1 + P_r) ,
\]

following Ref. 10. That paper investigates a schematic model of an almost forbidden state with \(v_0 = 90\) MeV, but this strength is too weak to give a bound state for the \(3d'\) system. We use the following parameter set in the present calculation: \(\nu = 0.12\) fm\(^{-2}\), \(\kappa = 0.46\) fm\(^{-2}\) and \(v_0 = 153\) MeV. With this value of \(v_0\), the \(2d'\) system is weakly bound.\(^{\ast}\)

In order to solve the three-cluster equation (3.1), we first prepare \([3]\) symmetric translationally invariant h.o. bases according to Moshinsky’s method.\(^{11,\ast\ast}\) The \([3]\) symmetric Pauli-allowed states, which we denote by \(\varphi^{[3]}_{a,n}(\rho, \lambda)\), can be explicitly constructed using the diagonalization procedure in Eq. (3.3). Here, \(\rho = (X_1 - X_2)/\sqrt{2}\) and \(\lambda = (X_1 + X_2 - 2X_3)/\sqrt{6}\) are the Jacobi coordinates for the center-of-mass coordinates \(X_i\) \((i = 1, 2\) and \(3)\) of the three \(d'\) clusters. These eigenstates are specified by the \(SU_3\) quantum number \((\lambda\mu)\) and the set of the other quantum numbers \(n\), which includes the total number of h.o. quanta, \(N\). We then perform a variational calculation using these basis states. We begin this calculation by expanding \(P\Psi\) as

\[
P\Psi = \sum_{(\lambda, \mu), n} c_n^{(\lambda, \mu)} \varphi^{[3]}_{a,n}(\rho, \lambda) .
\]

In the following, we use \(n\) to represent the set of both \((\lambda\mu)\) and \(n\) (and also the possible \(K\) quantum number if the total orbital angular momentum \(L\) does not vanish). Since \(\Psi\) is \([3]\) symmetric, the three interaction terms in Eq. (3.1) give the same contribution. This leads to the eigenvalue equation

\[
\sum_{n'} (E \delta_{nn'} - H_{nn'}) c_{n'} = 0 ,
\]

\[
H_{nn'} = (H_0)_{nn'} + 3 [ (V_D)_{nn'} + G_{nn'} + \varepsilon K_{nn'} ] .
\]

Here the \(K_{nn'}\) term does not contribute in the \(3d'\) problem, since the h.o. matrix elements of \(K\) are 1 for the \((0s)\) state and 0 otherwise, and the \([3]\) symmetric allowed basis does not contain the \((0s)\) component. This implies that our \(d'd'\) interaction is energy independent and the self-consistency condition for \(\varepsilon\) is unnecessary.

\(^{\ast}\) The S-wave phase shift for the \(2d'\) scattering indicates that the \(2d'\) system becomes bound between \(v_0 = 151\) MeV and \(152\) MeV.

\(^{\ast\ast}\) This process is most easily formulated using the theory of double Gel’fand polynomials.\(^{12}\)
Contrastingly, the $3\alpha$ system, which will be discussed in the next section, requires determination of $\varepsilon$ through

$$\varepsilon = \frac{\sum_{n,n'} [h_{nn'} + (V_D)_{nn'} + G_{nn'}] c_n c_{n'}}{1 - \sum_{n,n'} K_{nn'} c_n c_{n'}}.$$  \hspace{1cm} (4.4)

The two-body matrix elements in the three-body space, $O_{nn'} = \langle \varphi_n^3 | O | \varphi_n^{3} \rangle$, can be calculated using a power series expansion of the complex GCM kernel and $SU_3$ Clebsch-Gordan (C-G) coefficients of the type $\langle (N_f)\ell (N_2)\ell | (\lambda\mu)00 \rangle$. Fortunately, a concise expression is given by Hecht and Suzuki \(^{13}\) for this particular type of $SU_3$ C-G coefficients with $L = 0$.

