Quantum Radion on de Sitter Branes

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The quantum fluctuation of the relative location of two \((n - 1)\)-dimensional de Sitter branes (which have \(n\) spacetime dimensions) embedded in an \((n + 1)\)-dimensional anti-de Sitter bulk, which we call the “quantum radion”, is investigated at the linear order in perturbation theory. The quantization of the radion is carried out by deriving the effective action of the radion. Assuming that the positive tension brane represents our universe, the effect of the quantum radion is evaluated by using the effective Einstein equations on the brane, in which the radion contributes to the effective energy momentum tensor at the linear order of the radion amplitude. Specifically, the rms effective energy density arising from the quantum radion is compared with the background energy density. It is found out that this ratio remains small for reasonable values of the parameters of the model even without introducing a stabilizing mechanism for the radion, although the radion itself has a negative mass squared and is unstable. The reason behind this phenomenon is also discussed.

§1. Introduction

On the basis of the idea of a brane-world suggested by string theory,¹) Randall and Sundrum proposed an interesting scenario that either of two boundary 3-branes, one with positive and one with negative tension, may be our universe in a 5-dimensional anti-de Sitter space (AdS).²),³) One of the attractive features of the Randall-Sundrum (RS) scenario is that the gravity on the brane is confined to a short distance from the brane even for an infinitely large extradimension.³),⁴) This applies to the positive tension brane, and it is due to the fact that the AdS bulk on both sides of the positive tension brane shrinks exponentially as a function of the distance from the brane.

Because the RS scenario provides an exciting, new picture of our universe, it is clearly important to study various aspects of this scenario to test it and to determine constraints on its parameters. One good example of such a study is an analysis carried out by Garriga and Tanaka,⁵) in which they showed that the radion in the original RS scenario acts like a Brans-Dicke scalar on the branes at linear order in a perturbative analysis, and the effective gravity is that of the Brans-Dicke theory with positive and negative Brans-Dicke parameters on the positive and negative tension branes, respectively, where the values of the Brans-Dicke parameters are determined by the distance between the two branes. In a previous paper, we showed that essentially the same situation arises in the case of two de Sitter (dS) branes embedded in the AdS bulk. However, we have also shown that the radion effectively
has a negative mass squared with its absolute value proportional to the curvature of the dS brane, and hence is unstable if it can fluctuate independently without the matter energy momentum tensor.

The phenomenon we study in this paper is the quantum fluctuation of this mode, called the quantum radion. To make clear what we mean by the quantum radion, let us describe the radion mode in more detail. Our brane universe may be displaced from the 0th order trajectory of a homogeneous and isotropic brane. By appropriately fixing the coordinate gauge, the displacement perpendicular to the brane can be described by a scalar function on the brane. Assuming there are two branes that are fixed points of the $Z_2$ symmetry, it can be shown that only the relative displacement of the two branes is physical. We call this relative displacement the radion. There are two distinctively different kinds of displacement of a brane, the “bend” and “fluctuation.” A bend is a type of displacement resulting from inhomogeneities in the matter energy-momentum tensor on the brane. The trace of the energy-momentum acts as an additional tension of the brane, and the brane must “bend” accordingly. This relative bend is described by the mode of radion that couples with the source on the branes, and it can be written as a functional of the energy-momentum tensor. In the context of the quasi-localized gravity discussed by Gregory, Rubakov and Sibiryakov, the role of this type of radion has been extensively studied. Contrastingly, a fluctuation is a type of displacement that is purely geometrical, which obeys a free wave equation without a source. A relative displacement of this kind is the mode of the radion that we study in this paper. This mode of the radion was first studied by Charmousis, Gregory and Rubakov for RS branes, whose effective radion mass is zero, by solving the field equations for the RS branes and by Chacko and Fox for dS and AdS branes, whose effective radion masses squared are negative and positive, respectively.

A negative (or zero) radion mass squared suggests the (marginal) instability of the two brane system. In fact, in the case of the original RS flat two brane model, the negativity of the Brans-Dicke parameter on the negative tension brane can be regarded as a result of this marginal instability. To recover stability, Goldberger and Wise introduced a bulk scalar field that couples to the branes in such a way that the distance between the two branes is stabilized. However, it should also be noted that the effective Brans-Dicke parameter on a positive tension brane is positive, and it can be large enough to be consistent with experiments on the separation of branes larger than the AdS curvature radius, at least at linear order in the perturbative analysis. Hence, a stabilization mechanism may be unnecessary if we exist on a positive tension brane. We therefore do not introduce a stabilization mechanism.

We consider a system that consists of an $(n+1)$-dimensional AdS bulk space-time bounded by two branes of constant curvature that are fixed points of the $Z_2$-symmetry. The zero-curvature branes correspond to flat RS branes, while the positive-constant curvature branes correspond to dS branes. Our main concern is, of course, the dS brane case, but we treat the flat brane case simultaneously, to make clear the similarities and differences of the two cases.

The dS brane case is of particular interest, because it gives a good model of braneworld inflation. In the standard 4-dimensional inflation, quantum vacuum
fluctuations play a very important role. It is therefore natural to ask if the quantum radion fluctuations play an important (if not disastrous) role in the braneworld inflation.

