N-Point Vertex Functions, Ward-Takahashi Identities and Dyson-Schwinger Equations in Thermal QCD/QED in the Real Time Hard-Thermal-Loop Approximation

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In this paper we calculate the $n$-point hard-thermal-loop (HTL) vertex functions in QCD/QED for $n=2, 3$ and $4$ in the physical representation using the real time formalism (RTF). The result reveals that the $n$-point HTL vertex functions can be classified into two groups, a) those with odd numbers of external retarded indices, and b) those with even numbers of external retarded indices. The $n$-point HTL vertex functions with one retarded index, which obviously belong to group a), are identical to the HTL vertex functions that appear in the imaginary time formalism (ITF). All the HTL vertex functions belonging to group a) are $O(g^2 T^2)$ and satisfy among them the simple QED-type Ward-Takahashi identities, as in the ITF. Those vertex functions belonging to group b) never appear in the ITF, their existence being characteristic of the RTF, and their HTLs exhibit high temperature behavior that is $O(g^2 T^3)$, one power of $T$ higher than usual. Despite this difference, we were able to verify that those HTL vertex functions belonging to group b) also satisfy among themselves the QED-type Ward-Takahashi identities, thus guaranteeing the gauge invariance of the HTLs in the real time thermal QCD/QED.

As an application, we explicitly derive the HTL resummed Dyson-Schwinger equations for the physical fermion mass function and for the gauge boson polarization tensor in thermal QCD/QED.

§1. Introduction and summary

The hard-thermal-loop (HTL) resummed effective perturbation theory of Braaten and Pisarski provides a procedure to extract the dominant temperature effect due to semi-classical thermal fluctuations. HTLs were originally determined in the imaginary time formulation of thermal field theory by computing the one-loop diagrams. For calculating the Feynman diagrams in thermal field theory, there are essentially two different formulations, the imaginary time formalism (ITF) and the real time formalism (RTF). The ITF is restricted to calculating static quantities in equilibrium, while the RTF is necessary for analyzing dynamical quantities and for investigating out-of-equilibrium systems, and thus for studying the physics of quark-gluon plasmas.

The relations between quantities calculated in the ITF and in the RTF have
been studied. It has been found that the $n$-point function in the ITF corresponds to a specific sum of those in the RTF, and that the same is true for the $n$-point HTL functions. In the ITF, all the $n$-point HTL functions have been obtained and have been shown to be ultraviolet finite, to be gauge invariant and to satisfy simple Ward-Takahashi identities.\(^1\)\(^,\)\(^2\) Also in the RTF, the $n$-point HTL functions ($n=2, 3, 4$) and their spectral functions have been partly determined,\(^7\)\(^-\)\(^9\) but explicit expressions of the full $n$-point HTL functions have not yet been consistently obtained.\(^\ast\) Even with such a limited knowledge of the $n$-point HTL functions in the RTF, analyses of the 3-point functions have shown\(^8\),\(^10\) that there exist HTL functions proportional to $g^2 T^3$ (where $g$ is the coupling constant and $T$ is the temperature of the environment) that do not exist in the ITF, where all the $n$-point HTL functions are proportional to $g^2 T^2$. Here, it is worth noting that in the RTF, the Ward-Takahashi identities have been confirmed to hold only among the ITF counterparts.

Recent progress in the understanding of the physics of quark-gluon plasmas and the development of a calculational framework for non-equilibrium systems have motivated us to determine explicitly all the HTLs in the RTF, especially in QCD, along with relations such as the Ward-Takahashi identities they might satisfy.

In this paper we determine explicitly in the RTF, and in particular in the “physical representation” using the closed-time-path (CTP) or Keldysh formalism,\(^4\),\(^5\) the HTL contributions to the $n$-point vertex functions in thermal QCD/QED with $n=2, 3, 4$, which are needed. We also study the structure of the Ward-Takahashi identities that might be satisfied by them. As an application, we derive the Dyson-Schwinger (DS) equations in the HTL approximation for the fermion mass function and for the gauge boson polarization tensor. The results are summarized below.\(^{\ast\ast}\)

i) In the RTF there are two types of $n$-point vertex functions that give HTL contributions, a vertex function with $n$ external gauge bosons (the $ng$ vertex function) and a vertex function with a pair of external fermions and $(n-2)$ external gauge bosons (the $2f-(n-2)g$ vertex function). There are no other $n$-point HTL functions, and thus there are no $n$-point HTL functions with external ghost lines. This situation is precisely the same as that in the ITF.\(^1\)

ii) All the $n$-point HTL vertex functions with one retarded index ($V_{RAA}$, $V_{ARA}$, $V_{RAAA}$, $V_{ARAA}$, etc.) in the physical representation in the CTP formalism have exactly the same expressions as those in the ITF analytically continued through simple but corresponding continuation paths (and thus are proportional to $g^2 T^2$). They satisfy the same Ward-Takahashi identities as in the ITF.\(^1\),\(^2\)

iii) The $n$-point ($n=2, 3, 4$) vertex functions with two retarded indices ($V_{RRA}$, $V_{RRRA}$, etc.) are characteristic of the RTF, and do not appear in the ITF. The

\(^{\ast}\) Defu et al.\(^7\) calculated the four-point HTL vertex functions in QED. Unfortunately, however, their results for the two-fermion-two-gauge-boson (photon) HTL vertex function with two retarded indices (defined below in the main text) are not correct. Aurenche et al.\(^9\) calculated in QED the $n$-point vertex functions and verified the QED-type Ward-Takahashi identities using a different formalism, the so-called retarded-advanced formalism, in which full use was made of the equilibrium condition.

\(^{\ast\ast}\) Our notation is the same as that used in Ref. 7), and differs slightly from the original notation in Ref. 4). Details regarding the definitions and notation are given in §2.
corresponding $n$-point HTL vertex functions must be classified into two groups: the $ng$ vertex functions and the $2f-(n-2)g$ vertex functions. Other types of $n$-point vertex functions in general never have HTL contributions, as mentioned in i) above. It is quite remarkable that in QCD, the $ng$ vertex functions with two retarded indices are proportional to $g^2T^3$, in contrast to the $O(g^2T^2)$ behavior of the usual HTL vertex functions. Thus the $ng$ vertex functions are expected to play important roles in the study of temperature effects. Nevertheless, it can be verified that the QED-type Ward-Takahashi identities between the corresponding $ng$- and $(n-1)g$ HTL vertex functions are satisfied, guaranteeing the gauge invariance of the HTL approximation. Thus we can prove, through explicit calculations of the $n$-point HTL vertex functions, the gauge invariance of the real time thermal QCD/QED in the HTL approximation. It is also worth noting that the HTL contributions to the $2f-(n-2)g$ vertex functions with two retarded indices vanish.$^\ast$ Thus, in QED there are no additional HTL functions other than those appearing in the ITF.

iv) In carrying out the calculation of the Feynman diagrams in the RTF, we should be very careful in the treatment of the singular functions (the Dirac $\delta$ function and the principal part) appearing in the free propagators. We must use properly regularized forms in the calculation and should take the limit $\varepsilon \to 0$ at the end of all calculations, as noted by Landsman and van Wheert.$^4$  

v) We explicitly write down the HTL resummed DS equations for the physical fermion mass function $\Sigma_R$ and for the gauge boson polarization tensor $\Pi_{\mu\nu}^R$ in thermal QCD/QED. The DS equation for $\Sigma_R$ can be used to investigate the nature of the chiral phase transition at finite temperature, and the equation for $\Pi_{\mu\nu}^R$ can be used to study the thermal photon production rate and the magnetic mass generation. A comment on the double counting problem is also given.

As noted in the first footnote of this paper, the same results for QED were obtained with a different formalism,$^9$ the retarded-advanced formalism, in which full use was made of the equilibrium condition. We determine the vertex functions and DS equations in QCD and in QED using the real time CTP formalism, which can be easily extended to the non-equilibrium situation.

This paper is organized as follows. In §2 we give a brief review of the “physical representation” in the CTP or Keldysh formalism. In §3 the $n$-point HTL vertex functions ($n=2, 3, 4$) in QCD and QED are explicitly determined in the physical representation. The necessity of the regularized forms of the singular functions is demonstrated. The HTL Ward-Takahashi identities between the four- and three-point functions and between the three- and two-point functions are explicitly verified. As an application, the DS equations in the HTL approximation for the physical fermion mass function $\Sigma_R$ and for the gauge boson polarization tensor $\Pi_{\mu\nu}^R$ are derived in §5. Conclusions and some discussion are given in §6.

$^\ast$ This result differs from that in Ref. 7), in which it is claimed that there exist non-vanishing HTL contributions, proportional to $g^2T^2$, in the $2f-2g$ 4-point vertex functions with two retarded indices. The corresponding Ward-Takahashi identities they claim to hold are simply $0 = 0$ in our case.


§2. The real time closed-time-path or Keldysh formalism

We use the closed-time-path (CTP) or Keldysh formalism of the RTF throughout this paper. In this formalism there are two familiar representations, the “single time” representation and the “physical” representation (using the terminology of Ref. 4)). In the single-time representation, the single-particle propagator for free bosons has the $2 \times 2$ matrix form

$$
\hat{D}(K) = \begin{pmatrix}
D_{11}(K) & D_{12}(K) \\
D_{21}(K) & D_{22}(K)
\end{pmatrix},
$$

(2.1)

where $D_{ij}$ ($i,j=1,2$) are given in momentum space by

$$
D_{11}(K) = \frac{1 + n_B(k_0)}{K^2 - m^2 + i\varepsilon} - \frac{n_B(k_0)}{K^2 - m^2 - i\varepsilon},
$$

(2.2a)

$$
D_{12}(K) = \left[\theta(-k_0) + n_B(k_0)\right] \left( \frac{1}{K^2 - m^2 + i\varepsilon} - \frac{1}{K^2 - m^2 - i\varepsilon} \right),
$$

(2.2b)

$$
D_{21}(K) = \left[\theta(k_0) + n_B(k_0)\right] \left( \frac{1}{K^2 - m^2 + i\varepsilon} - \frac{1}{K^2 - m^2 - i\varepsilon} \right),
$$

(2.2c)

$$
D_{22}(K) = \frac{n_B(k_0)}{K^2 - m^2 + i\varepsilon} - \frac{1 + n_B(k_0)}{K^2 - m^2 - i\varepsilon}.
$$

(2.2d)

Here, $K^\mu = (k_0, k)$, $\theta$ denotes the step function, and the equilibrium distribution function is given by $n_B(k_0) = 1/\left[\exp(|k_0|/T) - 1\right]$. For fermions, the bare propagator can also be written as the $2 \times 2$ matrix

$$
\hat{S}(K) = \begin{pmatrix}
S_{11}(K) & S_{12}(K) \\
S_{21}(K) & S_{22}(K)
\end{pmatrix},
$$

(2.3)

with

$$
S_{ij} = (K - m)\bar{D}_{ij}(K),
$$

(2.4a)

$$
\bar{D}_{ij} = D_{ij}\{n_B(k_0) \rightarrow -n_F(k_0)\}, \ (i,j = 1, 2)
$$

(2.4b)

where the Fermi-Dirac distribution is given by $n_F(k_0) = 1/\left[\exp(|k_0|/T) + 1\right]$. The components of these propagators are not independent, but satisfy the relation

$$
G_{11} + G_{22} = G_{12} + G_{21},
$$

(2.5)

where $G$ stands for $D$ or $S$.

