Forced Entrainment and Elimination of Spiral Waves
for the FitzHugh-Nagumo Equation

Hidetsugu SAKAGUCHI and Takefumi FUJIMOTO

Department of Applied Science for Electronics and Materials,
Interdisciplinary Graduate School of Engineering Sciences,
Kyushu University, Kasuga 816-8580, Japan

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We study some effects of spatial inhomogeneity and external forcing on spiral waves in
the FitzHugh-Nagumo equation and attempt in a numerical investigation to eliminate spiral
patterns using “entrainment” and “trapping”. We find that a target pattern generated by
a circular fast pacemaker region can overwhelm a spiral pattern and eliminate spiral waves.
We also find that a spiral pattern can be trapped in a less excitable region. Through control
of this less excitable region, a pair of spirals can be annihilated. In addition, it is shown
that spiral waves are entrained and disappear under a spatially uniform periodic forcing if
the forcing is sufficiently strong. We discuss two mechanisms causing complete entrainment.

§1. Introduction

The control of chaotic dynamics and pattern formation in spatially extended
systems have been investigated with several methods.\textsuperscript{1)–4)} Some types of cardiac
arrhythmia are related to rotating waves, which are similar to spiral waves found in
excitable media.\textsuperscript{5)} The control and elimination of spiral waves are important as a
problem in medical treatment. The control of spiral waves in excitable media has also
been studied with several methods. For example, it has been found that meandering
of the spiral core can be controlled by a periodic parameter modulation,\textsuperscript{6)} impulses
have been applied to suppress traveling waves and spiral waves,\textsuperscript{7)} and local and
global feedback have been applied to excitable systems to eliminate spiral waves.\textsuperscript{8)}

In this paper, we report on a numerical study in which we attempt to control
and eliminate spiral waves by applying slightly different types of perturbations to
excitable media. We use the FitzHugh-Nagumo equation in the following form, which
we consider to describe a typical excitable medium:\textsuperscript{9)}

\[
\frac{\partial u}{\partial t} = u - u^3/3 + \nabla^2 u - v,
\]
\[
\frac{\partial v}{\partial t} = \epsilon(u - \gamma v + \beta).
\]

(1)

Here $\epsilon$ is a parameter controlling the relaxation time, and $\gamma$ and $\beta$ are parameters
that determine the excitability in this model system. In our numerical simulations,
we used $\gamma = 1/2$. 

\[
\frac{\partial u}{\partial t} = u - u^3/3 + \nabla^2 u - v,
\]
\[
\frac{\partial v}{\partial t} = \epsilon(u - \gamma v + \beta).
\]
§2. Competition of a target pattern and a spiral pattern

Spiral waves appear even in a spatially uniform medium. However, it is believed that target waves need a pacemaker region exhibiting self-generated oscillation. The one-dimensional analogue of a target pattern can be generated with the model equation

\[
\frac{\partial u}{\partial t} = u - u^3/3 + \frac{\partial^2 u}{\partial x^2} - v,
\]

\[
\frac{\partial v}{\partial t} = \epsilon(x)(u - \gamma v + \beta(x)),
\]  

(2)

where we consider the case in which \( \epsilon(x) = \epsilon_1 \), \( \beta(x) = 0 \) for \( 0 < x < l \) and \( \epsilon(x) = \epsilon_2 = 0.02 \), \( \beta(x) = 0.7 \) for \( l < x < L \). We also employ no-flux boundary conditions \( \partial u / \partial x = 0 \) at \( x = 0 \) and \( L \), and the initial condition is a uniform state: \( u(x) = 0.1 \) and \( v(x) = 0 \). For this equation, a limit cycle oscillation appears in the region \( 0 < x < l \), and pulses generated in the pacemaker region propagate in the excitable region, \( l < x < L \). The phase of the limit cycle oscillation is not uniform even in the oscillatory region, \( 0 < x < l \), because the state variables \( u(x) \) and \( v(x) \) need to be connected continuously to the excitable region. The relaxation parameter \( \epsilon_1 \) controls the frequency of the limit cycle oscillation. When the frequency of the pacemaker region becomes too large, the pulse propagation in the excitable region cannot follow the pacemaker, and as a result, a kind of desynchronization occurs. Figure 1(a) displays the ratio \( \omega_1/\omega_2 \) of the two frequencies in the pacemaker region and the excitable region and the frequency \( \omega_2 \) in the excitable region. The numerical simulation was performed using a second-order Runge-Kutta method (the Heun method) with timestep 0.01 and grid interval 2/3. The system size is \( L = 200 \) and the size of the pacemaker region is \( l = 40 \). The desynchronization transition occurs at \( \epsilon_1 = 0.036 \), and 2:1 entrainment appears for \( \epsilon_1 > 0.036 \). The pacemaker can