It should be mentioned that a very similar situation was analyzed by Garriga and Vilenkin, in which they considered the fluctuations of a thin domain wall in \((N+1)\) spacetime dimensions. Although they assumed a Minkowski background, many of their results apply equally to the present case. In particular, they showed that the wall fluctuation mode is represented by a scalar field existing in the \(N\)-dimensional de Sitter space that describes the internal metric on the domain wall, and the scalar field has a negative mass squared of \(-NH^2\).

This paper is organized as follows. In §2, we describe the background spacetime and define our notation. In §3, we solve the perturbation equation in the bulk that describes the radion mode. We find that there is a gauge degree of freedom that should be carefully treated in the dS brane case, in contrast to the flat brane case, where no such subtlety arises. In §4, assuming a dependence on the extra dimensional coordinate that solves the perturbation equation in the bulk, we derive the effective action for the radion and quantize it. In §5, based on the result obtained in §4, we evaluate the effective energy density of the radion on the brane which is linear in the radion amplitude and estimate its effect by calculating the rms value. We find this effect remains small for reasonable values of the model parameters, even though the radion itself is unstable. In §6, we summarize our results and discuss their implications.

§2. Background

First, we summarize the basic equations of the system and define our notation. The system we consider is a \(Z_2\) symmetric \((n+1)\)-dimensional bulk spacetime with two \((n-1)\)-dimensional branes as the fixed points of the symmetry. The bulk metric \(g_{ab}\) obeys the \((n+1)\)-dimensional Einstein equation,

\[
(n+1)G_{ab} + \Lambda_{n+1} g_{ab} = \kappa^2 T_{ab},
\]

where \(\kappa^2\) is the \((n+1)\)-dimensional gravitational constant, \(\Lambda_{n+1}\) is the \((n+1)\)-dimensional cosmological constant, which we assume to be negative, and \(T_{ab}\) is localized on the branes. We use Latin indices for \((n+1)\)-dimensional tensor fields in the bulk and Greek indices for \(n\)-dimensional tensor fields on the brane. Denoting the \(n\)-dimensional metric on the brane by \(q_{\mu\nu}\), we decompose the localized energy momentum tensor \(T_{ab}\) into the tension part, \(-\sigma q_{\mu\nu}\), and the matter part, \(\tau_{\mu\nu}\). Then, Eq. (2.1) reduces to

\[
(n)G_{\mu\nu} + \Lambda_n q_{\mu\nu} = \kappa_n^2 \tau_{\mu\nu} + \kappa_n^4 \pi_{\mu\nu} - E_{\mu\nu}
\]

on the brane, where \(\pi_{\mu\nu}\) is a tensor field quadratic in \(\tau_{\mu\nu}\), and \(E_{\mu\nu}\) is the projected \((n+1)\)-dimensional Weyl tensor defined by \(E_{\mu\nu} := (n+1)C^a_{\mu\nu}n_a n^b\), with \(n^a\) being the unit vector normal to the brane. The constants \(\Lambda_n\) and \(\kappa_n^2\) are related to the
basic constants of the systems as

\[ \Lambda_n := \frac{n-2}{n} \kappa^2 \left( \Lambda_{n+1} + \frac{n}{8(n-1)} \kappa^2 \sigma^2 \right), \quad \kappa_n^2 := \frac{n-2}{4(n-1)} \sigma \kappa^4. \]  

(2.3)

It should be noted that although the decomposition of \( T_{ab} \) is not unique, there is no arbitrariness in the definition of the metric on the brane.

We consider branes with \( \tau_{\mu\nu} = 0 \). From Eq. (2.2), we see that the \( E_{\mu\nu} \) term can be considered the effective energy-momentum tensor on the brane induced by the bulk gravitational field,

\[ \tau^E_{\mu\nu} := - \frac{1}{\kappa^2_n} E_{\mu\nu}. \]  

(2.4)

The background we consider is the RS brane system and the dS brane system. To be more precise, the bulk is the \((n+1)\)-dimensional anti-de Sitter spacetime (AdS\(_{n+1}\)), whose metric can be written

\[ ds^2 = \tilde{g}_{ab} dx^a dx^b = b^2(z) \left[ dz^2 + \gamma_{\mu\nu} dx^\mu dx^\nu \right] ; \]

\[ b(z) = \frac{\ell \sqrt{K}}{\sinh \sqrt{K} z}, \quad \ell = \left[ -\frac{n(n-1)}{2 \Lambda_{n+1}} \right]^{1/2}, \]  

(2.5)

where \( \gamma_{\mu\nu} \) is the metric of a Lorentzian \( n \)-dimensional constant curvature space with curvature \( K = 0 \) or \( K = 1 \), and \( \ell \) is the curvature radius of AdS\(_{n+1}\). The function \( b(z) \) is called the warp factor. The two branes are placed at the coordinates \( z = z_+ \) and \( z = z_- (z_+ < z_-) \), with tensions and cosmological constants given by

\[ \sigma_\pm = \pm \frac{2(n-1)}{\kappa^2 \ell} \cosh(\sqrt{K} z_\pm) \quad \text{and} \quad \Lambda_{n,\pm} = \frac{1}{2} \frac{(n-1)(n-2) K}{b^2(z_\pm)}, \]  

(2.6)

which satisfy the effective Einstein equations with \( \tau_{\mu\nu,\pm} = 0 \) and \( E_{\mu\nu,\pm} = 0 \). The choice \( K = 0 \) corresponds to the RS brane system, while \( K = 1 \) corresponds to the dS brane system. Note that the \( n \)-dimensional gravitational constant \( \kappa_n^2 \) on the positive tension dS brane is related to that on the positive tension RS brane, say \( \kappa_n^{2,\text{RS}} \), according to \( \kappa_n^2 = \kappa_n^{2,\text{RS}} \cosh z_+ \).