Using the orthogonal transformation defined by

$$
\hat{G} = Q^{-1}\tilde{G}Q, \ \tilde{G} = Q\hat{G}Q^{-1},
$$

(2.6)

where $G = D$ or $S$ and

$$
Q = \frac{1}{\sqrt{2}}(1 - i\sigma_2),
$$

(2.7)

we arrive at the propagator in the physical representation,
\[ \tilde{D}(K) = \begin{pmatrix} D_{AA}(K) & D_{AR}(K) \\ D_{RA}(K) & D_{RR}(K) \end{pmatrix}, \] (2.8)

where

\[ D_{AA}(K) = 0, \] (2.9a)

\[ D_{AR}(K) = \frac{1}{K^2 - m^2 - i \text{sgn}(k_0)\varepsilon}, \] (2.9b)

\[ D_{RA}(K) = \frac{1}{K^2 - m^2 + i \text{sgn}(k_0)\varepsilon}, \] (2.9c)

\[ D_{RR}(K) = (1 + 2n_B(k_0))\text{sgn}(k_0)[D_{RA}(K) - D_{AR}(K)] \] (2.9d)

for bosons, and

\[ \tilde{S}(K) = \begin{pmatrix} S_{AA}(K) & S_{AR}(K) \\ S_{RA}(K) & S_{RR}(K) \end{pmatrix}, \] (2.10)

\[ S_{AA}(K) = 0, \] (2.11a)

\[ S_{AR}(K) = (K - m)\frac{1}{K^2 - m^2 + i \text{sgn}(k_0)\varepsilon}, \] (2.11b)

\[ S_{RA}(K) = (K - m)\frac{1}{K^2 - m^2 - i \text{sgn}(k_0)\varepsilon}, \] (2.11c)

\[ S_{RR}(K) = (1 - 2n_F(k_0))\text{sgn}(k_0)[S_{RA}(K) - S_{AR}(K)] \] (2.11d)

for fermions, where the last equation in both Eqs. (2.9) and (2.11) is a consequence of the fluctuation-dissipation theorem.

The propagator (connected 2-point Green function) \( G \) and the 1-particle irreducible 2-point vertex function \( \Sigma \) satisfy the Dyson equations

\[ \int \tilde{\Sigma} \sigma_3 \tilde{G} = \sigma_3, \quad \int \tilde{G} \sigma_3 \tilde{\Sigma} = \sigma_3, \] (2.12a)

\[ \int \tilde{\Sigma} \sigma_1 \tilde{G} = \sigma_1, \quad \int \tilde{G} \sigma_1 \tilde{\Sigma} = \sigma_1, \] (2.12b)

where \( G = D \) or \( S \) and, correspondingly, \( \Sigma \) denotes the bosonic or fermionic 2-point vertex, i.e. the self-energy function. In the above, the relations (2.1), (2.3), (2.5)–(2.8), (2.10) and (2.12) also hold for full propagators and full 2-point vertex functions.

It is also worth noting here that in the above equations for the free propagators, Eqs. (2.1)–(2.4) and (2.8)–(2.11), we do not use expressions that explicitly contain the singular functions themselves, but, rather, use functions with the regularization parameter \( \varepsilon \), e.g.,

\[ \frac{1}{[K^2 - m^2 + i\varepsilon]} = \text{PP} \frac{1}{[K^2 - m^2]} - i\pi \delta(K^2 - m^2), \] (2.13)
where

\[
PP_1 \equiv \frac{x}{x^2 + \varepsilon^2}; \quad \text{(2.14a)}
\]

and

\[
\pi \delta(x) \equiv \frac{\varepsilon}{x^2 + \varepsilon^2}. \quad \text{(2.14b)}
\]

Keeping \( \varepsilon \) finite (i.e., \( \varepsilon \neq 0 \)) until the end of all calculations and then taking the limit \( \varepsilon \to 0 \) at the end is extremely important, as we see explicitly in the next section.

§3. The \( n \)-point (\( n=2, 3, 4 \)) vertex functions in QCD/QED in the HTL approximation

In this section we calculate the HTL contributions to the one-particle irreducible \( n \)-point vertex functions (\( n=2, 3, 4 \)) in the physical representation. In general, the \( n \)-point vertex functions can be constructed in the physical or the retarded-advanced representation from the components of the real-time \( n \)-point functions in the single-time representation.\(^4,6,7,11,12\) (Some care must be taken when consulting these references, because the notation varies among them.) Throughout this paper, we use the following notation. A given one-particle irreducible \( n \)-point Feynman diagram in the single-time representation contains a Keldish index at each end (external vertex) taking the values 1 and 2, corresponding to the two branches of the closed-time-path contour, while in the physical representation it contains a retarded-advanced index at each end that is either \( R \) or \( A \), corresponding to the retarded and advanced prescriptions. Our notation follows that of Ref. \(^7\).

3.1. The \( n \)-point vertex functions (\( n=2, 3, 4 \)) in the physical representation

For convenience, we here present explicitly the 2-, 3- and 4-point vertex functions in the physical representation constructed from the components in the single-time representation. Generally speaking, the \( n \)-point function has \( 2^n \) components. These components obey one constraint equation, which reduces the number of independent components to \( 2^n - 1 \). In the physical representation, this is expressed by the fact that the \( n \)-point vertex function with \( n \) external advanced indices automatically vanishes (see Eqs. (3.1a), (3.3a) and (3.4a) below). In equilibrium, the Kubo-Martin-Schwinger conditions impose additional constraints, reducing the number of independent components to \( 2^{(n-1)} - 1 \). (For details of the construction, see Refs. \(^4\) and \(^7\).) Since the transformation formulas between the vertex functions in the two representations do not depend on the external particle species (fermion or gauge boson), here we simply write \( V_{\alpha \beta ... \delta}, \{ \alpha, \beta, ..., \delta = R, A \} \) and \( V_{ij ... l}, \{ i, j, ..., l = 1, 2 \} \) for vertex functions in the physical and the single-time representations, respectively.

3.1.1. 2-point vertex functions

The 2-point vertex functions are as follows:

\[
V_{AA} = 0, \quad \text{(3.1a)}
\]

\[
V_{RA} = V_{11} + V_{12}, \quad \text{(3.1b)}
\]

\[
V_{AR} = V_{11} + V_{21}, \quad \text{(3.1c)}
\]
Among them, we have the relations
\begin{align}
V_{AR}(-K,K) &= [V_{RA}(-K,K)]^*, \\
V_{RR}(-K,K) &= \{1 + \eta 2n(k_0)\} \text{sgn}(k_0) \{V_{RA}(-K,K) - V_{AR}(-K,K)\},
\end{align}
where \(\eta = +1(-1)\) for a boson (fermion) and \(n(k_0)\) is the corresponding equilibrium distribution function.

### 3.1.2. 3-point vertex functions

The 3-point vertex functions are the following:
\begin{align}
V_{AAA} &= 0, \\
V_{RAA} &= V_{111} + V_{112} + V_{121} + V_{122}, \\
V_{ARA} &= V_{111} + V_{112} + V_{211} + V_{212}, \\
V_{AAR} &= V_{111} + V_{121} + V_{211} + V_{221}, \\
V_{RRA} &= V_{111} + V_{112} + V_{221} + V_{222}, \\
V_{RAR} &= V_{111} + V_{121} + V_{212} + V_{222}, \\
V_{ARR} &= V_{111} + V_{211} + V_{122} + V_{222}, \\
V_{RRR} &= V_{111} + V_{122} + V_{212} + V_{221}.
\end{align}

The Kubo-Martin-Schwinger conditions impose four additional constraints, reducing the number of independent components to three.

### 3.1.3. 4-point vertex functions

The 4-point vertex functions are
\begin{align}
V_{AAAA} &= 0, \\
V_{RAAA} &= V_{1111} + V_{1112} + V_{1121} + V_{1211} + V_{1122} + V_{1212} + V_{1221} + V_{1222}, \\
V_{ARAA} &= V_{1111} + V_{1112} + V_{1121} + V_{2111} + V_{1122} + V_{2112} + V_{2121} + V_{2122}, \\
V_{AARA} &= V_{1111} + V_{1112} + V_{1211} + V_{1212} + V_{1221} + V_{2211} + V_{2212} + V_{2221}, \\
V_{AAAR} &= V_{1111} + V_{1121} + V_{1211} + V_{2111} + V_{1212} + V_{2121} + V_{2122} + V_{2212}, \\
V_{RRAA} &= V_{1111} + V_{1112} + V_{1212} + V_{2112} + V_{2212} + V_{2221} + V_{2222}, \\
V_{RAR} &= V_{1111} + V_{1112} + V_{1212} + V_{2112} + V_{2212} + V_{2221} + V_{2222}, \\
V_{RAAR} &= V_{1111} + V_{1121} + V_{1211} + V_{1221} + V_{2112} + V_{2121} + V_{2122} + V_{2212} + V_{2222}.
\end{align}

There are seven independent vertex functions, Eqs. (3.4a)–(3.4h), in this case, and the other eight vertex functions can be obtained from them using the Kubo-Martin-Schwinger conditions.

