Phase-Locked and Phase-Drift Solutions of Phase Oscillators with Asymmetric Coupling Strengths

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Phase-locked solutions of coupled oscillators are studied with asymmetric coupling strengths and inhomogeneous natural frequencies. The solutions exhibit interesting profiles of the phase lags resulting from the pacemaker corresponding to the ratio of upward and downward coupling strengths. Using the existence condition of phase-locked solutions, the transition points from phase-locked to phase-drift states are evaluated. The application of the existence condition to the case of the linear gradient of the natural frequency illustrates some scaling properties in the frequency diagrams.

§1. Introduction

Coupled oscillators have been treated in diverse contexts associated with oscillation and synchronization phenomena. The phenomenon of collective synchronization of phases and frequencies has attracted attention not only in biology but also in physics and engineering.1), 2) Theoretical studies have been extensively carried out for the case of global coupling, and they have clarified properties of phase transitions of entrained oscillators.3)–7) The case of local coupling has also been studied in various contexts motivated by rhythmic phenomena in biological systems. They include animal gaits,8) fish swimming,9), 10) and the peristaltic movement of gastrointestinal tracts.1)–13) In recent years, studies of coupled oscillators have been extended to the case of non-local (but not global) coupling14) and more complicated network systems.15)

Since such complicated networks have, in general, intricate boundaries and inhomogeneous couplings, we have been found few clues to the analysis of their dynamical behaviour. We can, however, analyse some phase-locked solutions for more simple and symmetric trees of couplings. In the case of \( m \)-ary trees with \( m^n \) leaves of height \( n \), for instance, the system of coupled oscillators has some particular solutions with identical phases at each level, and these solutions can be reduced to phase-locked solutions for chains of oscillators with asymmetric coupling strengths.17) In addition to complicated network structures, intricate dynamical processes appear in biological systems whose inherent and isolated parts are difficult to extract. One way to study
such systems is to construct models from phenomenological evidence and thereby elucidate qualitative features. When we compare models with data taken from real systems, such as efficiency of movement and transport volume, we need to know the dynamical behaviour and solutions of the models in detail.

In this paper, we study chains of coupled phase oscillators. It is noteworthy that we can derive phase-locked solutions of the systems with algebraic calculations, even if the coupling strengths are asymmetric or the natural frequencies are inhomogeneous. Moreover, the analytical forms of the solutions are not overly cumbersome, and therefore the profiles of solutions are easy to observe. From the existence conditions of the phase-locked solutions, we can compute transition points of phase-locked to phase-drift states.

We present a system of coupled phase oscillators with asymmetric coupling strengths in the following section and derive the phase-locked solutions in §3. The profile of the phase-locked solutions and bifurcation phenomena are studied for the cases of asymmetric coupling strengths in §4 and of a linear frequency gradient in §5. In §6, we study continuous systems corresponding to the discrete oscillator systems with asymmetric coupling strengths.

§2. Coupled phase oscillators

We consider a linear chain of phase oscillators with asymmetric couplings governed by the following equations for the phases \( \theta_j \) (\( j = 0, \cdots, n \)):

\[
\begin{align*}
\dot{\theta}_0 &= \omega_0 + a_u^* h(\theta_1 - \theta_0), \\
\dot{\theta}_j &= \omega_j + a_d h(\theta_{j-1} - \theta_j) + a_u h(\theta_{j+1} - \theta_j), \quad (j = 1, \cdots, n-1) \\
\dot{\theta}_n &= \omega_n + a_d^* h(\theta_{n-1} - \theta_n).
\end{align*}
\] (2.1)

Here, \( \omega_j \) is the natural frequency of the \( j \)th oscillator, \( a_u \) and \( a_d \) are coupling coefficients, and \( h \) is a periodic function. For the 0th and \( n \)th oscillators, the coupling coefficients \( a_u^* \) and \( a_d^* \) are set according to the boundary conditions.