Table I lists the lowest eigenvalues of Eq. (4.3) with an increasing number of total h.o. quanta $N$ included in the calculation. The number of basis states rapidly increases as the maximum $N$ becomes larger. The list in this table is given up until the number of basis states $n_{\text{Max}}$ is over 1000, which is reached near $N = 60$. The convergence of the $3d'$ system is rather slow, since the bound-state energy is especially small in this particular system. The best value obtained in the variational calculation is $E_{3d'} = -0.4341$ MeV, using 3,964 basis states with $N \leq 98$. We have also solved the Faddeev equation (3.19) and obtained $E_{3d'} = -0.4375$ MeV and $-0.4378$ MeV, when the partial waves up to $\ell = 4$ and 6, respectively, are included in the calculation. The final value $E_{3d'} = -0.438$ MeV can also be compared with $E_{3d'}^{\text{RGM}} = -1.188$ MeV, which is obtained with the stochastic variational method \(^{14}\) for the $3d'$ RGM. It is thus found that our result obtained using the three-cluster equation (3.1) gives $3d'$ ground-state energy that is 0.75 MeV higher than that obtained with the full microscopic $3d'$ RGM calculation.

§5. $3\alpha$ System

In this system, the structure of the $2\alpha$ normalization kernel $K$ is more complicated. In the relative $S$-wave we have two Pauli-forbidden states, $(0s)$ and $(1s)$, while in the $D$-wave, only one $(0d)$ h.o. state is forbidden. The relative motion between the two $\alpha$ clusters are Pauli allowed for $N \geq 4$ h.o. quanta. The eigenvalue $\gamma_N$ for $K$ is given by $\gamma_N = 2^{2-N} - 3\delta_{N,0}$, which is 1 ($N = 0$ or 2), 1/4 ($N = 4$), 1/16 ($N = 6$), $\ldots$. The rather large value $\gamma_4 = 1/4$ makes the self-consistent procedure through Eq. (4.4) very important. For the two-body effective interaction, we use the Volkov No. 2 force with $m = 0.59$, following the $3\alpha$ RGM calculation of Fukushima and Kamimura. \(^{15}\) The h.o. constant for the $\alpha$ cluster is $\nu = 0.275$ fm\(^{-2}\), which gives the $\alpha$ cluster internal energy $E_{\alpha} = -27.3$ MeV for the $(0s)^4$ configuration, if the Coulomb interaction is included. (The experimental value is $E_{\alpha}^{\text{exp}} = -28.3$ MeV.) We have carried out the $2\alpha$ RGM calculation by using this parameter set, and found that the present $2\alpha$ system is bound with the binding energy $E_{2\alpha} = -0.245$ MeV (while the experimental value is $E_{2\alpha}^{\text{exp}} = 92$ keV.)

Table I elucidates the convergence of the lowest $3\alpha$ eigenvalues with respect to the maximum total h.o. quanta $N$. We find that the final values of $E_{3\alpha}$ and $\varepsilon$ are $E_{3\alpha} = -5.97$ MeV and $\varepsilon = 9.51$ MeV. If we compare this with $E_{3\alpha}^{\text{RGM}} = -7.5$ MeV
Table I. The lowest eigenvalues for 3d′ and 3α systems, obtained by diagonalization using [3] symmetric translationally invariant h.o. basis. The orthogonality condition due to the projection operator $P$ in Eq. (3.4) is imposed. $N$ represents the maximum total number of h.o. quanta included in the calculation, and $n_{\text{Max}}$ is the number of basis states with orbital angular momentum $L = 0$. The three-cluster equation (3.1) is used for $E_{3d'}$ and $E_{3\alpha}$, while the energy-independent renormalized interaction Eq. (5.2) and $v = \lambda(V_D + V_D^{\text{CL}})A$ are used in Eq. (3.6) for $E_{3\alpha}^{\text{RN}}$ and $E_{3\alpha}^{\text{OCM}}$, respectively.

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(cf. $E_{3\alpha}^{\text{exp't}} = -7.3$ MeV) obtained from the 3α RGM calculation, we find that our result represents a binding that is weaker by 1.5 MeV. The amplitude of the lowest shell-model component with the $SU_3$ (04) representation is $c_{(04)} = 0.790$. We believe that the underbinding compared to the three-cluster RGM calculation is reasonable, since our three-cluster equation does not include some attractive effect due to the genuine three-cluster exchange kernel. Oryu et al. carried out a 3α Faddeev calculation using the 2α RGM kernel. They obtained very low 3α energy $E_{3\alpha} = -28.2$ MeV with no Coulomb force. Since the effect of the Coulomb force is
at most 6 MeV, this 3α energy is extremely low. This may result from the fact that they did not treat the εK term in the RGM kernel properly, and the effect of P in Eq. (3.1) is not fully taken into account in their Faddeev formalism. In order to see the importance of the εK term in Eq. (4.3), it is useful to single out the contribution of this term in the lowest h.o. (04) configuration. The decomposition of $E_{3α}^{(04)} = 12.634$ MeV with $N = 8$ in Table I is