![Fig. 1. (a) The ratio \( \omega_1/\omega_2 \) of the two frequencies in the pacemaker region and the excitable region for the one-dimensional model (2) with \( \epsilon(x) = \epsilon_1 \) and \( \beta(x) = 0 \) for \( 0 < x < 40 \), and \( \epsilon(x) = \epsilon_2 = 0.02 \) and \( \beta(x) = 0.7 \) for \( 40 < x < L \). (b) Time evolution of the pulse propagation for \( \epsilon_1 = 0.04 \).](https://academic.oup.com/ptp/article-abstract/108/2/241/1869745)
excite pulses every other period. Figure 1(b) displays the time evolution of the pulse propagation in the 2:1 entrainment state for $\epsilon_1 = 0.04$. The frequency $\omega_2$ generally increases as a function of $\epsilon_1$, but it decreases discontinuously at the transition point.

Two-dimensional simulations were performed using a model equation

$$\frac{\partial u}{\partial t} = u - u^3/3 + \nabla^2 u - v,$$

$$\frac{\partial v}{\partial t} = \epsilon(x, y)(u - \gamma v + \beta(x, y)),$$

where we set $\epsilon(x, y) = \epsilon_1$, $\beta(x, y) = 0$ for $|x - x_0| < r_0$ and $\epsilon(x, y) = 0.02$, $\beta(x, y) = 0.7$ for $|x - x_0| > r_0$. We consider the case of a small circular pacemaker region with center $x_0 = (100/3, 100)$ and radius $r_0 = 80/3$. A spiral pattern whose core is located near $(L/2, L/2) = (100, 100)$ is prepared as the initial conditions. The natural frequency of a single spiral pattern without the pacemaker region was numerically evaluated to be $\omega_0 = 0.0115$. The frequency of the pacemaker region is about 0.0131 for $\epsilon_1 = 0.03$ if no spiral pattern exists. Therefore, the natural frequency of the pacemaker region is faster than the natural frequency of spiral waves. In general, the fastest pacemaker dominates the dynamics of the whole system. Figure 2(a), (b)
and (c) display three snapshot patterns of the contour $u = 0$ at $t = 800, 2400, 4000$ for $\epsilon_1 = 0.03$. It is seen that the spiral pattern is overwhelmed by the target pattern emitted from the pacemaker region. Figure 2(d) displays the trajectory of the spiral core, where the position of the spiral core is defined as the position at which $u = v = 0$. The spiral core is carried away toward the boundary and disappears there. For $\epsilon_1 = 0.05$, the frequency of the pacemaker region is $\omega_1 = 0.0202$. This frequency is faster than the natural frequency of the spiral waves. However, 1:1 entrainment cannot be realized for the parameter values, as shown in Fig. 1(a) for the one-dimensional case; that is, fast pulses with frequency $\omega_1$ cannot propagate in the region $|x - x_0| > r_0$. The 2:1 entrainment state is expected to appear for such parameter values, if the spiral pattern does not exist. The frequency for the 2:1 entrainment state is 0.0101, which is slower than the natural frequency of the spiral waves. For this reason, the spiral pattern survives, as shown in Fig. 3.