Introducing new variables defined by neighbouring sites,

\[
\psi_j := \theta_j - \theta_{j-1}, \quad d_j := \omega_j - \omega_{j-1},
\]

we obtain \( n \) equations for the phase differences:

\[
\dot{\psi} = d + A_u h_+ - A_d h_-,
\]

\[
\psi := \begin{bmatrix} \psi_1 \\
\psi_2 \\
\vdots \\
\psi_{n-1} \\
\psi_n \end{bmatrix}, \quad d := \begin{bmatrix} d_1 \\
d_2 \\
\vdots \\
d_{n-1} \\
d_n \end{bmatrix}, \quad h_\pm := \begin{bmatrix} h(\pm \psi_1) \\
h(\pm \psi_2) \\
\vdots \\
h(\pm \psi_{n-1}) \\
h(\pm \psi_n) \end{bmatrix},
\]

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A := 
\begin{bmatrix}
-a_u^* & a_u \\
-a_u & a_u \\
\ddots & \ddots \\
-a_u & a_u \\
\end{bmatrix},
\begin{bmatrix}
a_d \\
a_d \\
\ddots \\
a_d \\
\end{bmatrix}
\begin{bmatrix}
a_u \\
a_d \\
\ddots \\
a_d \\
\end{bmatrix}
(2.2)

In the analysis of phase locked/drift solutions given below, we assume h is an odd function \[ h(-\psi) = -h(\psi) \], and therefore \[ h_- = -h_+ \].

§3. Phase-locked solutions

For a phase-locked solution of Eq. (2.2), all the oscillators have the same period, and the phase difference is constant, i.e. \[ \dot{\psi}_j = 0 \]. Thus they are solutions of

\[ Ah + d = 0 \]

where \[ A := A_u + A_d \] and \[ h := h_+ \]. We obtain formal solution of the above equations under free boundary conditions (mutual entrainment), and one-sided coupling conditions (forced entrainment).

First, we study the case of free boundary conditions, \[ a_u^* = a_u, \ a_d^* = a_d \], where the boundary oscillators and their neighbours are coupled in a bi-directional manner. Under these conditions, all of the oscillators have the same period,

\[ \dot{\theta}_j = \Omega_n/\Lambda_n, \]

where \[ \Omega_n \] and \[ \Lambda_n \] are defined below. From Eq. (3.1), we obtain the solution for \[ h_j \] \((j = 1, \cdots, n)\), the jth element of \( h \), as

\[ a_d h_j = \Delta_{n-j} - (\Delta_n/\Lambda_n) A_{n-j} \]

or

\[ a_u h_j = \tilde{\Delta}_{j-1} - (\tilde{\Delta}_n/\tilde{\Lambda}_n) \tilde{A}_{j-1} \]

where

\[ \Omega_j := \sum_{k=0}^{j} \omega_k \left( \frac{a_u}{a_d} \right)^k \], \[ A_j := \sum_{k=0}^{j} \left( \frac{a_u}{a_d} \right)^k \], \[ \tilde{\Lambda}_j := \sum_{k=0}^{j} \left( \frac{a_d}{a_u} \right)^k \],

\[ \Delta_j := \sum_{k=0}^{j} (\omega_{n-j+k} - \omega_0) \left( \frac{a_u}{a_d} \right)^k \], \[ \tilde{\Delta}_j := \sum_{k=0}^{j} (\omega_n - \omega_{j-k}) \left( \frac{a_d}{a_u} \right)^k \].

The phase difference \( \psi_j \) is obtained by inverting Eq. (3.3). When the coupling function \( h \) is continuous and bounded, the existence of solutions implies \( \min h < \]
h_j < \max h \text{ for all } j. \text{ The linear stability conditions for the solutions are obtained from the eigenvalue problem of the linearized matrix } A\mathbf{h}', \text{ where the } j\text{th element of } \mathbf{h}' \text{ is } h'(\psi_j).

We comment on the case of forced oscillations, that is, the case in which one of the terminals is a forced oscillator. Suppose that the 0th oscillator experiences one-sided coupling to its neighbour, \( a_u^* = 0 \), but the opposite end satisfies the free boundary condition \( a_d^* = a_d \). Under these conditions, \( h = \Delta_{n-j}/a_d \) is a solution of Eq. (3.1). If the boundary conditions are exchanged, that is, if the \( n \)th oscillator is a forced oscillator, \( a_u^* = a_u \), \( h = \tilde{\Delta}_{j-1}/a_u \) is a solution of Eq. (3.1).