### 3.2. The \(n\)-point vertex functions \((n=2, 3, 4)\) in the HTL approximation

Now we calculate the \(n\)-point vertex functions \((n=2, 3, 4)\) in the physical representation in the HTL approximation. In the following we consider massless QCD/QED, that is, high temperature hot QCD/QED, in which all the fermions and gauge bosons are massless.
Because we are accustomed to the Feynman rules in the single-time representation, we calculate explicitly the right-hand sides of Eqs. (3.1), (3.3) and (3.4). As stressed above, we use Eqs. (2.13) and (2.14) for propagators. In other words, we do not use expressions that explicitly contain the singular functions themselves, but, rather, use functions with the regularization parameter $\varepsilon$ and keep $\varepsilon$ finite until the end of all calculations. The limit $\varepsilon \to 0$ is taken at the end.

3.2.1. 2-point vertex functions: the fermion self-energy and the gauge boson polarization tensor

\[ \hat{=}-i \delta \Sigma(P, Q) \quad P + Q = 0 \]

Fig. 1. One-loop diagram for the fermion self-energy and the definition of its HTL, $\delta \Sigma$.

Here, $\hat{=}$ represents equality in the HTL approximation.

The 2-point vertex function in QCD/QED is usually called the self-energy part $\Sigma$ for fermions and the (vacuum) polarization tensor $\Pi^{\mu\nu}$ for gauge bosons (gluons and photons). Although the HTL results for $\Sigma$ and $\Pi^{\mu\nu}$ in the single-time representation and in the ITF are well known,\(^1\),\(^13\) for the sake of completeness here we present the results in the physical representation.

The HTL contribution to the fermion self-energy, $\delta \Sigma$, in the single-time representation is obtained in QCD by calculating the diagram shown in Fig. 1 (in QED, $g^2 C_F$ should read $e^2$),

\[ -i \delta \Sigma_{ij}(P, Q) = 2g^2 C_F \int \frac{d^4 K}{(2\pi)^4} K D_{ij}(K + Q) D_{ji}(K), \quad P + Q = 0, \quad (3.5) \]

where $D_{ij}$ and $\bar{D}_{ij}$ are given in Eqs. (2·2) and (2·4). With the use of Eq. (3·1) we obtain the HTL fermion self-energy in the physical representation. There are nothing new in the results obtained for QCD. We find only the previous results,

\[ \delta \Sigma_{RA}(P, Q) = -\frac{m_f^2}{4\pi} \int d\Omega \frac{\hat{K}}{Q \cdot \hat{K} + i\varepsilon}, \quad (3.6a) \]
\[ \delta \Sigma_{AR}(P, Q) = -\frac{m_f^2}{4\pi} \int d\Omega \frac{\hat{K}}{Q \cdot \hat{K} - i\varepsilon}, \quad (3.6b) \]
\[ \delta \Sigma_{RR}(P, Q) = 0, \quad (3.6c) \]

where

\[ m_f^2 \equiv \begin{cases} \frac{1}{8} e^2 T^2, & \text{(QED)} \\ \frac{1}{8} g^2 C_F T^2, & \text{(QCD)} \end{cases} \quad (3.7) \]

and $\hat{K}^{\mu} = (1, \hat{k})$ with $\hat{k} \equiv k/k$, $k = \sqrt{k^2}$.

As expressed in Eq. (3·1), we usually denote the RA component as the R (i.e. retarded) component and the AR component as the A (i.e. advanced) component of the corresponding quantity. In other words, $\Sigma_{RA} \equiv \Sigma_R$ is the fermion self-energy
Fig. 2. One-loop diagram for the gauge boson polarization tensor and the definition of its HTL, \(\delta \Pi^{\mu \nu}\). In QED, only the fermion loop exists.

part of the inverse retarded fermion propagator, and has a definite physical meaning, i.e., the physical fermion mass function.

For the HTL contribution to the gauge boson polarization tensor, \(\delta \Pi^{\mu \nu}\), the diagrams to be calculated in the single-time representation are shown in Fig. 2 \((P + Q = 0)\). We have

\[
i \delta \Pi_{ij}^{\mu \nu}(P, Q) = \begin{cases} 
-4e^2(-1)^{i+j-2} \int \frac{d^4K}{(2\pi)^4} \left[ 2K^\mu K^\nu - g^{\mu \nu} K^2 \right] \\
\times & \mathcal{D}_{ij}(K)\mathcal{D}_{ji}(K + P), \quad \text{(QED)} \\
2g^2(-1)^{i+j-2} \int \frac{d^4K}{(2\pi)^4} \left[ 2K^\mu K^\nu - g^{\mu \nu} K^2 \right] \\
\times & \{N_c \mathcal{D}_{ij}(K)\mathcal{D}_{ji}(K + P) \\
& - N_f \mathcal{D}_{ij}(K)\mathcal{D}_{ji}(K + P) \}, \quad \text{(QCD)}.
\end{cases}
\]

from which we obtain the HTL gluon polarization tensor,

\[
\delta \Pi_{RA}^{\mu \nu}(P, Q) = -\frac{m_g^2}{4\pi} \int d\Omega \left( \hat{K}^{\mu} \hat{K}^{\nu} \frac{q_0}{\hat{K} \cdot Q + i\varepsilon} - g^{\mu \nu} g^{00} \right), \quad (3.9a)
\]

\[
\delta \Pi_{AR}^{\mu \nu}(P, Q) = -\frac{m_g^2}{4\pi} \int d\Omega \left( \hat{K}^{\mu} \hat{K}^{\nu} \frac{q_0}{\hat{K} \cdot Q - i\varepsilon} - g^{\mu \nu} g^{00} \right), \quad (3.9b)
\]

\[
\delta \Pi_{RR}^{\mu \nu}(P, Q) = -\frac{m_g^2 T}{2\pi} \int d\Omega \hat{K}^{\mu} \hat{K}^{\nu} \left( \frac{1}{\hat{K} \cdot Q + i\varepsilon} - \frac{1}{\hat{K} \cdot Q - i\varepsilon} \right), \quad (3.9c)
\]

where

\[
m_g^2 = \begin{cases} 
1 \frac{e^2 T^2}{3}, \quad \text{(QED)} \\
\frac{1}{3} g^2 T^2 \left( N_c + \frac{1}{2} N_f \right). \quad \text{(QCD)}
\end{cases}
\]

The quantity \(\Pi_{RA}^{\mu \nu} = \Pi_{AR}^{\mu \nu} = \Pi_{RR}^{\mu \nu}\) is the gauge boson polarization tensor in the inverse retarded (advanced) gauge boson propagator, and thus has a definite physical meaning.
It should be noted that the RR component of the polarization tensor, $\delta \Pi_{RR}^{\mu\nu}$, is proportional to $g^{2}T^{3}$. This should be compared to the ordinary retarded or advanced polarization tensor, $\delta \Pi_{RA}^{\mu\nu}$ or $\delta \Pi_{AR}^{\mu\nu}$, which is proportional to $g^{2}T^{2}$. Also, note that all the HTL contributions in the ITF are of order $O(g^{2}T^{2})$. Thus the above results show that completely new vertex functions appear in the RTF.

3.2.2. 3-point vertex functions

There are two types of 3-point vertex functions, the fermion-gauge-boson vertex function and the 3-gauge-boson vertex function. In QED, only the fermion-gauge-boson vertex function exists.

- The fermion-gauge-boson vertex functions

We define the fermion-gauge-boson vertex function in the HTL approximation as in Fig. 3, where

$$^*\Gamma^\mu = \gamma^\mu + \delta \Gamma^\mu, \quad (3.11a)$$

with $\gamma^\mu$ representing the tree vertex,

$$\gamma^\mu_{ijk} \equiv \begin{cases} (-)^{(i-1)}\gamma^\mu & \text{for } i = j = k, \\ 0 & \text{otherwise}, \end{cases} \quad (3.11b)$$

Fig. 3. Definition of the fermion-gauge boson HTL resummed vertex function $^*\Gamma^\mu$.

Fig. 4. One-loop diagrams for the fermion-gauge boson vertex function. In QED, only diagram (a) exists.
and the HTL contribution to the fermion-gauge-boson vertex function, \( \delta \Gamma^\mu \), is obtained by calculating the one-loop diagrams, shown in Fig. 4:

\[
\delta \Gamma^\mu_{ijk}(P, Q, R) = \begin{cases} 
4ie^2 \int \frac{d^4K}{(2\pi)^4} K^\mu K \tilde{V}_{ijk}, & (\text{QED}) \\
4ig^2 C_F \int \frac{d^4K}{(2\pi)^4} K^\mu K \tilde{V}_{ijk} \\
-2ig^2 N_c \int \frac{d^4K}{(2\pi)^4} K^\mu K (\tilde{V}_{ijk} + V_{ijk}), & (\text{QCD}) 
\end{cases}
\] (3.12a)

where

\[
V_{ijk} \equiv (-1)^{i+j+k-3}D_{ij}(K)D_{jk}(K-Q)D_{ki}(K+P),
\] (3.12b)

\[
\tilde{V}_{ijk} \equiv (-1)^{i+j+k-3}D_{ij}(K)\bar{D}_{jk}(K-Q)\bar{D}_{ki}(K+P).
\] (3.12c)

The results in the physical representation are given as follows:

\[
\delta \Gamma^\mu_{RAA}(P, Q, R) = -\frac{m_f^2}{4\pi} \int d\Omega \frac{\hat{K}^\mu \hat{K}}{(\hat{K} \cdot P - i\varepsilon)(\hat{K} \cdot Q + i\varepsilon)},
\] (3.13a)

\[
\delta \Gamma^\mu_{ARA}(P, Q, R) = -\frac{m_f^2}{4\pi} \int d\Omega \frac{\hat{K}^\mu \hat{K}}{(\hat{K} \cdot P + i\varepsilon)(\hat{K} \cdot Q - i\varepsilon)},
\] (3.13b)

\[
\delta \Gamma^\mu_{AAR}(P, Q, R) = -\frac{m_f^2}{4\pi} \int d\Omega \frac{\hat{K}^\mu \hat{K}}{(\hat{K} \cdot P + i\varepsilon)(\hat{K} \cdot Q + i\varepsilon)},
\] (3.13c)

\[
\delta \Gamma^\mu_{RRR}(P, Q, R) = -\frac{m_f^2}{4\pi} \int d\Omega \frac{\hat{K}^\mu \hat{K}}{(\hat{K} \cdot P - i\varepsilon)(\hat{K} \cdot Q - i\varepsilon)},
\] (3.13d)

\[
\delta \Gamma^\mu_{RRA}(P, Q, R) = \delta \Gamma^\mu_{RAR}(P, Q, R) = \delta \Gamma^\mu_{ARR}(P, Q, R) = \delta \Gamma^\mu_{AAA}(P, Q, R) = 0.
\] (3.13e)

We can see that all the HTL terms of the fermion-gauge-boson vertex function, \( \delta \Gamma^\mu \), are proportional to \( g^2 T^2 \).