§3. Controlling the position of the spiral core

We studied the effect of a faster pacemaker on the spiral pattern in the previous section. We consider in this section another effect that of inhomogeneity of excitability. The spiral core is usually stationary or meanders around a certain position in a homogeneous medium. It can drift in inhomogeneous media, and has a tendency to drift toward a region with lower excitability. We performed simulations to elucidate the drift motion. In these simulations we used the FitzHugh-Nagumo equation in the form

$$\frac{\partial u}{\partial t} = u - u^3/3 + \nabla^2 u - v + a(x, y),$$
$$\frac{\partial v}{\partial t} = \epsilon(u - \gamma v + \beta),$$

(4)
Forced Entrainment and Elimination of Spiral Waves

Fig. 5. Four snapshot patterns at (a) $t = 0$, (b) 400, (c) 1400, and (d) 1600, together with the two regions of smaller excitability for the model equation (4) with $\epsilon = 0.02$, $\beta = 0.7$ and $a(x, y) = -0.4$ for $|x-x_01|, |x-x_02| < 15$, where $x_{01}(t) = (L/4+0.12t, 80)$ and $x_{02} = (3L/4, 80)$.

with $\epsilon = 0.02$ and $\beta = 0.7$. The parameter $a(x, y)$ is set as $a(x, y) = -0.25 + 0.0025|x - x_0|$, where $x_0 = (53, 100)$. This parameter modifies the effective excitability; that is, a region with smaller $a$ is less excitable. In our simulation, the excitability is the lowest at $x_0$ and it increases slowly in proportion to the distance $|x - x_0|$. Initially, the spiral core is located near $(L/2, L/2)$. It then drifts toward $x_0$ with some meandering motion, as shown in Fig. 4. Finally, it rotates around $x_0$.

Steinbock and Müller found in an experiment on the BZ reaction and a numerical simulation that the meandering core of spiral can be trapped in a small unexcitable spot region. We have confirmed that this is the case for the FitzHugh-Nagumo model with $a(x, y) = -0.4$ in the circular region satisfying $|x-x_0| < 15$ and $a(x, y) = 0$ elsewhere. The excitable medium is less excitable in this circular region. We have further found that the spiral core moves with this region, if it is slowly moved. This numerical result suggests that the position of the spiral core can be controlled by slowly moving the region with smaller excitability. This method to control the core position can be used to eliminate spiral pairs. We have performed numerical simulation in which we eliminate a spiral pair using this trapping method. A pair of clockwise and counterclockwise rotating spirals is prepared as the initial conditions. Initially, the spiral cores are located at $x_{01} = (L/4, 80)$ and $x_{02} = (3L/4, 80)$. The parameter $a(x, y)$ has a value of $-0.4$ inside the two circular regions satisfying $|x -$
$x_{01} < 15$ and $|x - x_{02}| < 15$. The center $x_{02}$ of the second such region is fixed, and the center $x_{01}$ is moved slowly as $x_{01}(t) = (L/4 + 0.12t, 80)$ until $x_{01}$ coincides with $x_{02}$. Figure 5 displays four snapshot patterns at $t = 0, 400, 1400$ and $1600$ together with the two regions of small excitability. The left spiral moves toward the right spiral, and the two spirals collide and disappear.

§4. Forced entrainment of spiral waves

We considered mutual entrainment in a diffusively coupled system in §2. In this section we consider an externally forced model. The model equation we consider is

$$\frac{\partial u}{\partial t} = u - u^3/3 + \nabla^2 u - v + K(u_0 - u),$$
$$\frac{\partial v}{\partial t} = \epsilon_1(u - \gamma v + \beta),$$
$$\frac{du_0}{dt} = u_0 - u_0^3/3 - v_0,$$
$$\frac{dv_0}{dt} = \epsilon_0(u_0 - \gamma v_0).$$

In this system, there is an external oscillator, which also obeys the FitzHugh-Nagumo equation in the oscillatory regime, and the spiral pattern is perturbed by this external oscillator. We investigate the forced entrainment of the spiral pattern by the external relaxation oscillator. We could investigate the forced entrainment using a simpler form of the external oscillation, such as a sinusoidal force, but we investigate the above model, since entrainment is expected to occur easily when the external oscillation is similar to the entrained oscillation.