§4. Asymmetric coupling strengths

We study mutual entrainment to a pacemaker with asymmetric coupling strengths in the present section. We assume that all oscillators other than the 0th have the same natural frequency, while the 0th oscillator (the pacemaker) has a high frequency. If we choose \( \omega_0 > 0 \) and \( \omega_1 = \cdots = \omega_n = 0 \), Eq. (3.3) becomes

\[
h_j = -\frac{\omega_0 A_{n-j}}{a_d A_n}.
\]

(4.1)

The entrained frequency is then obtained from Eq. (3.2) as

\[
\bar{\omega}_n = \begin{cases} 
\omega_0/(1 + \cdots + \lambda^n), & (a_u > a_d) \\
\omega_0 \lambda^n/(1 + \cdots + \lambda^n), & (a_u < a_d)
\end{cases}
\]

(4.2)

where \( \lambda := \max(a_u, a_d)/\min(a_u, a_d) \). We set \( a_u, a_d, \) and \( h'(0) \) to positive values below. For the numerical calculations, whose results are given in the figures, we set \( \omega_0 = 1.0 \) and chose the coupling function as \( h(\psi) = \sin \psi + 0.1 \sin 2\psi - 0.03 \sin 3\psi \) to obtain generic results. This choice is the same as that in Ref. 12). This is done to make it easy to compare our analytical results in the next section with numerical calculations in Ref. 12).

When the coupling strengths are asymmetric, i.e. \( a_u \neq a_d \), the profile of \( h_j \) given in Eq. (4.1) has an exponential dependence on \( j \). In Fig. 1, we illustrate some features of phase-locked solutions: the profile of the coupling interaction, the phase difference and the phase lag from the pacemaker. Calculations of these quantities were carried out for the system of 20 oscillators \( (n = 19) \) and for three ratios of the couplings, \( a_u/a_d = 2, 1, \) and \( 1/2 \). Since the analytical solution displayed in Fig. 1 satisfies \( h'(\psi_j) > 0 \) for all \( j \), it is linearly stable. If the coupling function \( h(\psi) \) is approximated by a linear function for small \( \psi \), then the dependence of the phase lag, \( \theta_j - \theta_0 \), on \( j \) is almost (i) exponential for \( a_u > a_d \), (ii) parabolic for \( a_u = a_d \), and (iii) linear for \( a_u < a_d \). Such dependence is inherited by more complicated network systems on trees.\(^{17}\)

For small coupling strengths, the phase-locked state is broken, and the phase of each oscillator drifts from the phase of the pacemaker. Such a phase drift solution has two regions of frequency entrainment, since the pacemaker and other oscillators
Fig. 1. Profiles of phase-locked solutions for phase oscillators with asymmetric couplings: the coupling interaction $h_j$, phase difference $\psi_j$, and phase lag from the pacemaker ($\theta_j - \theta_0$), where $\psi_j$ and ($\theta_j - \theta_0$) are plotted in units of $\pi$. The left, centre, and right columns correspond to the cases of (i) $a_u/a_d = 2$ ($a_u = 1.0, a_d = 0.5$), (ii) $a_u/a_d = 1$ ($a_u = 1.5, a_d = 1.5$), and (iii) $a_u/a_d = 1/2$ ($a_u = 1.5, a_d = 3.0$), respectively. The solid curves connect the points derived from Eq. (4.1), while open circles were obtained from numerical calculations of the phase equation (2.1).

have different natural frequencies. Phase drift occurs between these two regions. Figure 2 depicts the transition between the phase-locked and phase-drift solutions for three ratios of $a_u$ and $a_d$. Each system has 10 oscillators ($n = 9$). The average frequency of the $j$th oscillator and the coupling strength are defined by

\[
\langle \omega_j \rangle := \lim_{t \to \infty} \left( \frac{\theta_j(t) - \theta_j(0)}{t} \right), \quad \epsilon := \min(a_u, a_d) \max h.
\]

In Fig. 2, values of $\langle \omega_j \rangle$ were obtained from numerical integration of Eq. (2.1) with step width $\Delta \epsilon = 10^{-3} \times \max h$. We can estimate the transition point of the phase-locked and phase-drift solutions from the necessary condition for the existence of the phase-locked solutions,

\[
\epsilon > \epsilon_n := \frac{\omega_0}{1 + \cdots + \lambda^{n-1}}.
\]

The critical points ($\epsilon_n, \bar{\omega}_n$) are plotted by the open circles in Fig. 2.