In obtaining the above results, it is important to use from the beginning the regularized expressions for free propagators with the regularization parameter \( \varepsilon (\neq 0) \), Eqs. (2.1)–(2.4) and (2.13)–(2.14), not the expressions in terms of the explicit singular functions. To see this point more clearly, let us calculate the above HTL vertex function \( \delta \Gamma^\mu_{RAA} \) in Eq. (3.13a) explicitly in QED. In each of the integrals in (3.13a)–(3.13d), we must evaluate four diagrams in the single-time representation, given in Eq. (3.3). Thus, for example, we must perform the integration for \( \delta \Gamma^\mu_{111}(P, Q, R) \),

\[
\delta \Gamma^\mu_{111}(P, Q, R) = 4ie^2 \int \frac{d^4K}{(2\pi)^4} D_{11}(K)\overline{D}_{11}(K-Q)\overline{D}_{11}(K+P).
\] (3.14)
using residue analysis. By ignoring the $O(e^2T)$ contributions, we obtain, after some
manipulations (here we set $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon$),

$$\delta \Gamma_{111}^\mu(P, Q, R) = -\frac{e^2}{(2\pi)^3} \int kdk \ n(k) d\Omega \frac{K_+ P}{(K_+ P)^2 + (\epsilon)^2} \frac{K_+ Q}{(K_+ Q)^2 + (\epsilon)^2},$$

where $n(k) = n_B(k) + n_F(k)$ and $K_+^\mu = k(1, \hat{k}) = k\hat{K}^\mu$ with $\hat{k} = k/k$. A similar
calculation gives

$$\delta \Gamma_{112}^\mu(P, Q, R) = -\frac{e^2}{(2\pi)^3} \int kdk \ n(k) d\Omega \frac{\epsilon}{(K_+ P)^2 + (\epsilon)^2} \frac{\epsilon}{(K_+ Q)^2 + (\epsilon)^2},$$

(3.16a)

$$\delta \Gamma_{121}^\mu(P, Q, R) = -i \frac{e^2}{(2\pi)^3} \int kdk \ n(k) d\Omega \frac{K_+ P}{(K_+ P)^2 + (\epsilon)^2} \frac{\epsilon}{(K_+ Q)^2 + (\epsilon)^2},$$

(3.16b)

$$\delta \Gamma_{122}^\mu(P, Q, R) = i \frac{e^2}{(2\pi)^3} \int kdk \ n(k) d\Omega \frac{\epsilon}{(K_+ P)^2 + (\epsilon)^2} \frac{K_+ Q}{(K_+ Q)^2 + (\epsilon)^2}.$$  

(3.16c)

Adding Eqs. (3.15) and (3.16a)–(3.16c), we obtain Eq. (3.13a).

If we naively take the limit $\epsilon \to 0$ before the loop-momentum integration over $K^\mu$ and use in Eq. (3.14) the free propagators explicitly containing the singular Dirac $\delta$ function and the principal part, then the naive calculation gives, instead of Eq. (3.15),

$$\delta \Gamma_{111}^\mu(P, Q, R) = -\frac{e^2}{(2\pi)^3} \int kdk \ n(k) d\Omega \frac{K_+ P}{(K_+ P)^2 + (\epsilon)^2} \frac{K_+ Q}{(K_+ Q)^2 + (\epsilon)^2} \frac{\epsilon}{(K_+ P)^2 + (\epsilon)^2} \frac{(K_+ Q)^2 + (\epsilon)^2}{\delta(K_+ P)\delta(K_+ Q)}. $$

(3.15')

The other three vertex functions coincide with the $\epsilon \to 0$ limit of Eqs. (3.16a)–
(3.16c), and we cannot get Eq. (3.13a). The additional term in Eq. (3.15'), being
proportional to the product of Dirac $\delta$ functions, may have its origin in an integral
whose integrand is the product of two principal parts:

$$\int dy \int dx PP \left[ \frac{1}{x - a(y)} \right] PP \left[ \frac{1}{x - b(y)} \right].$$

(3.17)

If at some $y = y_0$ inside the integration range over $y$, the relation $a(y) = b(y)$ holds,
then we must perform the integration over $x$ of the square of the principal part,
which cannot be well-defined. This is exactly what happens in the above calculation
of Eq. (3.14) if the free propagators explicitly contain the singular Dirac $\delta$
function and the principal part. Without concrete and consistent prescriptions stipulating
how to treat a product of singular functions, we cannot obtain a definite result.

The $\epsilon$-regularization method gives such a prescription. As we have already noted,
throughout this paper we use $\epsilon$-regularized singular functions.
Three gauge-boson (three gluon) vertex functions

We define the three gluon vertex function in the HTL approximation as in Fig. 5, where

\[ * \Gamma^{\mu\nu\rho} = V^{\mu\nu\rho} + \delta \Gamma^{\mu\nu\rho}, \]  

(3.18)

with \( V^{\mu\nu\rho} \) representing the tree vertex, which does not exist when the number of external retarded indices is zero or two. The HTL contribution to the three gluon vertex function, \( \delta \Gamma^{\mu\nu\rho} \), is obtained by calculating the one-loop diagrams shown in Fig. 6:

\[
\delta \Gamma^{\mu\nu\rho}_{ijk}(P, Q, R) = -\frac{ig^2}{2\pi^4}(-1)^{i+k+j-3} \int d^4 K \ K^\mu K^\nu K^\rho \\
\times \left[ N_c D_{ij}(K) D_{jk}(K-Q) D_{ki}(K+P) - N_f \overline{D}_{ij}(K) \overline{D}_{jk}(K-Q) \overline{D}_{ki}(K+P) \right].
\]  

(3.19)

By using Eq. (3.3) we obtain the HTL results in the physical representation,

\[
\delta \Gamma^{\mu\nu\rho}_{RRA}(P, Q, R) = -\frac{m_a^2}{4\pi} \int d\Omega \hat{K}^\mu \hat{K}^\nu \hat{K}^\rho \\
\times \left[ \frac{p_0}{(\hat{K} \cdot P - i\varepsilon)(\hat{K} \cdot R + i\varepsilon)} - \frac{q_0}{(\hat{K} \cdot Q + i\varepsilon)(\hat{K} \cdot R + i\varepsilon)} \right],
\]  

(3.20a)

\[
\delta \Gamma^{\mu\nu\rho}_{ARA}(P, Q, R) = -\frac{m_a^2}{4\pi} \int d\Omega \hat{K}^\mu \hat{K}^\nu \hat{K}^\rho \\
\times \left[ \frac{p_0}{(\hat{K} \cdot P + i\varepsilon)(\hat{K} \cdot R + i\varepsilon)} - \frac{q_0}{(\hat{K} \cdot Q - i\varepsilon)(\hat{K} \cdot R - i\varepsilon)} \right],
\]  

(3.20b)

\[
\delta \Gamma^{\mu\nu\rho}_{AAR}(P, Q, R) = -\frac{m_a^2}{4\pi} \int d\Omega \hat{K}^\mu \hat{K}^\nu \hat{K}^\rho \\
\times \left[ \frac{p_0}{(\hat{K} \cdot P + i\varepsilon)(\hat{K} \cdot R - i\varepsilon)} - \frac{q_0}{(\hat{K} \cdot Q + i\varepsilon)(\hat{K} \cdot R - i\varepsilon)} \right],
\]  

(3.20c)

Fig. 6. One-loop diagrams for the 3-gauge boson vertex function. (d) represents all possible diagrams with distinct configurations of the three external legs with momenta \( P, Q \) and \( R \).
\[ \delta \Gamma_{RRR}^{\mu \nu \rho}(P, Q, R) = -\frac{m_2^2}{4\pi} \int d\Omega \hat{K}^{\mu} \hat{K}^{\nu} \hat{K}^{\rho} \times \left[ \frac{p_0}{(\hat{K} \cdot P - i\epsilon)(\hat{K} \cdot R - i\epsilon)} - \frac{q_0}{(\hat{K} \cdot Q - i\epsilon)(\hat{K} \cdot R - i\epsilon)} \right], \quad (3.20d) \]

\[ \delta \Gamma_{RRA}^{\mu \nu \rho}(P, Q, R) = -\frac{m_2^2 T}{2\pi} \int d\Omega \hat{K}^{\mu} \hat{K}^{\nu} \hat{K}^{\rho} \times \left[ \frac{1}{(\hat{K} \cdot P - i\epsilon)(\hat{K} \cdot Q + i\epsilon)} - \frac{1}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot Q - i\epsilon)} \right], \quad (3.20e) \]

\[ \delta \Gamma_{RAR}^{\mu \nu \rho}(P, Q, R) = -\frac{m_2^2 T}{2\pi} \int d\Omega \hat{K}^{\mu} \hat{K}^{\nu} \hat{K}^{\rho} \times \left[ \frac{1}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot R - i\epsilon)} - \frac{1}{(\hat{K} \cdot P - i\epsilon)(\hat{K} \cdot R + i\epsilon)} \right], \quad (3.20f) \]

\[ \delta \Gamma_{ARR}^{\mu \nu \rho}(P, Q, R) = -\frac{m_2^2 T}{2\pi} \int d\Omega \hat{K}^{\mu} \hat{K}^{\nu} \hat{K}^{\rho} \times \left[ \frac{1}{(\hat{K} \cdot Q - i\epsilon)(\hat{K} \cdot R + i\epsilon)} - \frac{1}{(\hat{K} \cdot Q + i\epsilon)(\hat{K} \cdot R - i\epsilon)} \right], \quad (3.20g) \]

\[ \delta \Gamma_{AAA}^{\mu \nu \rho}(P, Q, R) = 0. \quad (3.20h) \]

It should be noted that, depending on the number of external retarded indices, the HTL three-gluon vertex functions can be classified into two groups, that for which the number of retarded indices is 1 or 3 and that for which the number of retarded indices is 0 or 2. Any vertex belonging to the first group has a tree vertex and is proportional to \( g^2 T^2 \), while any vertex belonging to the second group does not have a tree term and is proportional to \( g^2 T^3 \), having no counterpart in the ITF.