Figure 6(a) displays a snapshot of the spiral pattern and a trajectory of the core for the parameter values $\epsilon_1 = 0.02$, $\epsilon_0 = 0.04$, $\beta = 0.7$, $K = 0.082$. It is seen that the spiral core exhibits a meandering motion. Figure 6(b) displays time sequences of $v_0(t)$ and $v(x, y, t)$ at $(x, y) = (96, 84)$ (inside the core region) and $(96, 180)$ (far from the core region), with the initial transient time removed. Here, 2:1 entrainment is observed for the motion of $v(x, y, t)$ inside the core region. The motion of $v(x, y, t)$ outside of the core region, however, is not completely entrained to the external force. As the coupling constant $K$ is increased, the completely entrained region (core region) grows. Figure 6(c) displays the radius of the core region. We have defined the core radius as the radius of the minimum circle that can contain the entire core trajectory. Near $K = 0.08$, the core radius increases rapidly and seems to diverge near $K = 0.0835$. After the divergence of the core radius, the spiral pattern disappears, and the excitable medium is uniformly entrained to the external oscillation. This is one typical route to forced entrainment. This type of forced entrainment has also observed in the complex Ginzburg-Landau equation.\(^{11}\)

As $\epsilon_0$ is decreased, the frequency of the external oscillation becomes small. There is a region of values $\epsilon_0$, in which the oscillation of the spiral core tends to be entrained to the external force with a 1:1 period ratio but the uniform oscillation tends to be entrained to the external force with a 2:1 period ratio. (The spiral core is a kind
Forced Entrainment and Elimination of Spiral Waves

Fig. 6. (a) Snapshot of the spiral pattern and the trajectory of the core for $\epsilon_1 = 0.02$, $\epsilon_0 = 0.04$, $\beta = 0.7$ and $K = 0.082$ for the model equation (5). (b) Time sequences of $v_0(t)$ (top) and $v(x, y, t)$ at $(x, y) = (96, 84)$ (bottom, solid curve) and $(96, 180)$ (bottom, dashed curve), with the initial transient time removed. The solid curve and the dashed curve are nearly coincident for $t > 200$ in this plot. However, later they separate again. (c) Radius $R$ of the core region as a function of $K$.

of pacemaker region and has an effectively faster frequency.) Figure 7(a) displays a snapshot of the spiral pattern and a trajectory of the core for $\epsilon_1 = 0.02$, $\epsilon_0 = 0.025$, $\beta = 0.7$, $K = 0.06$. Figure 7(b) displays time sequences of $v_0(t)$ and $v(x, y, t)$ at $(x, y) = (104, 130)$ (inside the core region) and $(104, 20)$ (far from the core region), with the initial transient time removed. The oscillation of the core region is entrained to the external force with a 1:1 period ratio. If the core radius becomes sufficiently large for larger $K$, the state in this region becomes close to a uniform state. However, the uniform state is entrained to the external force with a 2:1 period ratio. Thus, there is a kind of frustration for the entrainment. In numerical simulations, the core radius does not increase rapidly, as in Fig. 6(c) for $\epsilon_0 = 0.04$, but it increases slowly from 22 to 26 when $K$ is increased step by step from 0.04 to 0.12. For $K > 0.12$, the spiral core collapses, and spatio-temporal chaos appears. However, if the external oscillation is applied suddenly for $t > 0$ to the unperturbed spiral pattern (that is, $K = 0$ for $t \leq 0$ and $K = K > 0$ for $t > 0$), the spiral pattern disappears for $K > 0.087$. Figure 8(a) displays the time evolution of the ratio of the area with positive $u(x, y)$, i.e., $p = \int \int dx dy \theta(u(x, y))/L^2$, for $K = 0.09$, $\epsilon_1 = 0.02$ and
$\epsilon_0 = 0.025$. It is seen that the spiral pattern collapses and a uniformly oscillating state appears, which is completely entrained to the external force with a 2:1 period.