The numerical calculations show that the phase drift state always consists of two regions: the pacemaker and the other oscillators. This implies that if the pacemaker can entrain its neighbouring oscillator, all oscillators will become entrained to the pacemaker, but if cannot do this, no oscillators become entrained to the pacemaker.
Fig. 2. Frequency diagrams for phase oscillators with the asymmetric couplings. The left, middle, and right diagrams correspond to the cases (i) \(a_u/a_d = 2\), (ii) \(a_u/a_d = 1\), and (iii) \(a_u/a_d = 1/2\), respectively. The average frequencies \(\langle \omega \rangle\) are plotted against the coupling strength \(\epsilon\). The dots were obtained from numerical calculations of Eq. (2-1). The open circles denote the critical points of the existence condition for phase-locked solutions.

It is also possible to understand this entrainment feature from the existence condition of the phase-locked solutions. We assume that the pacemaker entrains \(j\) oscillators. Then we divide the system into two sets of coupled oscillators: (i) the upper \((j+1)\) oscillators, including the pacemaker, and (ii) the lower \((n-j)\) oscillators. For set (i), the existence of a phase-locked solution implies

\[
\omega_0 \frac{1 + \cdots + \lambda^{j-1}}{1 + \cdots + \lambda^{j}} < \epsilon. \tag{4.4}
\]

For set (ii), we add one oscillator with the entrainment frequency of set (i) to the upper side as the pacemaker. If the coupling strength \(\epsilon\) satisfies

\[
\epsilon < \bar{\omega}_j \frac{1 + \cdots + \lambda^{n-j-1}}{1 + \cdots + \lambda^{n-j}}, \tag{4.5}
\]

then a phase drift solution exists for the set (ii) with the pacemaker. Since no value of the coupling strength \(\epsilon\) satisfies both conditions (4.4) and (4.5), either all or none of the oscillators are entrained to the pacemaker.

§5. Linear frequency gradient

Daido\(^{12}\) has numerically studied chains of coupled phase oscillators with a linear gradient of natural frequencies. He has demonstrated some scaling properties in the frequency diagram, a plot of the average frequency against the coupling strength. In the present section, we attempt to derive some of them from Eq. (3-3).

We assume that the natural frequency of the \(j\)th oscillator is \(\omega_j := j/n\) and that the coupling strengths are symmetric, i.e. \(a := a_u = a_d\). Then a phase-locked solution is obtained from Eq. (3-3) in the form

\[
h_j = \frac{\delta j(n+1-j)}{2}, \tag{5.1}
\]

where \(\delta := 1/n\) is the difference between the natural frequencies of neighbouring oscillators. The entrained frequency of this solution is

\[
\bar{\omega}_n = \frac{n}{2} \left( = \frac{1}{2} \right). \tag{5.2}
\]
We chose \( \alpha \) and \( h'(0) \) to be positive and chose the coupling function as in the previous section for the numerical calculations whose results are displayed in Figs. 3 and 4.

We note that \( h_j \) takes its maximal value at the centre of the chain of entrained oscillators from Eq. (5.1). If the phase difference \( \psi_j \) is sufficiently small that we can approximate the coupling function \( h(\psi) \) as having a linear dependence on \( \psi \), then \( \psi_j \), as well as \( h_j \), has a convex dependence on \( j \). In this case, the phase lag \( (\theta_j - \theta_0) \) has an inflection point at the centre of the chain. Figure 3 displays the profile of phase lags and other variables. The analytical solution plotted in Fig. 3 (solid lines) is linearly stable, because it satisfies \( h'(\psi_j) > 0 \) for all \( j \).