### 3.2.3. 4-point vertex functions

There are two types of 4-point vertex functions, the fermion-pair 2-gauge boson vertex functions and the 4-gauge boson vertex functions. In QED, as we see below, all the HTL contributions to the 4-photon vertex functions vanish.

\[ \{ QCD \} \]

\[ \{ QED \} \]

\[ P + Q + R + S = 0 \]

Fig. 7. Definition of the fermion-pair 2-gauge boson HTL resummed vertex function \( \Gamma^{\mu \nu} \). The color indices of the external gluons with momenta \( R \) and \( S \) are summed over. This is indicated by the fact that they have the same color index, \( c \).
Fig. 8. One-loop diagrams for the fermion-pair 2-gauge boson vertex function. The diagrams obtained by exchanging the external gauge boson legs with momenta \( R \) and \( S \) should also be added. In QED, only the diagram (a) and its exchanged diagram exist.

- **One fermion-pair 2-gauge boson vertex functions**

  We define the fermion-pair 2-gauge boson vertex functions in the HTL approximation as in Fig. 7, where the summation over the color indices of gluons with momenta \( R \) and \( S \) is understood

\[
\delta \Gamma_{\mu\nu} = \delta \Gamma_{\mu\nu}^\text{\ast}. \tag{3.21}
\]

The HTL contribution to the fermion-pair 2-gauge boson vertex function, \( \delta \Gamma_{\mu\nu} \), is obtained by calculating the one-loop diagrams shown in Fig. 8 and diagrams obtained by exchanging the external legs with momenta \( R \) and \( S \). We find

\[
\delta \Gamma_{ijkl}^{\mu\nu}(P, Q, R, S) = \begin{cases} 
- \frac{i8\epsilon^2}{m^2} \int \frac{d^4 K}{(2\pi)^4} K^\mu K^\nu K \\
\times D_{ij}(K) \bar{D}_{jk}(K - Q) \bar{D}_{kl}(K + P + S) \bar{D}_{li}(K + P), \\
\text{QED} \\
- \frac{i8g^2}{m^2} C_F (-1)^{i+j+k+l} \int \frac{d^4 K}{(2\pi)^4} K^\mu K^\nu K \\
\times \left[ C_F D_{ij}(K) \bar{D}_{jk}(K - Q) \bar{D}_{kl}(K + P + S) \bar{D}_{li}(K + P) \\
- N_c \bar{D}_{ij}(K) D_{jk}(K - Q) D_{kl}(K + P + S) D_{li}(K + P) \\
+ \frac{1}{2} N_c D_{ij}(K + P) D_{jl}(K - P - S) \bar{D}_{lk}(K + R) \bar{D}_{ki}(K) \right], \\
\text{QCD} \\
\end{cases}
\]  

(3.22)

The results in the physical representation are, with Eq. (3.4), given as follows:

\[
\delta \Gamma_{RAAA}^{\mu\nu}(P, Q, R, S) = \frac{m^2}{4\pi} \int d\Omega \hat{K}^\mu \hat{K}^\nu \hat{K} \\
\times \frac{1}{\hat{K} \cdot P - i\epsilon} \frac{1}{\hat{K} \cdot Q + i\epsilon} \left[ \frac{1}{\hat{K} \cdot (P + S) - i\epsilon} + \frac{1}{\hat{K} \cdot (P + R) - i\epsilon} \right], \tag{3.23a}
\]

\[
\delta \Gamma_{ARAA}^{\mu\nu}(P, Q, R, S) = \frac{m^2}{4\pi} \int d\Omega \hat{K}^\mu \hat{K}^\nu \hat{K}
\]

\[\ast \text{ Conversely, when taking the trace in color space over the fermion loop, the color factor } C_F \text{ should be considered } \frac{1}{2} N_f.\]
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\[ S, c, \sigma \quad R, c, \rho \]

\[ P, a, \mu \quad Q, b, \nu \]

\[ = \quad -i g^2 \delta^{ab} N_c \Gamma^{\mu\nu\rho\sigma}(P, Q, R, S); \]

\[ P + Q + R + S = 0 \]

Fig. 9. Definition of the 4-gluon HTL resummed vertex function \( \Gamma^{\mu\nu\rho\sigma} \). Color indices of the external gluons with momenta \( R \) and \( S \) are summed over. This is indicated by the fact that they have the same color index, \( c \).

\[
\delta \Gamma_{AAAR}^{\mu\nu}(P, Q, R, S) = \frac{m_f^2}{4\pi} \int d\hat{K} \hat{K}^{\mu} \hat{K}^{\nu} \hat{K} \\
\times \frac{1}{\hat{K} \cdot P + i\varepsilon} \hat{K} \cdot Q - i\varepsilon \left[ \frac{1}{\hat{K} \cdot (P + S) + i\varepsilon} + \frac{1}{\hat{K} \cdot (P + R) + i\varepsilon} \right]
\]

\[
\delta \Gamma_{AAAR}^{\mu\nu}(P, Q, R, S) = \frac{m_f^2}{4\pi} \int d\hat{K} \hat{K}^{\mu} \hat{K}^{\nu} \hat{K} \\
\times \frac{1}{\hat{K} \cdot P + i\varepsilon} \hat{K} \cdot Q + i\varepsilon \left[ \frac{1}{\hat{K} \cdot (P + S) + i\varepsilon} + \frac{1}{\hat{K} \cdot (P + R) - i\varepsilon} \right]
\]

\[
\delta \Gamma_{RRAA}^{\mu\nu}(P, Q, R, S) = \delta \Gamma_{RARA}^{\mu\nu}(P, Q, R, S) = 0
\]

The other eight vertex functions can be obtained from Eqs. (3.23a)–(3.23e) using the KMS conditions. Again, all the HTL contributions to the vertex functions with external fermion legs are \( O(g^2 T^2) \).

A comment on Eq. (3.23e) is in order. Hou Defu et al.\(^{12}\) calculated the same HTL vertex functions and claimed the existence of non-zero HTL contributions, with only the first function \( \delta \Gamma_{RRAA}^{\mu\nu}(P, Q, R, S) \) vanishing in equilibrium. However, as we explicitly show, all three functions vanish in the HTL approximation, as long as the propagation of thermal “quasi-particles” can be described by the form of the free particle propagators Eqs. (2.1)–(2.4), whether the system is in equilibrium or just out of equilibrium.

**4-gauge boson vertex functions**

The HTL contribution to the 4-photon vertex function in QED vanishes. For this reason, we confine our interest to the 4-gluon vertex functions in QCD. Because of the complexity of their tensor structure and our interest in their application, we calculate the HTLs in the case that the vertex functions are summed over the color indices of the two external gluon legs.
Defining the 4-gluon vertex functions in the HTL approximation as in Fig. 9, where the summation over the color indices of the gluons with momenta $R$ and $S$ is understood and

$$^* \Gamma^{\mu\nu\rho\sigma} = W^{\mu\nu\rho\sigma} + \delta \Gamma^{\mu\nu\rho\sigma}, \quad (3.24a)$$

with $W^{\mu\nu\rho\sigma}$ representing the tree vertex,

$$W^{\mu\nu\rho\sigma}_{ijkl} \equiv \begin{cases} (-1)^i(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) & \text{for } i = j = k = l, \\ 0 & \text{otherwise}. \end{cases} \quad (3.24b)$$

The HTL contribution to the 4-gluon vertex function, $\delta \Gamma^{\mu\nu\rho\sigma}$, is obtained by calculating the one-loop diagrams shown in Fig. 10 and all possible diagrams obtained by exchanging the external legs with momenta $Q$, $R$ and $S$. (Note that the color indices of the gluons with momenta $R$ and $S$ are summed over.) We have

$$\delta \Gamma_{ijkl}^{\mu\nu\rho\sigma}(P, Q, R, S) = -g^2 \frac{32}{N_c} \int \frac{d^4 K}{(2\pi)^4} K^\mu K^\nu K^\rho K^\sigma$$

$$\times \left[ N_c^2 \left\{ V_{ijkl}(P, Q, R, S) + V_{ijk}(P, Q, S, R) \right\} - C_F N_F \left\{ \tilde{V}_{ijkl}(P, Q, R, S) ight\} + \tilde{V}_{ijk}(P, Q, S, R) \right] + \frac{N_c^2}{2} \tilde{V}_{ikjl}(P, R, Q, S) + \frac{N_f}{2N_c} \tilde{V}_{ikjl}(P, R, Q, S), \quad (3.25a)$$
where
\[ V_{ijkl} \equiv (-1)^{i+j+k+l} D_{ij}(K)D_{jk}(K-Q)D_{kl}(K+P+S)D_{li}(K+P), \]
\[ \tilde{V}_{ijkl} \equiv (-1)^{i+j+k+l} \tilde{D}_{ij}(K)\tilde{D}_{jk}(K-Q)\tilde{D}_{kl}(K+P+S)\tilde{D}_{li}(K+P). \]