Although we carry out most of our calculations in the coordinate system with \( z \), sometimes it is more convenient to use the coordinate system with the proper distance coordinate \( r \) defined by

\[ dr = -b(z) dz. \]  

(2.7)

It is also useful to introduce the rate of change of the warp factor \( b(z) \) with \( r \), which we shall denote by \( J(z) \):

\[ J(z) := \frac{\partial_r b}{b} = - \frac{\partial_z b}{b^2} = \frac{\cosh \sqrt{K} z}{\ell}. \]  

(2.8)
§3. Radion mode in the bulk

In this section, we consider the gravitational perturbation on the background described in the previous section. Our analysis is a generalization and a reformulation of Ref. 12) which treats the RS brane system ($K = 0$). The 4-dimensional dS and AdS brane system is investigated in Ref. 13).

We denote the gravitational perturbation to the background metric $\tilde{g}_{ab}$ by $h_{ab}$, i.e., $g_{ab} = \tilde{g}_{ab} + h_{ab}$, where $g_{ab}$ is the full metric. We choose the RS gauge with respect to the positive or negative tension brane and denote the gravitational perturbation in this gauge by $h_{\mu\nu}^{[\pm]}$ and the coordinates by $\{r_{[\pm]}, x_{[\pm]}^\mu\}$. The RS gauge condition is

$$h_{55}^{[\pm]} = h_{55}^{[\pm]} = h_{[\pm]5\mu} = h_{[\pm]\mu5} = 0. \quad (3.1)$$

In the RS gauge, the Einstein equations (2.1) simplify to

$$\left[ \frac{1}{b^{n-1}} b \partial_r b^{n-1} b \partial_r + \Box_n - 2K \right] h_{[\pm]5\mu\nu} = 0. \quad (3.2)$$

Since $h_{[\pm]5a} = 0$, we can use Eq. (A-23) of Ref. 5) to obtain the boundary condition on $h_{[\pm]}$. Expressing the perturbed locations of the branes in terms of $\phi^{[\pm]}(x)$ as

$$r_{[+]} = r_+ + \phi^{[+]}(x) \quad \text{and} \quad r_{[-]} = r_- + \phi^{[-]}(x), \quad (3.3)$$

where $r_\pm$ denote the positions of the background branes, we have

$$b \partial_r \left[ \frac{h_{[\pm]5\mu}}{b^2} \right] = \frac{2}{b} \mathcal{L}_{\mu\nu} \phi^{[\pm]} \quad \text{at} \quad r_{[\pm]} = r_\pm \mp 0, \quad (3.4)$$

where

$$\mathcal{L}_{\mu\nu} := D_\mu D_\nu + K \gamma_{\mu\nu}. \quad (3.5)$$

Note that $\tau_{\mu\nu,\pm}$ is set to zero, as we are interested in a gravitational perturbation that does not couple to the matter on the brane. It should be noted here that Eq. (A-23) of Ref. 5) is defined in terms of the smaller value of $r$, while in this paper, the boundary condition at the negative tension brane is given in terms of the larger value of $r$, $r_{[-]} = r_- + 0$. For this reason, the signature is reversed. In passing, we also note that Eq. (3.4) may be derived by first introducing Gaussian normal coordinates with respect to the branes and transforming them to the RS gauge, as done in Ref. 12).

With the junction condition (3.4), we consider a particular solution of the form

$$h_{[\pm]5\mu\nu} = b^2 u(z) \mathcal{L}_{\mu\nu} \phi^{[\pm]}(x^\mu). \quad (3.6)$$

The traceless condition on the metric perturbation in the RS gauge requires $\phi^{[\pm]}$ to satisfy

$$[-\Box_n - nK] \phi^{[\pm]} = 0, \quad (3.7)$$
which implies

\[
[-\Box_n + n\mathcal{K}]\mathcal{L}_{\mu\nu}\varphi^{[\pm]} = 0.
\]  

(3.8)

Then, Eq. (3.2) reduces to

\[
\left[\frac{1}{b^{n-1}}\partial_z b^{n-1}\partial_z + (n - 2)\mathcal{K}\right] u(z) = 0.
\]  

(3.9)

The general solution to this equation is obtained as follows. First, it is easy to see that \(u^{(1)}(z) = \ell J(z)\) is a solution. [Note here the identity \(\partial_z J(z) = \mathcal{K}b^{-1}(z)\).] Then, the other independent solution, \(u^{(2)}(z)\), may be found using the Wronskian of the above equation,

\[
u^{(2)}_n(z) = \ell^{n-2} J(z) \int^z dz' J^{-2}(z') b^{1-n}(z').
\]  

(3.10)

Thus, the general solution of the form (3.6) is

\[
h^{[\pm]}_{\mu\nu} = b^2 \ell^{-1} \left(C^{[\pm]} u^{(1)}(z) + D^{[\pm]} u^{(2)}(z)\right) \mathcal{L}_{\mu\nu}\varphi^{[\pm]},
\]  

(3.11)

where \(C^{[\pm]}\) and \(D^{[\pm]}\) are constants.