Fig. 7. (a) Snapshot of the spiral pattern and the trajectory of the core for $\epsilon_1 = 0.02$, $\epsilon_0 = 0.025$, $\beta = 0.7$ and $K = 0.06$ for the model equation (5). (b) Time sequences of $v(x, y, t)$ at $(x, y) = (104, 130)$ (solid curve) and $(104, 20)$ (dashed curve), and $v_0(t)$ (dotted curve), with the initial transient time removed.

Fig. 8. (a) Time evolution of $u_0(t)$ and the ratio of the area with positive $u(x, y, t)$ at $K = 0.09$, $\epsilon_1 = 0.02$ and $\epsilon_0 = 0.025$ for the model equation (5). Two snapshot patterns of $u(x, y)$ at $t = 129$ and 131 just before the collapse of the spiral. The curves corresponding to $u(x, y, t) = 0$ are drawn.
Forced Entrainment and Elimination of Spiral Waves

Figures 8(b) and (c) display two snapshot patterns of \( u(x, y) \) at \( t = 129 \) and \( 131 \) just before the collapse of the spiral. The “front” and “back” of the locally excited region collide and disappear at the instant of spiral collapse.

Figure 9 displays a phase diagram in the \( \epsilon_0 - K \) plane for the forced entrainment. The initial conditions consist of an unperturbed spiral with \( \epsilon_1 = 0.02 \) and \( \beta = 0.7 \), and the external force is applied for \( t > 0 \). In the parameter region denoted by triangles the \( \triangle \), the spiral core drifts toward the boundary and disappears there. It can be interpreted that the core radius diverges or it is sufficiently large compared to the system size in this parameter region. The uniformly entrained state with a 2:1 period ratio is obtained after the disappearance of the spiral core, as the simulation shown in Fig. 6. A similar type of core drift occurs in the parameter region denoted by rhombi \( \lozenge \). A uniformly entrained state with a 1:1 period ratio appears after the elimination of the spiral core in this parameter region. Parameter regions in which the forced entrainment occurs due to the drift motion of the spiral core are not large. On the other hand, forced entrainment that is caused by the collision of the “front” and “back” is observed if the coupling constant \( K \) is sufficiently large for any value of \( \epsilon_0 \). We have plotted the curve above which a single spiral pattern collapses due to the collision of the “front” and “back” in Fig. 9. This collapse line does not always correspond to the transition to complete entrainment. The collision does not occur uniformly, and a few irregular spirals are created for a certain parameter range just above the critical curve. Above this range, the collision occurs more clearly, and the spiral evolves into a uniformly entrained state. The critical value of \( K \) for spiral

![Fig. 9. Phase diagram for the forced entrainment. The triangles denote the 2:1 entrainment region as a result of the core drift, and the rhombuses denote the 1:1 entrainment region as a result of the core drift. The marked curve represents the boundary above which a single spiral collapses.](https://academic.oup.com/ptp/article-abstract/108/2/241/1869745)
collapse is $K_c = 0.083$ for $\epsilon_0 = 0.025$, and the critical value of $K$ for the spiral pattern to evolve into a uniformly entrained state is 0.087 for $\epsilon_0 = 0.025$, as stated in the previous paragraph. Further, the critical value of $K$ for the spiral to collapse through the step by step increase of $K$ is 0.12 for $\epsilon_0 = 0.025$. The wavelength of the spiral increases gradually when $K$ is increased step by step. Contrastingly, when the external force is applied suddenly for $t \geq 0$, the wavelength of the spiral is almost the same as the wavelength for $K = 0$, and the spiral collapses after a few periods, before the wavelength of the spiral is modified. The critical value for spiral collapse depends strongly on the wavelength. This is because the critical value for the collapse of the spiral depends on the manner in which the external force is applied.