The results in the physical representation are, with the use of Eq. (3.4), given as follows:

\[
\delta \Gamma^{\mu\nu\rho\sigma}_{RAAA}(P,Q,R,S) = -\frac{m^2_q}{2\pi} \int d\Omega \hat{K}^\mu \hat{K}^\nu \hat{K}^\rho \hat{K}^\sigma \frac{1}{\hat{K} \cdot S + i\varepsilon} \frac{1}{\hat{K} \cdot R + i\varepsilon} \\
\times \left[ \frac{p_0 + r_0}{\hat{K} \cdot (P+R) - i\varepsilon} - \frac{q_0 - r_0}{\hat{K} \cdot Q - i\varepsilon} \frac{p_0}{\hat{K} \cdot P - i\varepsilon} + \frac{q_0 + r_0}{\hat{K} \cdot (Q+R) + i\varepsilon} \right], \tag{3.26a}
\]

\[
\delta \Gamma^{\mu\nu\rho\sigma}_{ARAA}(P,Q,R,S) = -\frac{m^2_q}{2\pi} \int d\Omega \hat{K}^\mu \hat{K}^\nu \hat{K}^\rho \hat{K}^\sigma \frac{1}{\hat{K} \cdot S + i\varepsilon} \frac{1}{\hat{K} \cdot R - i\varepsilon} \\
\times \left[ \frac{p_0 + r_0}{\hat{K} \cdot (P+R) + i\varepsilon} - \frac{q_0 - r_0}{\hat{K} \cdot Q + i\varepsilon} \frac{p_0}{\hat{K} \cdot P + i\varepsilon} + \frac{q_0 + r_0}{\hat{K} \cdot (Q+R) - i\varepsilon} \right], \tag{3.26b}
\]

\[
\delta \Gamma^{\mu\nu\rho\sigma}_{AARA}(P,Q,R,S) = -\frac{m^2_q}{2\pi} \int d\Omega \hat{K}^\mu \hat{K}^\nu \hat{K}^\rho \hat{K}^\sigma \frac{1}{\hat{K} \cdot S - i\varepsilon} \frac{1}{\hat{K} \cdot R + i\varepsilon} \\
\times \left[ \frac{p_0 + r_0}{\hat{K} \cdot (P+R) - i\varepsilon} - \frac{q_0 - r_0}{\hat{K} \cdot Q - i\varepsilon} \frac{p_0}{\hat{K} \cdot P - i\varepsilon} + \frac{q_0 + r_0}{\hat{K} \cdot (Q+R) + i\varepsilon} \right], \tag{3.26c}
\]

\[
\delta \Gamma^{\mu\nu\rho\sigma}_{AAAR}(P,Q,R,S) = -\frac{m^2_T}{2\pi} \int d\Omega \hat{K}^\mu \hat{K}^\nu \hat{K}^\rho \hat{K}^\sigma (P - Q) \cdot \hat{K} \\
\times \left[ \frac{1}{\hat{K} \cdot (P+R) + i\varepsilon} \frac{1}{\hat{K} \cdot P + i\varepsilon} \frac{1}{\hat{K} \cdot (Q+R) - i\varepsilon} \frac{1}{\hat{K} \cdot Q - i\varepsilon} - c.c. \right], \tag{3.26d}
\]

\[
\delta \Gamma^{\mu\nu\rho\sigma}_{RRAA}(P,Q,R,S) = -\frac{m^2_T}{2\pi} \int d\Omega \hat{K}^\mu \hat{K}^\nu \hat{K}^\rho \hat{K}^\sigma \\
\times \left[ \frac{1}{\hat{K} \cdot P + i\varepsilon} \frac{1}{\hat{K} \cdot (Q+R) - i\varepsilon} \frac{1}{\hat{K} \cdot R - i\varepsilon} - c.c. \right], \tag{3.26e}
\]

\[
\delta \Gamma^{\mu\nu\rho\sigma}_{RARA}(P,Q,R,S) = -\frac{m^2_T}{2\pi} \int d\Omega \hat{K}^\mu \hat{K}^\nu \hat{K}^\rho \hat{K}^\sigma \\
\times \left[ \frac{1}{\hat{K} \cdot P - i\varepsilon} \frac{1}{\hat{K} \cdot (P+R) - i\varepsilon} \frac{1}{\hat{K} \cdot S + i\varepsilon} - c.c. \right]. \tag{3.26f}
\]

\[
\delta \Gamma^{\mu\nu\rho\sigma}_{RRAAR}(P,Q,R,S) = -\frac{m^2_T}{2\pi} \int d\Omega \hat{K}^\mu \hat{K}^\nu \hat{K}^\rho \hat{K}^\sigma \\
\times \left[ \frac{1}{\hat{K} \cdot P - i\varepsilon} \frac{1}{\hat{K} \cdot (P+R) - i\varepsilon} \frac{1}{\hat{K} \cdot S + i\varepsilon} - c.c. \right]. \tag{3.26g}
\]

The other eight vertex functions can be obtained from Eqs. (3.26a)–(3.26g) using the KMS conditions. It is worth noting that in this case, again depending on the number of external retarded indices, the HTL four-gluon vertex functions can be classified into two groups, that for which the number of retarded indices is 1 or 3.
and that for which the number of retarded indices is 0 or 2. Any vertex belonging to the first group is \(O(g^2 T^2)\), and any vertex belonging to the second group is \(O(g^2 T^3)\), having no counterpart in the ITF. The existence of the \(O(g^2 T^3)\) HTL contribution is characteristic of the \(n\)-gauge boson vertex functions.

§4. Ward-Takahashi identities in the HTL approximation

Having determined the 2-, 3- and 4-point vertex functions in QCD and QED in the HTL approximation, in this section we verify that all the Ward-Takahashi identities satisfied among them are simple QED-type identities. As we have shown, a HTL \(n\)-gauge boson vertex function with an even number of external retarded indices exhibits, in contrast to the ordinary \(O(g^2 T^2)\) behavior of other vertex functions, \(O(g^2 T^3)\) behavior, which does not exist in any amplitude in the ITF. We here verify that they also satisfy the simple QED-type Ward-Takahashi identities, thus guaranteeing the gauge invariance of the HTL approximation in the real-time thermal QCD/QED.

4.1. Ward-Takahashi identities between the three- and four-point vertex functions in QCD/QED

4.1.1. Between the 3- and 4-point HTL vertex functions with a fermion pair

With the exception of the group factor, the 3- and 4-point HTL vertex functions with a fermion pair are the same in QCD and QED. For this reason, we study the Ward-Takahashi identities between them in QCD. Comparing the results in Eqs. (3.23) and (3.13), we obtain the Ward-Takahashi identities between the vertices shown in Figs. 7 and 3:

\[
\begin{align*}
R_\mu^\ast \Gamma^\mu_{RAA}(P, Q, R, S) &= \delta \Gamma^\nu_{RRA}(P + R, Q, S) - \delta \Gamma^\nu_{RRA}(P, Q + R, S), \\
R_\mu^\ast \Gamma^\mu_{ARA}(P, Q, R, S) &= \delta \Gamma^\nu_{ARA}(P + R, Q, S) - \delta \Gamma^\nu_{ARA}(P, Q + R, S), \\
R_\mu^\ast \Gamma^\mu_{AAR}(P, Q, R, S) &= \delta \Gamma^\nu_{ARA}(P + R, Q, S) - \delta \Gamma^\nu_{ARA}(P, Q + R, S), \\
R_\mu^\ast \Gamma^\mu_{AAR}(P, Q, R, S) &= \delta \Gamma^\nu_{AAR}(P + R, Q, S) - \delta \Gamma^\nu_{AAR}(P, Q + R, S).
\end{align*}
\]  

When we operate with \(S_\mu\) in place of \(R_\mu\), then obviously the roles of \(S\) and \(R\) are exchanged in Eqs. (4.1), and also the roles of Eqs. (4.1c) and (4.1d) are exchanged.

4.1.2. Between the 3- and 4-gluon HTL vertex functions

In QED there are no 3-photon vertex functions, and the 4-photon vertex functions vanish in the HTL approximation. For this reason, we study the relation between the 3- and 4-gluon HTL vertex functions in QCD. Comparing the results in Eqs. (3.26) and (3.20), we obtain the following Ward-Takahashi identities between vertices shown in Figs. 9 and 5:

\[
\begin{align*}
R_\mu \delta \Gamma^\mu_{RAA}(P, Q, R, S) &= \delta \Gamma^\nu_{RRA}(P + R, Q, S) - \delta \Gamma^\nu_{RRA}(P, Q + R, S), \\
R_\mu \delta \Gamma^\mu_{ARA}(P, Q, R, S) &= \delta \Gamma^\nu_{ARA}(P + R, Q, S) - \delta \Gamma^\nu_{ARA}(P, Q + R, S), \\
R_\mu \delta \Gamma^\mu_{AAR}(P, Q, R, S) &= \delta \Gamma^\nu_{ARA}(P + R, Q, S) - \delta \Gamma^\nu_{ARA}(P, Q + R, S), \\
R_\mu \delta \Gamma^\mu_{AAR}(P, Q, R, S) &= \delta \Gamma^\nu_{AAR}(P + R, Q, S) - \delta \Gamma^\nu_{AAR}(P, Q + R, S).
\end{align*}
\]
When we operate with $S_\mu$ in place of $R_\mu$, then obviously the roles of $S$ and $R$ are exchanged in Eqs. (4.2), and also the roles of Eqs. (4.2f) and (4.2g) are exchanged, together with the roles of Eqs. (4.2c) and (4.2d). The situation is somewhat complicated in this case, and we give the identities in explicit form below:

\begin{align*}
S_\mu \delta \Gamma^{\mu\nu\rho}_{RAAA}(P, Q, R, S) &= \delta \Gamma^{\mu\nu\rho}_{RRAA}(P + S, Q, R) - \delta \Gamma^{\mu\nu\rho}_{RRAA}(P, Q + S, R), \quad (4.3a) \\
S_\mu \delta \Gamma^{\mu\nu\rho}_{ARAAR}(P, Q, R, S) &= \delta \Gamma^{\mu\nu\rho}_{ARAAR}(P + S, Q, R) - \delta \Gamma^{\mu\nu\rho}_{ARAAR}(P, Q + S, R), \quad (4.3b) \\
S_\mu \delta \Gamma^{\mu\nu\rho}_{AARAR}(P, Q, S, R) &= \delta \Gamma^{\mu\nu\rho}_{AARAR}(P + S, Q, R) - \delta \Gamma^{\mu\nu\rho}_{AARAR}(P, Q + S, R), \quad (4.3c) \\
S_\mu \delta \Gamma^{\mu\nu\rho}_{ARARA}(P, Q, R, S) &= \delta \Gamma^{\mu\nu\rho}_{ARARA}(P + S, Q, R) - \delta \Gamma^{\mu\nu\rho}_{ARARA}(P, Q + S, R), \quad (4.3d) \\
S_\mu \delta \Gamma^{\mu\nu\rho}_{RAARA}(P, Q, R, S) &= \delta \Gamma^{\mu\nu\rho}_{RAARA}(P + S, Q, R) - \delta \Gamma^{\mu\nu\rho}_{RAARA}(P, Q + S, R), \quad (4.3e) \\
S_\mu \delta \Gamma^{\mu\nu\rho}_{RRAAR}(P, Q, R, S) &= \delta \Gamma^{\mu\nu\rho}_{RRAAR}(P + S, Q, R) - \delta \Gamma^{\mu\nu\rho}_{RRAAR}(P, Q + S, R).
\end{align*}

It is worth noting that Eqs. (4.2e)–(4.2g) and (4.3e)–(4.3g) are the QED-type Ward-Takahashi identities satisfied by the vertices with $O(g^2 T^3)$ behavior. Thus, these are absent in the ITF.