Note that there is an ambiguity in \(u^{(2)}(z)\), which depends on the choice of the integration constant. Changing this causes a change of the coefficient \(C^{[\pm]}\). This implies that the solution \(u^{(1)}\) represents a gauge degree of freedom. In fact, if we evaluate \(E^{[\pm]}_{\mu\nu}\), we obtain

\[
E^{[\pm]}_{\mu\nu} = -\frac{1}{n - 1} (\partial_z^2 - \mathcal{K}) \left(\frac{h^{[\pm]}_{\mu\nu}}{b^2}\right) = -(b/\ell)^{2-n} \ell^{-1} D^{[\pm]} \mathcal{L}_{\mu\nu}\varphi^{[\pm]},
\]  

(3.12)

which is independent of \(C^{[\pm]}\). Note also that the \(z\)-dependence of \(E_{\mu\nu}\) is clearly consistent with the boundary condition at a brane with no source, \(\partial_z (b^{n-2} E_{\mu\nu}) = 0\).\(^5\)

Since \(E_{\mu\nu}\) is a gauge-invariant tensor field, we see that the displacement of the branes should be related as

\[
D^{[+]}\varphi^{[+]} = D^{-}\varphi^{-}.
\]  

(3.13)

With the above relation, we can use a residual gauge degree of freedom in the RS gauge to cause the two different sets of the coordinates \(\{x^a_{[\pm]}\}\) to coincide. Let us denote this unified set of coordinates by \(\{x^a\}\). This is carried out through the infinitesimal coordinate transformation

\[
r \rightarrow r + \hat{\xi}^r(x^\mu), \quad x^\mu \rightarrow x^\mu + \frac{\ell}{2} B^{-2} \hat{\xi}^r\hat{\xi}^\mu(x^\mu) + \hat{\xi}^\mu(x^\mu);
\]  

(3.14)

\[
\hat{\xi}^r = \frac{\mathcal{K}}{2} \left[C^{[\pm]} - \alpha D^{[\pm]}\right] \varphi^{[\pm]}, \quad \hat{\xi}^\mu = \frac{1}{2\ell} \left[C^{[\pm]} - \alpha D^{[\pm]}\right] \varphi_{[\mu]}^{[\pm]};
\]  

(3.15)
where $\alpha$ is the remaining gauge parameter, and the function $B(z)$ is defined as

$$B^{-2} := -\frac{\ell}{2} \left[ \int_0^z b^{-2} dr \right]^{-1} = \frac{2 \cosh \sqrt{K} z - 1}{\ell^2 \sqrt{K}},$$

where

$$B^{-2} = \begin{cases} b^{-2}, & (K = 0) \\ \frac{2}{\ell} (J - \ell^{-1}), & (K = 1) \end{cases}$$

(3.16)

The metric perturbation in the unified coordinates $\{x^a\}$ is

$$h_{\mu\nu} = b^2 \ell^{-1} \left( \alpha u^{(1)}(z) + u^{(2)}(z) \right) D^{[\pm]} \mathcal{L}_{\mu\nu} \varphi^{[\pm]},$$

(3.17)

and the branes are now positioned at

$$r = r_\pm + \left( 1 + \frac{\mathcal{K}}{2} \left[ C^{[\pm]} - \alpha D^{[\pm]} \right] \right) \varphi^{[\pm]}.$$  (3.18)

We note that the positions of the two branes cannot fluctuate independently in this unified gauge, because of the relation given in Eq. (3.13). This elucidates that the nature of the radion is to describe the relative displacement of the branes.

With the above form of the metric perturbation and the locations of the branes, we find that the boundary condition on $h_{ab}$ gives

$$D^{[\pm]} = -\frac{2\ell}{\alpha \mathcal{K} \ell + b_\pm \partial_z u^{(2)}(z_\pm)} \left( 1 + \frac{\mathcal{K}}{2} \left[ C^{[\pm]} - \alpha D^{[\pm]} \right] \right),$$

(3.19)

which implies

$$1 + \frac{\mathcal{K}}{2} C^{[\pm]} + \frac{b_\pm \partial_z u^{(2)}(z_\pm)}{2\ell} D^{[\pm]} = 0.$$  (3.20)

An apparent complication in the case $\mathcal{K} = 1$ is that this seems to constrain the values of $C^{[\pm]}$, which are merely unphysical gauge parameters. This complication can be resolved by considering the integration constant term in $u^{(2)}$ defined in Eq. (3.10). With a change of $C^{[\pm]}$ given by

$$C^{[\pm]} \rightarrow C^{[\pm]} + \beta,$$

(3.21)

we consider a simultaneous change of the integration constant term in $u^{(2)}$ given by

$$u^{(2)} \rightarrow u^{(2)} - (D^{[\pm]} \beta) u^{(1)},$$

(3.22)

which results in

$$\partial_z u^{(2)} \rightarrow \partial_z u^{(2)} - D^{[\pm]} \beta \frac{\mathcal{K} \ell}{b}.$$  (3.23)

This combined change of $C^{[\pm]}$ and the integration constant term in $u^{(2)}(z)$ leaves Eq. (3.20), as well as $h^{[\pm]}_{\mu\nu}$, unchanged. The reason for the simplification in the case of $\mathcal{K} = 0$ can also be seen from Eq. (3.23): The change of $u^{(2)}$ does not affect $\partial_z u^{(2)}$ if $\mathcal{K} = 0$. 