When the coupling strength is decreased, the entrained region is divided into clusters of frequency-entrained oscillators, called frequency plateaus.\(^{1,11,12}\) In the limiting case of vanishing coupling strength, each such plateau comes to possess only a single oscillator.

Although the frequency diagram exhibits fine and complicated bifurcation structure,\(^{12}\) we can determine the approximate arrangement of transition points by using the necessary conditions for the existence of phase-locked solutions. To illustrate the frequency diagram, we define the average frequency of the \( j \)th oscillator and the coupling strength as in Eq. (4.3). We pick out a frequency plateau with \( m \) oscillators from the \((n+1)\)-oscillator system, and consider these oscillators as an isolated system, with free boundary conditions in both sides. The existence of phase-locked solutions of the \( m \)-oscillator system implies

\[
\epsilon > \epsilon_m := \begin{cases} 
(\delta/8)m^2, & (m : \text{even}) \\
(\delta/8)(m^2 - 1), & (m : \text{odd}) 
\end{cases} \tag{5.3}
\]

The entrained frequency is the average of the natural frequencies:

\[
\bar{\omega}_{k,k+m-1} = (\omega_k + \cdots + \omega_{k+m-1})/m
\]
\[ = \delta(m + 2k - 1)/2. \quad (k = 0, \cdots, n - m) \quad (5.4) \]

In Fig. 4, we plot approximate values for the critical points \((\bar{\omega}_{k,k+m-1}, \epsilon_m)\) in the frequency diagram derived in Ref. 12) for a system of 10 oscillators \((n = 9)\). Some of these points are located near the transition points, but others have no transition points around them. The existence of the latter points seems to indicate that it is not possible for the system to have states such as small plateaus located near the boundaries or extremely asymmetric arrangements of plateaus. We obtain the relation between \(\bar{\omega}_{0,m-1}\) and \(\epsilon_m/\epsilon_{n+1}\) from Eqs. (5.3) and (5.4) as

\[ \bar{\omega}_{0,m-1} \sim \frac{1}{2} \left( \frac{\epsilon_m}{\epsilon_{n+1}} \right)^{1/2} \quad (5.5) \]

in the limit of \(n \to \infty\). However, in the approximate computation of the critical points given above, we do not know how to determine the boundaries of the frequency plateaus. More general methods taking such a point into consideration are presented in Ref. 13).

If the coupling strengths become asymmetric in a system with a linear frequency gradient, the results given in Figs. 3 and 4 become deformed. We discuss such cases in the next section using the analogous continuous system.

Fig. 4. The frequency diagram for phase oscillators whose natural frequencies possess a linear gradient. The diagram plots the average frequency \(\langle \omega_j \rangle\) against the coupling strength \(\epsilon\). The average frequencies plotted by the dots represent the numerically calculated values. The open circles denote critical points for the existence condition of phase-locked solutions with \(m\) oscillators \((m = 1, \cdots, n + 1)\).
§6. Discussion

We can consider the phase-difference system (2.2) to be derived from some continuous partial differential equation by means of a discretization procedure. If the coupling function $h$ is an odd function, the continuous system corresponding to Eq. (2.2) is

$$
\partial_t \psi + \Gamma \partial_x h(\psi) = d(x) + D \partial^2_x h(\psi),
$$

(6.1)

where $x$ is the spatial coordinate. The coefficients $\Gamma$ and $D$ are defined by

$$
\Gamma := (a_d - a_u) \Delta x, \quad D := \frac{a_d + a_u (\Delta x)^2}{2}
$$

for the difference interval $\Delta x$. Here, we used the central difference for the differential of first order. In the continuous system (6.1), entrained solutions are obtained from the ordinary differential equation

$$
D h'' - \Gamma h' + d(x) = 0,
$$

(6.2)

where $'$ is the derivative with respect to $x$.

When all of natural frequencies are identical [i.e. $d(x) \equiv 0$], the solutions of Eq. (6.2) are

$$
h(x) = \begin{cases} 
    c + b \exp(\Gamma x/D), & (\Gamma \neq 0) \\
    c + bx, & (\Gamma = 0)
\end{cases}
$$

where $b$ and $c$ are constants defined by the boundary conditions. These solutions are consistent with Eq. (4.1) and Fig. 1. In another case, corresponding to a linear frequency gradient [i.e. $d(x) = \delta$ (const)], a solution of Eq. (6.2) is

$$
h(x) = c + bx - (\delta/2D)x^2.
$$

This is also consistent with Eq. (5.1) and Fig. 3 if we set the arbitrary constants $b$ and $c$ to the appropriate values.