4.2. Ward-Takahashi identities between the two- and three-point vertex functions in QCD/QED

4.2.1. Between the 2- and 3-point HTL vertex functions with a fermion pair

In this case, again, because the only difference between the HTL vertex functions in QCD and QED is the group factor, by comparing the results Eqs. (3.13) and (3.6), it is easy to see that we can verify the same Ward-Takahashi identities in QCD and QED between the vertices shown in Figs. 3 and 1:

\begin{align*}
R_\mu \delta \Gamma^{\mu}_{RAA}(P, Q, R) &= \delta \Sigma_{RA}(P, Q + R) - \delta \Sigma_{RA}(P + R, Q), \quad (4.4a) \\
R_\mu \delta \Gamma^{\mu}_{ARA}(P, Q, R) &= \delta \Sigma_{AR}(P, Q + R) - \delta \Sigma_{AR}(P + R, Q), \quad (4.4b) \\
R_\mu \delta \Gamma^{\mu}_{AAR}(P, Q, R) &= \delta \Sigma_{AR}(P, Q + R) - \delta \Sigma_{RA}(P + R, Q), \quad (4.4c) \\
R_\mu \delta \Gamma^{\mu}_{RAR}(P, Q, R) &= \delta \Sigma_{RA}(P, Q + R) - \delta \Sigma_{AR}(P + R, Q). \quad (4.4d)
\end{align*}

4.2.2. Between the 2- and 3-gluon HTL vertex functions

In QED there are no 3-photon vertex functions. For this reason, we study the relation between the 2- and 3-gluon HTL vertex functions in QCD. Comparing the results in Eqs. (3.20) and (3.9), we obtain the following Ward-Takahashi identities between the vertices shown in Figs. 5 and 2:

\begin{align*}
R_\mu \delta \Gamma^{\mu\nu\rho}_{RRAA}(P, Q, R) &= \delta \Pi^{\mu\nu\rho}_{RRA}(P, Q + R) - \delta \Pi^{\mu\nu\rho}_{RRA}(P + R, Q), \quad (4.5a) \\
R_\mu \delta \Gamma^{\mu\nu\rho}_{ARAAR}(P, Q, R) &= \delta \Pi^{\mu\nu\rho}_{ARAAR}(P, Q + R) - \delta \Pi^{\mu\nu\rho}_{ARAAR}(P + R, Q), \quad (4.5b) \\
R_\mu \delta \Gamma^{\mu\nu\rho}_{AARAR}(P, Q, R) &= \delta \Pi^{\mu\nu\rho}_{AARAR}(P, Q + R) - \delta \Pi^{\mu\nu\rho}_{AARAR}(P + R, Q), \quad (4.5c) \\
R_\mu \delta \Gamma^{\mu\nu\rho}_{ARARA}(P, Q, R) &= \delta \Pi^{\mu\nu\rho}_{ARARA}(P, Q + R) - \delta \Pi^{\mu\nu\rho}_{ARARA}(P + R, Q), \quad (4.5d)
\end{align*}
\[ R_\mu \delta \Gamma_{R \mu \nu \rho}^{\mu \nu \rho}(P, Q, R) = \delta \Pi_{RR}^{\nu \rho}(P, Q + R), \] (4.5e)
\[ R_\mu \delta \Gamma_{ARR}^{\mu \nu \rho}(P, Q, R) = -\delta \Pi_{RR}^{\nu \rho}(P + R, Q), \] (4.5f)
\[ R_\mu \delta \Gamma_{RRR}^{\mu \nu \rho}(P, Q, R) = \delta \Pi_{RA}^{\nu \rho}(P, Q + R) - \delta \Pi_{AR}^{\nu \rho}(P + R, Q). \] (4.5g)

Here again Eqs. (4.5e)–(4.5f) are the QED-type Ward-Takahashi identities satisfied by vertex functions with \( O(g^2T^3) \) behavior.

In §3 we showed that in the physical representation in the real time CPT formalism there exist vertex functions with two (presumably an even number in the arbitrary \( n \)-point case) external retarded indices, having high temperature behavior that is \( O(g^2T^3) \). We should recall that, in the ITF, there are no \( n \)-point functions with \( O(g^2T^3) \) behavior. In this section we verify that those vertex functions with high temperature behavior that is \( O(g^2T^3) \) also satisfy the simple QED-type Ward-Takahashi identities in the HTL approximation. Thus we can show explicitly the gauge invariance of the HTLs in the real time thermal QCD/QED.

§5. Dyson-Schwinger equation in the HTL approximation

Having determined all possible ingredients, namely the HTL contributions to the 2-, 3- and 4-point vertex functions, we can now write down the Dyson-Schwinger (DS) equations in the HTL approximation. Before doing this, however, let us here recall the reason we have calculated the HTL vertex functions in the physical representation in the real time CTP formalism of thermal field theory.

As noted in §3, to investigate the consequences for the physical mass, we need to study the mass function of the inverse of the retarded propagator. To investigate the chiral phase transition, we need the DS equation for the retarded component of the fermion mass function, \( \Sigma_R \), and to study the magnetic screening, we need the DS equation for the retarded component of the gluon polarization tensor, \( \Pi_{RR}^{\mu \nu} \). Nevertheless, in many studies,\(^{14}\) the 11 component of the fermion self-energy in the single-time representation, \( \Sigma_{11} \) (not \( \Sigma_R \)), has been studied, with its imaginary part ignored. As is well known, \( \text{Re} \Sigma_R = \text{Re} \Sigma_{11} \), but \( \text{Im} \Sigma_R \neq \text{Im} \Sigma_{11} \). The lesson learned from the HTL resummed perturbation theory\(^{15,16}\) reminds us that it is in fact the imaginary parts of \( \Sigma_R \) and \( \Pi_{RR}^{\mu \nu} \) that experience the important thermal effects, with these effects communicated indirectly to the real parts. In this sense, it is important to construct the DS equations for \( \Sigma_R \) and for \( \Pi_{RR}^{\mu \nu} \).

Now we are ready to write down the DS equations. The DS equation for the fermion self-energy \( \Sigma_R \) in the HTL approximation can be obtained by applying the following approximation to the full DS equation:

i) Replace the full gauge boson propagator with the HTL resummed propagator.

ii) Approximate the full vertex functions with the HTL resummed vertex functions.

Following this procedure, we obtain in QCD the desired DS equation,

\[ -i \Sigma_R(P) = -i \Sigma_{RA}(-P, P) = -g^2 C_F 2 \int \frac{d^4 K}{(2\pi)^4} \]
\[ \times \left( \Gamma_{R AA}^{\mu}(-P, K, P - K)S_{RA}(K) \Gamma_{R AA}^{\nu}(-K, P, K - P) G_{RR, \mu \nu}(P - K) \right) \]
\[ + \Gamma^\mu_{RAA}(-P,K,P-K)S_{RR}(K)\Gamma^\nu_{RAA}(-K,P,K-P)G_{RA,\mu\nu}(P-K), \] (5.1)

(while for QED, \(g^2C_F\) here should read \(e^2\)), where \(\Gamma^\mu\) is the HTL resummed quark-gluon vertex, defined in Eq. (3.11) and given in Eq. (3.12) with \(\gamma^\mu_{RAA} = \gamma^\mu_{AAR} = \gamma^\mu\). The quantity \(G^{\mu\nu}\) is the HTL resummed gluon propagator whose retarded/advanced form is given by

\[
*G^\mu_{RA/AR}(K) = \frac{1}{\Pi^R_{T/A} - K^2 \mp \text{sgn}(k_0)\varepsilon} A^\mu + \frac{1}{\Pi^R_{T/A} - K^2 \mp \text{sgn}(k_0)\varepsilon} B^\mu
- \frac{\xi}{K^2 \mp \text{sgn}(k_0)\varepsilon} D^\mu, \tag{5.2a}
\]

where \(\xi\) is the gauge-fixing parameter (\(\xi = 0\) in the Landau gauge), and the \(RR\) component is given by

\[
*G^\mu_{RR}(K) = (1 + 2n_B(k_0))\text{sgn}(k_0)\{ *G^\mu_{RA}(K) - *G^\mu_{AR}(K) \}, \tag{5.2b}
\]

with \(*\Pi^R_{T/A}\) and \(*\Pi^R_{L/A}\) being the HTL contributions to the retarded/advanced gluon self-energy of the transverse and longitudinal modes, respectively. \(^{13}\) In Eq. (5.2a), \(A^\mu, B^\mu\) and \(D^\mu\) are the projection tensors \(^{17}\)

\[
A^\mu(K) \equiv g^\mu - B^\mu(K) - D^\mu(K), \tag{5.3a}
\]

\[
B^\mu(K) \equiv -\frac{\vec{K}^\mu \vec{K}^\nu}{K^2}, \tag{5.3b}
\]

\[
D^\mu(K) \equiv \frac{K^\mu K^\nu}{K^2}, \tag{5.3c}
\]

where \(\vec{K}^\mu \equiv (k,k_0\hat{k})\), with \(\hat{k} \equiv k/k\). The quantities \(S_{RA}\) and \(S_{AR}\) are the retarded and advanced full fermion propagators,

\[
S_{RA/AR}(P) = \frac{1}{P \pm i\varepsilon - \Sigma_{R/A}}, \tag{5.4a}
\]

and \(S_{RR}\) is the \(RR\) component, or the correlation,

\[
S_{RR}(P) = (1 - 2n_F(p_0))\text{sgn}(p_0)\{S_{RA}(P) - S_{AR}(P)\}. \tag{5.4b}
\]