Introducing $\tilde{\varphi}(x)$ defined as

$$\tilde{\varphi} := -\frac{D^{[+]}\varphi^{[+]} - D^{[-]}\varphi^{[-]}}{2\ell},$$

the results are summarized as

$$\left\{ \begin{array}{ll}
\bullet & h_{\mu\nu} = -2b^2(\alpha u^{(1)} + u^{(2)})\mathcal{L}_{\mu\nu}\tilde{\varphi} \\
\bullet & (\text{positive negative}) \text{ brane at } r = r_\pm + L(z_\pm; \alpha)\tilde{\varphi},
\end{array} \right. \tag{3.25}$$

where $L(z; \alpha) := \alpha\ell K + b(z)\partial_z u^{(2)}(z)$ and $\tilde{\varphi}$ satisfies

$$[-\Box_n - nK]\tilde{\varphi}(x^\mu) = 0. \tag{3.26}$$

As should be clear by now, the parameter $\alpha$ describes the residual freedom of the RS gauge in the unified coordinates, corresponding to the coordinate transformation $r \to r + Kf(x^\mu)$ and $x^\mu \to x^\mu + Jf|\mu(x^\mu)$, with $f(x^\mu)$ satisfying $L^\mu\mu f(x^\mu) = 0$.

### §4. Quantum radion

Given the initial data for $\tilde{\varphi}$, we obtain the full time evolution of the fluctuations of the brane by solving Eq. (3.26). As the initial data, it is natural to assume that $\varphi$ is classically null, and that only the quantum vacuum fluctuations are present.

To consider the quantum fluctuations of the brane, we need the action for $\tilde{\varphi}$. We obtain this by substituting the classical solution (3.25) into the original action of the system, but without constraining $\tilde{\varphi}(x)$ to satisfy Eq. (3.26). This is because the action thus obtained is already maximized, except for the degree of freedom corresponding to the fluctuation of the brane, and hence it is appropriate for the quantization of such a degree of freedom.\(^{19}\)

Our system consists of identical bulk spacetimes $\mathcal{M}_1$ and $\mathcal{M}_2$ with branes $\Sigma_+$ and $\Sigma_-$ that are the boundary hypersurfaces of $\mathcal{M}_1$ and $\mathcal{M}_2$ with tensions of $\sigma_+$ and $\sigma_-$, respectively: $\partial \mathcal{M}_1 = \Sigma_+ \cup \Sigma_-$ and $\partial \mathcal{M}_2 = \Sigma_+ \cup \Sigma_-$. The action of this system is given by

$$I = I_{R-2\Lambda,1} + I_{R-2\Lambda,2} + I_{K,1} - I_{K,2} + I_{\sigma} + I_{\sigma};$$

$$I_{R-2\Lambda,1} := \frac{1}{2\kappa^2} \int_{\mathcal{M}_1} \sqrt{-g} \, dr \, d^n x \left[ (n+1)R - 2\Lambda n + 1 \right],$$

$$I_{K,1} := \frac{1}{\kappa^2} \int_{\partial \mathcal{M}_1} \sqrt{-g} \, d^n x \, K, \quad I_{\sigma} := -\int_{\Sigma_\pm} \sqrt{-g} \, d^n x \, \sigma_\pm, \tag{4.1}$$

where $q_{ab} := g_{ab} - n_a n_b$, $K_{ab} := g_a^c q_b^d \nabla_c n_d$, and $n^a$ is the unit vector normal to the branes, pointing from $\mathcal{M}_1$ to $\mathcal{M}_2$. Substituting $h_{\mu\nu}$ and the locations of the branes given by Eq. (3.25) into the action (4.1), we find

$$I^{(2)}[\tilde{\varphi}] = \frac{(n-1)(n-2)\ell^{n-1}}{\kappa^2} N(n-2)K \int d^n x \sqrt{-\gamma} \left[ \Box_n + nK \right] \tilde{\varphi}, \tag{4.2}$$
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where
\[ N_{(n-2)K} := \ell^{n-3} \int_{z_+}^{z_-} dz' \ell^{3-n} \left( b_- \partial_z u^{(2)}(z\_+ - b_+ \partial_z u^{(2)}(z\_+) \right). \] (4.3)

The normalization factor \( N_{(n-2)K} \) is found to be identical to that introduced in Ref. 5) for the normalization of the radion mode of \( E_{\mu\nu} \). Details of the derivation and its comparison with those of Chiba\(^{20}\) and of Chacko and Fox\(^{13}\) are given in Appendix A.

Since \( N_{(n-2)K} > 0 \), we may normalize \( \tilde{\varphi} \) as
\[ \psi(x^\mu) := \sqrt{\frac{2(n-1)(n-2)N_{(n-2)K}}{\kappa^2}} \left( \frac{\ell}{b_+} \right)^{n-2} \tilde{\varphi}(x^\mu), \] (4.4)
so that the action reduces to that of a scalar field \( \psi(x) \) with mass squared \( m^2 := -nKb_{+}^{-2} \) on the positive tension brane. We obtain
\[ I^{(2)}[\psi] = \int d^n x \sqrt{-q_+} \left[ -\frac{1}{2} q_+^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} m^2 \psi^2 \right], \] (4.5)
where \( q_+^{\mu\nu} := b_+^2 \gamma_{\mu\nu} \) is the intrinsic metric of the positive tension brane. Here and below, we focus on the positive tension brane, but similar arguments apply to the negative tension brane, with only a difference in the normalization factor.