Here we give some more examples. When the two cases mentioned above are combined, that is, when the natural frequency has a linear gradient in space [$d(x) = \delta$] and the coupling strengths are asymmetric [$\Gamma \neq 0$], a solution of Eq. (6.2) is

$$
h(x) = c + (\Gamma \delta/D)x + b \exp(\Gamma x/D).
$$

Even if $\Gamma = 0$, we can obtain similar dependence of $h$ on $x$ from a system in which the natural frequency has an exponential gradient in space [$d(x) = \gamma \exp(\gamma (x - L))$], where $L$ is the system size. A solution of this system is

$$
h(x) = c + bx - (1/D\gamma) \exp(\gamma (x - L)).
$$

Finally, we comment on the case in which the coupling functions are divided into odd and even parts. In general, the division of a function $h(x)$ is carried out as

$$
h(x) = h_{\text{odd}}(x) + h_{\text{even}}(x),
$$

$$
h_{\text{odd}}(x) := \frac{h(x) - h(-x)}{2}, \quad h_{\text{even}}(x) := \frac{h(x) + h(-x)}{2}.
$$
The system of coupled phase oscillators (2.2) is derived from the following continuous system using a spatial discretization:

\[
\partial_t \psi + \Gamma_{\text{odd}} \partial_x h_{\text{odd}}(\psi) - \Gamma_{\text{even}} \partial_x h_{\text{even}}(\psi) = d(x) + D_{\text{odd}} \partial_x^2 h_{\text{odd}}(\psi) - D_{\text{even}} \partial_x^2 h_{\text{even}}(\psi),
\]

(6.3)

where the coefficients are defined as

\[
\Gamma_{\text{odd}} := (a_d - a_u) \Delta x, \quad \Gamma_{\text{even}} := (a_d + a_u) \Delta x,
D_{\text{odd}} := \frac{a_d + a_u}{2} (\Delta x)^2, \quad D_{\text{even}} := \frac{a_d - a_u}{2} (\Delta x)^2.
\]

We note that the coefficients of the advection and diffusion terms are related in the following interesting way:

\[
D_{\text{odd}} = (\Delta x/2) \Gamma_{\text{even}}, \quad D_{\text{even}} = (\Delta x/2) \Gamma_{\text{odd}}.
\]

It is also possible to obtain the phase-locked solutions from Eq. (6.3) as above, but we do not give the derivation here.

\section*{§7. Conclusions}

We obtained phase-locked solutions of coupled phase oscillators with asymmetric coupling strengths and inhomogeneous natural frequencies. These solutions exhibit different phase lag profiles for the three types of ratios of upward to downward coupling strength. The phase differences obtained in analytical form must be applicable to studies on the efficiencies of biological motion and transport based on collective oscillation. Moreover, the inherent structures of the real systems, asymmetry of couplings, for example, are suggested by the spatio-temporal patterns observed in such systems.

From the existence conditions of phase-locked solutions, we derived transition points from phase-locked to phase-drift states. We also found that the length of the region entrained to the pacemaker is either approximately zero or the system size. However, for discrete time-dependent Ginzburg-Landau equations, the number of oscillators entrained to the pacemaker depends on the coupling strength, and the sequence of bifurcations was observed numerically.\(^{17}\)

Next, we determined profiles of phase-locked solutions for the case of a linear frequency gradient. The phase difference has the maximal absolute value around the centre of the chain system. This implies that the phase drift tends to occur around the centre when the coupling strengths decrease. In addition to the case of asymmetric coupling strengths, we determined approximate transition points in the frequency diagram from the existence conditions of phase-locked solutions and derived some scaling properties.

\section*{References}

13) H. Daido, talk at 55th Annual Meeting of the Physical Society of Japan, September 22–25, 2000, Niigata University, Japan.