The DS equation for the gauge boson polariztion tensor \(\Pi_{R}^{\mu\nu}\) in the HTL approximation becomes (in QED, \(\frac{1}{2}g^2N_f \rightarrow e^2\))

\[
i\Pi_{R}^{\mu\nu}(P) = i\Pi_{RA}^{\mu\nu}(-P,P) = \frac{1}{4}g^2N_f \int \frac{d^4K}{(2\pi)^4} \left[ *\Gamma_{RAA}^{\mu}(-P,K,P-K)S_{RA}(K) *\Gamma_{RAA}^{\nu}(-K,P,K-P)S_{RR}(P-K) + *\Gamma_{RAA}^{\mu}(-P,K,P-K)S_{RR}(K) *\Gamma_{AAR}^{\nu}(-K,P,K-P)S_{RA}(P-K) \right]
\]
\[ + \frac{1}{4} g^2 N_c \int \frac{d^4 K}{(2\pi)^4} \]
\[ \left[ *\Gamma^{\mu \rho}_{RAA}(-P, K, P - K) G_{AR, \rho' \sigma'}(K) *\Gamma^{\rho' \sigma'}_{ARR}(-K, P, K - P) *G_{RA, \sigma' \sigma}(P - K) \right. \]
\[ + \left. *\Gamma^{\mu \sigma}_{RAA}(-P, K, P - K) G_{RR, \rho' \rho}(K) *\Gamma^{\rho' \mu}_{RAR}(-K, P, K - P) *G_{RA, \rho \sigma}(P - K) \right. \]
\[ + \left. *\Gamma^{\mu \sigma}_{RAA}(-P, K, P - K) G_{RA, \rho' \rho}(K) *\Gamma^{\rho' \mu}_{RAR}(-K, P, K - P) *G_{RR, \rho \sigma}(P - K) \right. \]
\[ + \left. *\Gamma^{\mu \sigma}_{RAR}(-P, K, P - K) G_{RA, \rho' \rho}(K) *\Gamma^{\rho' \mu}_{RAR}(-K, P, K - P) *G_{RR, \rho \sigma}(P - K) \right. \]
\[ + \int \frac{d^4 K}{(2\pi)^4} G_{RR, \rho \sigma}(K) , \] (5.5)

where \(*S(P)\) is the HTL resummed fermion propagator,\(^{13}\) and \(^{17}\)
\[ \Pi^\mu_R(K) = \Pi^R_T(K) A^\mu + \Pi^R_L(K) B^\mu + \Pi^R_C(K) C^\mu + \Pi^R_D(K) D^\mu , \] (5.6)
\[ G^\mu_{RA}(K) = \frac{1}{\Pi^R_T - K^2 - i\text{sgn}(k_0)\varepsilon} A^\mu + \frac{1}{\Pi^R_L - K^2 - i\text{sgn}(k_0)\varepsilon} B^\mu \]
\[ + \frac{1}{\Pi^R_L - K^2 - i\text{sgn}(k_0)\varepsilon} H^\mu_R \xi + \frac{1}{\Pi^R_C - K^2 + i\text{sgn}(k_0)\varepsilon} H^\mu_R , \] (5.7)

with \(A^\mu, B^\mu\) and \(D^\mu\) given in Eq. (5.3), and
\[ C^\mu(K) \equiv \frac{K^\mu \bar{K}^\nu + \bar{K}^\mu K^\nu}{K^2} , \] (5.8)
\[ H^\mu_R(K) \equiv K^\mu \sqrt{K^2 - \Pi^R_L(K)} + \bar{K}^\mu \sqrt{\Pi^R_D(K)} , \] (5.9)
\[ \Pi^R_C(K) = \sqrt{(K^2 - \Pi^R_L(K))\Pi^R_D(K)} . \] (5.10)

In the present HTL approximation, the DS equation (5.1) becomes an integral equation for the unknown fermion self-energy function \(\Sigma_R\), while the DS equation (5.5) becomes an integral equation for the unknown polarization tensor \(\Pi^\mu_R\). It is worth giving some comments on these DS equations.

i) The HTL resummed quark-gluon vertex function is substituted for both the vertices in Eq. (5.1) and for the HTL resummed 3-gluon vertex functions and the quark-gluon functions in Eq. (5.5). This may cause a double counting problem when the loop momentum becomes “hard”. For this reason, in the actual analysis, we should introduce an intermediate momentum scale to cut the loop-momentum integration. In the
hard loop-momentum region, the naive ladder approximation may work, while in the
soft loop momentum region, we must use Eqs. (5.1) and (5.5).

ii) There are no contributions from the 4-point fg-vertex function in Eq. (5.1). In
QCD, the contribution to $\Sigma_R$ from the diagrams shown in Fig. 11 are exactly
accounted for by the loop diagram with two 3-point quark-gluon vertices in Eq. (5.1).
Inclusion of the contribution from Fig. 11 causes the trouble of double counting of
the diagrams.

iii) In the present HTL resummed DS equation for $\Pi_R^{\mu\nu}$, the net contribution
from the 4-gluon vertex function comes only from the tree (point) 4-gluon vertex, the
last term in Eq. (5.5). Contributions from the HTLs, $\delta\Gamma^{\mu\nu\rho\sigma}$ [given in Eq. (3.26)] are
exactly accounted for by the loop diagram with two 3-point gluon vertices. Inclusion
of the full HTL resummed 4-gluon vertex function causes the trouble of double-counting of
the diagrams. There are also no contributions from the 4-point fg-vertex function in Eq. (5.5). Thus, in QED the DS equation for $\Pi_R^{\mu\nu}$ consists of only the
first two terms with 3-point fermion-photon vertices.

iv) In Eq. (5.1), the first term in the square brackets, which exists also in the limit
of the ladder approximation, has been dismissed in previous DS equation analyses.\textsuperscript{14}
As can be seen, this term could produce a dominant contribution, due to the presence
of the enhanced temperature dependence. The results of the analysis of Eqs. (5.1)
and (5.5) will be given elsewhere.\textsuperscript{18}

§6. Conclusions and discussion

In this paper we calculated the $n$-point HTL vertex functions in QCD and QED
for $n=2,3$ and 4 in the physical representation in the RTF. The result shows that
the $n$-point HTL vertex functions can be classified into two groups, a) those with
odd numbers of external retarded indices and b) those with even numbers of external
retarded indices. The $n$-point HTL vertex functions with one retarded index, which
belong to group a), are identical to the HTL vertex functions that appear in the
ITF.\textsuperscript{6} All the HTL vertex functions belonging to group a) are $O(g^2T^2)$ and satisfy
simple QED-type Ward-Takahashi identities, as in the ITF.\textsuperscript{1,2} All the HTL vertex
functions with a fermion pair belong to group a). Those vertex functions belonging
to group b) never appear in the ITF, their existence being characteristic of the RTF,
and their HTLs exhibit high temperature behavior that is $O(g^2T^3)$, one power of $T$
higher than usual. Despite this difference, we were able to verify that those HTL vertex
functions belonging to group b) also satisfy the QED-type Ward-Takahashi
identities, thus guaranteeing the gauge invariance of the HTLs in real time thermal
QCD/QED. The group b) HTLs consist of $n$-gluon vertex functions.

Comments and discussion are in order.

i) The present results support the validity of carrying out in the framework of
real time thermal field theories nonperturbative analyses in the HTL approximation
on dynamical phenomena, such as the chiral phase transition. As an application,
we derived the HTL resummed DS equation for the retarded fermion self-energy
function and for the gauge boson polarization tensor. These equations enable us to
study in QCD/QED the chiral phase transition at finite temperature and to study single-photon emission and gluon magnetic mass generation.

ii) It has been shown\(^6\) that the \(n\)-point functions in the ITF correspond to a special combination of those in the RTF, namely, the \(n\)-point functions in the real time physical representation with one external retarded index and \(n-1\) external advanced indices. This means that the \(n\)-point functions with more than two external retarded indices have no counterparts in the ITF. Furthermore, we have already obtained results indicating the existence of order \(O(g^2 T^3)\) contributions in the RTF.\(^8\),\(^10\) The results above clearly show that such \(O(g^2 T^3)\) contributions do indeed exist in the real time physical representation. These unusual HTLs appear only in the \(n\)-gluon vertex functions in QCD. In QED there are no 3-photon vertex functions and no 4-photon HTL vertex functions, as already shown.

iii) Another point to be noted is that the results, for example, Eqs. (3.9a)–(3.9c), have no trouble in regard to the number of independent components, which is in the present case 1. As we can easily see, the HTL contributions satisfy Eq. (3.2a),

\[
\delta\Pi_{\mu\nu}^{\mu\nu}(P, Q) = \left(\delta\Pi_{\mu\nu}^{\mu\nu}(P, Q)\right)^*.
\]

Recalling that Eq. (3.2b) holds for the full two-point vertex functions and that here we are studying the HTL contributions, we can see that the additional power of \(T\) in Eq. (3.6c) comes from the boson distribution function, guaranteeing that the number of independent components in the present case is 1. The same is true for 3- and 4-point vertex functions with more than two external retarded indices.

iv) Although the Ward-Takahashi identities (4.2f) and (4.3g) and also (4.5e) and (4.5f) seem to have a structure that differs somewhat from that of the others in Eqs. (4.2), (4.3) and (4.5), all the identities (4.2a)–(4.2g), (4.3a)–(4.3g) and (4.5a–4.5g) actually have the same structure. This fact can be clearly seen by noting that, e.g., the right-hand sides of Eqs. (4.5e) and (4.5f) are the HTL contributions to the \(RR\)-component of the gluon polarization tensor, which is actually the difference of two polarization tensors:

\[
\Pi_{\mu\nu}^{\mu\nu}(P, Q) = (1 + 2n_B(q_0))\text{sgn}(q_0)(\Pi_{\mu\nu}^{\mu\nu}(P, Q) - \Pi_{\mu\nu}^{\mu\nu}(P, Q)), \quad P + Q = 0. \quad (6.1)
\]

In the HTL approximation (the external momentum \(-P = Q\) must be soft), we have

\[
\delta\Pi_{\mu\nu}^{\mu\nu}(P, Q) = \frac{2T}{q_0} \left[\delta\Pi_{\mu\nu}^{\mu\nu}(P, Q) - \delta\Pi_{\mu\nu}^{\mu\nu}(P, Q)\right]. \quad (6.2)
\]

Equation (6.2) shows that the right-hand sides of Eqs. (4.5e) and (4.5f) are identical to the difference of two HTL gluon polarization tensors, and thus they have exactly the same structure as the others. The same is true for the right-hand sides of the identities (4.2f) and (4.3g).

v) We showed explicitly that in the physical representation in the RTF there exist vertex functions with two external retarded indices that have high temperature behavior that is \(O(g^2 T^3)\). This should be true for any \(n\)-gluon vertex functions with an even number of external retarded indices.
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References

   A. Ayala and A. Bashir, Phys. Rev. D 64 (2001), 025015.
   See also, M. LeBellac, in Ref. 3).
   See also, Y. Fueki, H. Nakagawa, H. Yokota and K. Yoshida, in preparation.