In terms of \( \psi(x^\mu) \), we see from Eq. (3.25) that \( E_{\mu\nu} \) is given by
\[ E_{\mu\nu} = N_E \left( \frac{b_+}{b} \right)^{n-2} L_{\mu\nu}^{TL} \psi(x), \] (4.6)
where
\[ N_E = \sqrt{\frac{2}{(n-1)(n-2)N_{(n-2)K}}} \frac{\kappa^2}{\ell} \left( \frac{\ell}{b_+} \right)^{n-2}, \]
\[ L_{\mu\nu}^{TL} = L_{\mu\nu} - \frac{\gamma_{\mu\nu}}{n} \gamma^{\alpha\beta} L_{\alpha\beta} = D_\mu D_\nu - \frac{\gamma_{\mu\nu}}{n} \Box_n. \] (4.7)

Here, it may be appropriate to make a comment on the form of Eq. (4.6), in which the operator \( L_{\mu\nu} \) is replaced by its traceless part. In the previous section, we considered only the classical solution for the radion fluctuations. Hence, there we were allowed to use the equation of motion to express \( E_{\mu\nu} \) in terms of the radion field, and the equation of motion for the radion was derived from the traceless condition of the RS gauge. When we consider the quantum radion, however, \( E_{\mu\nu} \) must be traceless at the off-shell level as well. This implies that the traceless condition of the RS gauge should not be used to express \( E_{\mu\nu} \) in terms of the radion field. Instead, it is necessary to go back to the original definition of the Weyl tensor in terms of the metric perturbation, whose trace is zero by construction. This is the reason for the appearance of the traceless part of \( L \) in Eq. (4.6).
Now that we have obtained the action for the fluctuation of the brane, which is of the form of the scalar field action, we can canonically quantize it in the ordinary manner. Employing a flat coordinate chart on the brane, which is specified by
\[ ds^2 = b_+^2 \gamma_{\mu\nu} dx^\mu dx^\nu = a^2(\eta) \left[ -d\eta^2 + (dx)^2 \right] \]
\[ \left( a(\eta) := \frac{b_+}{1 - K}, \quad H_+ := \frac{\partial_\eta a}{a^2} = Kb_+^{-1} \right), \quad (4.8) \]
we obtain
\[ \hat{\psi}(\eta, x) = \int d^{n-1}k \left[ \hat{a}_k \hat{\psi}_k(\eta)e^{ik \cdot x} + \hat{a}_k^\dagger \hat{\psi}_k^*(\eta)e^{-ik \cdot x} \right], \quad (4.9) \]
where \( \hat{a}_k \) and \( \hat{a}_k^\dagger \) satisfy the commutation relations
\[ [\hat{a}_k, \hat{a}_k^\dagger] = \delta(k - k'), \quad [\hat{a}_k, \hat{a}_k'] = [\hat{a}_k^\dagger, \hat{a}_k^\dagger] = 0, \quad (4.10) \]
and \( \{\psi_k(\eta)\} \) satisfies the equation of motion
\[ \left[ \frac{d^2}{d\eta^2} - \frac{(n - 2)K}{\eta} \frac{d}{d\eta} + k^2 - \frac{nK}{\eta^2} \right] \psi_k(\eta) = 0, \quad (4.11) \]
with the normalization condition
\[ \psi_k \frac{d}{d\eta} \psi_k^* - \psi_k^* \frac{d}{d\eta} \psi_k = \frac{2i}{a^{n-2}}. \quad (4.12) \]
As a natural choice for the vacuum, we choose the Bunch-Davis vacuum, for which we have
\[ \psi_k(\eta) = \begin{cases} \frac{e^{-ik\eta}}{\sqrt{2k}}, & (K = 0) \\ \frac{\sqrt{\pi}}{2} \frac{b_+^{-(n-2)/2}(-\eta)^{(n-1)/2}}{(-k\eta)^n} H_{(n+1)/2}^{(1)}(-k\eta), & (K = 1) \end{cases} \quad (4.13) \]
where \( H_{\lambda}^{(1)}(z) \) is the Hankel function of the first kind. For \( K = 1 \), this would correspond to choosing the de Sitter invariant vacuum if the mass squared were positive. Note that the rms amplitude of the vacuum fluctuations per logarithmic interval of \( k \) exhibits an infrared divergence in the limit \( \eta \to -0 \):
\[ \sqrt{k^{n-1} |\psi_k(\eta)|^2} \propto \frac{1}{|k\eta|}. \quad (4.14) \]
We now focus on the case \( K = 1 \), which is of cosmological interest, and consider the effect of the quantum radion on the brane. As noted, \( \psi \) exhibits an instability that grows as \( (-\eta)^{-1} \propto a(\eta) \). However, the quantity that can be detected on the brane is \( E_{\mu\nu} \) given by Eq. (4.6), which acts as the effective energy-momentum tensor,
Eq. (2.4). To see its behavior, let us give the components of $E_{\mu\nu}$ explicitly. The components for each $k$ mode are

\[
E_{00} = N_E \left[ \frac{n-1}{\eta} \left( \frac{\partial}{\partial \eta} + \frac{1}{\eta} \right) - k^2 \right] \psi_k Y, \\
E_{0j} = N_E (-k) \left[ \frac{\partial}{\partial \eta} + \frac{1}{\eta} \right] \psi_k Y_j, \\
E_{ij} = N_E k^2 \psi_k Y_{ij} + N_E \delta_{ij} \left[ \frac{1}{\eta} \left( \frac{\partial}{\partial \eta} + \frac{1}{\eta} \right) - k^2 \right] \frac{1}{n-1} \psi_k Y, 
\]

where Eq. (4.11) has been used, and $Y, Y_j$ and $Y_{ij}$ are defined as follows:

\[
Y = e^{ik \cdot x}, \quad Y_j = -i \frac{k_j}{k} Y, \quad Y_{ij} = \left( -\frac{k_i k_j}{k^2} + \frac{\delta_{ij}}{n-1} \right) Y.
\]