$$H = H_0 + 3(V_D + G_K + G^V + V_{DCL} + G_{CL} + εK)$$
$$= 125.4 + 3(-36.54 - 15.68 + 6.54 + 2.58 - 0.78 + 25.12/4)$$
$$= 125.4 - 3 \times 37.6 = 12.6 \text{ MeV}. \quad (5.1)$$

This example shows very clearly that a self-consistent procedure for the energy-dependence of the RGM kernel in the allowed model space is sometimes very important.

Since the present calculation employs the h.o. basis, it is very easy to examine another approximation that eliminates the explicit energy dependence of $v(ε)$. This approximation is related to the proper normalization of the two-cluster relative wave function $χ$ in Eq. (2.1) through $ψ = \sqrt{1 - K}χ$, and we call this approximation the renormalized RGM. In this formulation, the interaction generated from the RGM kernel is expressed as

$$v = \left( \frac{1}{\sqrt{1 - K}} \right)' (h_0 + V_D + G) \left( \frac{1}{\sqrt{1 - K}} \right)' - Λh_0Λ, \quad (5.2)$$

where the prime in $(1/\sqrt{1 - K})'$ indicates the inversion of $\sqrt{1 - K}$ in the allowed model space. (See Ref. 8 and the discussion in Ref. 6.) The column labeled "$E_{3α}^{RN}$" in Table I lists the results of this approximation. We find $E_{3α}^{RN} = -4.99$ MeV, which represents a binding that is 0.98 MeV weaker than that given by our result. This may indicate that this type of off-shell transformation for the two-cluster RGM kernel is not favorable for the present three-cluster problems. Table I also lists the results obtained from 3α OCM ($E_{3α}^{OCM}$), whose procedure is to use $v = Λ(V_D + V_{DCL})Λ$. We find $E_{3α}^{OCM} = -4.68$ MeV, which is 0.31 MeV greater than $E_{3α}^{RN}$. In this case, 2α OCM gives a stronger binding ($E_{2α}^{OCM} \leq -0.4$ MeV) than the 2α RGM. If we readjust the potential parameter to fit the 2α binding energy, we would apparently obtain an even worse result. It was realized a long time ago that a simple choice of the direct potential $V_D$ for the effective potential $V_{eff}$ in OCM gives a poor result. 17)

§6. Summary

The main purpose of this study is to find an optimum equation for three-cluster systems interacting via a pairwise two-cluster RGM kernel. This is a necessary first step to apply the quark-model baryon-baryon interactions to few-baryon systems. We have found that the orthogonality condition on the pairwise Pauli-forbidden

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1) In our calculation, $E_{3α} = -11.42$ MeV when the Coulomb interaction is not included, which implies that the effect of the Coulomb interaction is 5.45 MeV.
states is a necessary condition to insure a reasonable result. The inherent energy dependence of the two-cluster exchange RGM kernel should be treated self-consistently if the eigenvalues of the exchange normalization kernel $K$ are non-negligible in the allowed space. The proposed three-cluster equation has the nice feature that the equivalent three-cluster Faddeev equation can be straightforwardly formulated using the non-singular part of the full $T$-matrix derived from the RGM kernel. We have applied this equation to simple systems composed of three di-neutrons and three $\alpha$ clusters. The equivalent Faddeev equation was also solved for the three di-neutron system. The results of the Faddeev calculation for the $3\alpha$ system will be reported in a separate paper.\(^\text{18}\) For the ground state of the three $\alpha$ system, the obtained $3\alpha$ energy is 1.5 MeV higher than that obtained from the full microscopic three-cluster RGM calculation. We consider this satisfactory, since the complete three-cluster RGM calculation of the few-baryon systems using quark-quark interaction is still beyond the scope of feasibility. The application to the hypertriton of our quark-model nucleon-nucleon and hyperon-nucleon interactions\(^2\)\(^{-5}\) is now under way.

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References