Forced entrainment caused by the collision of the “front” and “back” can also be found in following one-dimensional FitzHugh-Nagumo model:

$$\frac{\partial u}{\partial t} = u - u^3/3 + \frac{\partial^2 u}{\partial x^2} - v + K(u_0 - u),$$
$$\frac{\partial v}{\partial t} = \epsilon_1(u - \gamma v + \beta),$$
$$\frac{du_0}{dt} = u_0 - u_0^3/3 - v_0,$$
$$\frac{dv_0}{dt} = \epsilon_0(u_0 - \gamma v_0).$$  (6)

We used periodic boundary conditions in the numerical simulation of this system. Figures 10(a) and (b) display the time evolution of $u(x, t)$ for $\epsilon_1 = 0.02$, $\epsilon_0 = 0.025$, $\gamma = 0.5$, $\beta = 0.7$ and $K = 0.095$. Figure 10(a) displays the time evolution of the forced entrainment of pulse trains. Initially, an unperturbed pulse-train of wavelength 80 is prepared. The external forcing starts at $t = 0$. The “fronts” and “backs” of the pulse train collide and disappear, and then a uniformly entrained state appears. The critical value of $K$ at which the pulse train disappears depends on the wavelength of the pulse train. In fact, one pulse cannot be completely entrained to the external force even at $K = 0.1$ for $\epsilon_1 = 0.02$, $\epsilon_0 = 0.025$, $\gamma = 0.5$, $\beta = 0.7$, as shown in Fig. 10(b).

The collapse of the spiral as a result of the collision of the “front” and “back” is expected to occur fairly generally as a result of the sudden application of the external force. If an abrupt single impulse is applied at $t = 0$, the model equation is written

$$\frac{\partial u}{\partial t} = u - u^3/3 + \nabla^2 u - v + K\delta(t),$$
$$\frac{\partial v}{\partial t} = \epsilon_1(u - \gamma v + \beta),$$  (7)

where $K$ represents the strength of the impulse. After the impulse is applied, the value of $u$ is shifted as $u(x, y, t = +0) = u(x, y, t = -0) + K$. The numerical simulation was performed using the initial conditions $u = u_-(x, y) + K$, $v = v_-(x, y)$, where $u_-$ and $v_-$ represent an unperturbed spiral solution. For $K > 1.49$, the spiral solution collapses as a result of the collision of the “front” and “back”. This leads to a uniform stationary state, since no external force is applied for $t > 0$. Two snapshot
Forced Entrainment and Elimination of Spiral Waves

Fig. 10. Time evolution of $u(x, t)$ for $\epsilon_1 = 0.02$, $\epsilon_0 = 0.025$, $\gamma = 0.5$, $\beta = 0.7$ and $K = 0.095$ for the one-dimensional model (6). (a) The time evolution of the pulse trains initially including 10 pulses. (b) Time evolution starting from one pulse.

Fig. 11. Two snapshot patterns of $u(x, y)$ at (a) $t = 20$ and (b) 28 just before the collapse of the spiral for $K = 1.6$, $\epsilon_1 = 0.02$ and $\beta = 0.7$ in Eq. (7).

patterns at $t = 20$ and 28 are displayed for $K = 1.6$, $\epsilon_1 = 0.02$ and $\beta = 0.7$ in Fig. 11. The elimination of spirals by a strong single electric impulse is used medically for defibrillation.

§5. Summary

We have performed several types of numerical simulations involving the entrainment and elimination of spiral patterns in the FitzHugh-Nagumo model. We have found a spiral pattern can be eliminated by a faster pacemaker region. We have also used a method to control the position of the spiral core to eliminate a pair of clockwise and counterclockwise rotating spirals. Finally, we observed the divergence of the core radius and the collision of the “front” and “back” of as different mechanisms of complete entrainment to an external force.
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References