Inserting into these expressions the explicit form of $\psi_k$ given in Eq. (4.13), we find

\[
E_{00} = \tilde{N}_E k^2 (-k\eta)^{(n-1)/2} H_{(n-3)/2}^{(1)} (-k\eta) Y, \quad E_{ij}^T = \frac{\delta_{ij}}{n-1} E_{00}, \\
E_{0j} = \tilde{N}_E k^2 (-k\eta)^{(n-1)/2} H_{(n-1)/2}^{(1)} (-k\eta) Y_j, \\
E_{ij}^T = \tilde{N}_E k^2 (-k\eta)^{(n-1)/2} H_{(n+1)/2}^{(1)} (-k\eta) Y_{ij},
\]

where $E_{ij}^T$ and $E_{ij}^T$ denote the trace and traceless parts of $E_{ij}$, respectively, and

\[
\tilde{N}_E = N_E \sqrt{\frac{\pi H_{(n-2)/2}^{(n-2)/2}}{2k}}.
\]

From the above, and noting $H_{(n)/2}^{(1)}(z) \sim z^{-\nu}$ for $z \to 0$, we see that $E_{ij}^T$, which represents the effective anisotropic stress, dominates the other components for $k|\eta| \to 0$, i.e., on super-horizon scales, though its physical amplitude ($E_{ij}^T/a^2$) decays as $1/a$.

The effective energy density induced by the radion decays rapidly as $a^{-3}$. However, it may be said that the decay rate is not sufficiently fast in the following sense. Since $E_{\mu\nu}$ is traceless and conserved at linear order in the perturbative analysis, it can be regarded as the energy momentum tensor of a radiation-like fluid. If it behaved exactly like radiation, the energy density would decay as $a^{-n}$ instead of $a^{-3}$. Of course the decay rate $a^{-3}$ does not imply that $E_{\mu\nu}$ behaves like dust matter. The fact that the decay rate is slower than that for the standard radiation fluid (for $n \geq 4$) is caused by the strong anisotropic stress present in the case of the radion. Thus, in this sense, the instability of the radion indeed is manifested on the brane, but its effect turns out to be rather mild, due to the nature of de Sitter space, in which the background energy density remains constant in time. We defer detailed investigation of this point to Appendix C.

To summarize, we conclude that the instability of the quantum radion induces a large anisotropic stress on the brane, but due to the rapid expansion of the brane, this anisotropic stress is not strong enough to give rise to a gravitational instability on the brane.
§5. Cosmological effect of quantum radion

In the previous section, it was seen that quantum radion fluctuations do not cause a gravitational instability on the brane. However, this does not necessarily mean their effect is always small. In particular, if the rms value of the effective energy density due to the radion field is comparable to the background vacuum energy density of the de Sitter brane, the evolution of the brane will be significantly affected. If we regard our background as representing a braneworld undergoing cosmological inflation, this would imply a significant modification of the braneworld inflationary scenario. In this section, assuming that a positive tension brane represents our universe, we consider the cosmological constraints arising from the effect of the radion quantum fluctuations.

To quantify the effect, we consider the power spectrum of the effective energy density,

\[ \rho_E := \frac{1}{\kappa_n^2} E^0_0. \]  

(5.1)

Focusing on the positive tension brane, we define the power spectrum \( P_n(k) \) of \( \rho_E \) normalized by the background energy density \( \rho_\Lambda = 3H_+^2/\kappa_n^2 \) as

\[ \frac{\langle (\hat{\rho}_E)^2 \rangle}{\rho_\Lambda^2} = \frac{S_{n-1}}{(2\pi)^n-1} \int \frac{dk}{k} k^{n-1} P_n(k), \]

(5.2)

where \( S_{n-1} \) is the surface area of the \((n-1)\)-dimensional unit sphere: \( S_{n-1} = 2\pi^{n/2}/\Gamma(n/2) \). We find

\[ k^{n-1} P_n(k) = \frac{2\pi\kappa_n^2 H_+^{n-2}(H_+\ell)^{n-2}}{9(n-1)(n-2)^2 N_{(n-2)\kappa} \cosh z_+} (-k\eta)^{n+3} |H^{(1)}_{(n-3)/2}(-k\eta)|^2 \]

\[ \rightarrow \frac{2\pi\kappa_n^2 H_+^{n-2}(H_+\ell)^{n-2}}{9(n-1)(n-2)^2 N_{(n-2)\kappa} \cosh z_+} \frac{2^n |k\eta|^6}{\Gamma(n-1/2)^2}. \]  

(5.3)

Next we define the density parameter \( \Omega_E \) for the rms effective energy density of the radion field as

\[ \Omega_E := \frac{\sqrt{\langle (\hat{\rho}_E)^2 \rangle}}{\rho_\Lambda}, \]

(5.4)

which is obtained by integrating \( k^{n-1} P_n(k) \) over \( k \) and taking its square root. When \( \Omega_E \) is of order unity or larger, the fluctuations of the brane are non-negligible. If we integrate Eq. (5.3) over \( k \), we encounter a divergence resulting from large values of \( k \). However, this ultraviolet divergence is the same as the usual one in the Minkowski background. Therefore, to remove this divergence, we cut off the integration at the Hubble horizon scale, \( k = k_H = 1/|\eta| \). The result is

\[ \Omega_E = \sqrt{\frac{1}{6(2\pi)^{n-1}} \left. k^{n-1} P(k) \right|_{k=1/|\eta|}} \]

\[ = \frac{2}{135} \sqrt{\frac{2}{\pi \cosh z_+ (\cosh z_- - \cosh z_+)} (\kappa_4 H_+)(H_+\ell)}. \]  

(5.5)
Here and below we fix the number of dimensions to \( n = 4 \). Note that \( H_+ \ell = \sinh z_+ \) is the ratio of the \( \text{AdS}_5 \) curvature radius to the \( \text{dS}_4 \) curvature radius, and \( \kappa_4^2 = \kappa_4^2 \cosh z_+ / \ell \). If we introduce the 4-dimensional Planck scale in the single flat brane limit, \( \ell_{\text{pl}} = \kappa_4 / \sqrt{\cosh z_+} \), we obtain the well-known relation \( \ell_{\text{pl}}^2 = \ell_5^2 / \ell^2 \). Below, we assume that \( \ell_{\text{pl}} \) is the present-day Planck length in our universe.

To discuss cosmological constraints, it is convenient to introduce the 5-dimensional gravitational length scale \( \ell_5 = \kappa_5^{2/3} \). In terms of \( \ell_5 \), Eq. (5.5) can be re-expressed as

\[
\Omega_E = \frac{2}{135} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\cosh z_+ - \cosh z_+}} (H_+ \ell)^2 \left( \frac{\ell_5}{\ell} \right)^{3/2}.
\]

(5.6)

We see that the quantum radion effect is negligible if

\[
\cosh z_+ - \cosh z_+ \gtrsim (H_+ \ell)^2 \left( \frac{\ell_5}{\ell} \right)^{3/2}.
\]

(5.7)

Given the ratios \( H_+ \ell \) and \( \ell_5 / \ell \), this constrains the value of \( H_- \) to be larger than a critical value,

\[
\sqrt{1 + (H_- \ell)^2} \gtrsim \sqrt{1 + (H_+ \ell)^2} + (H_+ \ell)^2 \left( \frac{\ell_5}{\ell} \right)^{3/2}.
\]

(5.8)

Since \( \ell_5 \) is the fundamental scale of the 5-dimensional theory, it is natural to assume \( \ell_5 \lesssim \ell \); otherwise the entire system will be in the quantum gravitational regime. Note that \( (\ell_5 / \ell)^3 = (\ell_{\text{pl}} / \ell)^2 \). With this assumption, let us consider the following two limiting cases.

(a) \( H_+ \ell \ll 1 \):

In this case, Eq (5.8) is satisfied for almost all values of \( H_- > H_+ \), except for values very close to \( H_+ \); that is, the constraint becomes

\[
\frac{H_-}{H_+} - 1 \gtrsim \left( \frac{\ell_5}{\ell} \right)^{3/2}.
\]

(5.9)

(b) \( H_+ \ell \gg 1 \):

In this case, we have

\[
\frac{H_-}{H_+} - 1 \gtrsim (H_+ \ell) \left( \frac{\ell_5}{\ell} \right)^{3/2}.
\]

(5.10)

Using the equality \( (\ell_5 / \ell)^3 = \ell_{\text{pl}} / \ell \), the right-hand side can be rewritten as \( H_+ \ell_{\text{pl}} \). For any reasonable scenario of inflation, we must have \( H_+ \ell_{\text{pl}} \ll 1 \).

Hence, the constraint on the value of \( H_- \) is very mild in this case as well. Thus, with the assumptions that \( H_+ \ell_{\text{pl}} \ll 1 \) and \( (\ell_5 / \ell)^3 = (\ell_{\text{pl}} / \ell)^2 \lesssim 1 \), the constraints obtained in the above two cases can be concisely expressed as

\[
\ln \left( \frac{H_-}{H_+} \right) \gtrsim (1 + H_+ \ell) \frac{\ell_{\text{pl}}}{\ell}.
\]

(5.11)

To summarize, as long as we focus on cosmological constraints from the effect of quantum radion fluctuations, the negative tension brane and positive tension brane can be very close, in fact even with distances smaller than the \( \text{AdS}_5 \) curvature radius.
§6. Summary and discussion

In this paper, considering the Randall-Sundrum type two-brane scenario in an $(n+1)$-dimensional spacetime, we have investigated the quantum fluctuations of the relative displacement of the branes, which we call the quantum radion. We have considered the cases of flat two-brane and de Sitter two-brane systems simultaneously. Adopting the so-called Randall-Sundrum gauge, we first solved the linear gravitational perturbation equations that describe the radion mode. Then assuming a perturbation of a form with a fixed $z$-dependence that solves the gravitational equation in the $z$-direction, where $z$ is the extra-dimensional coordinate orthogonal to the branes, we derived the effective action for the radion. With this effective action, we quantized the radion, assuming that the radion state is the Bunch-Davis vacuum.

We analyzed the effect of the quantum radion on the brane using the effective Einstein equations derived by Shiromizu, Maeda and Sasaki,\cite{18} in which the radion adds an effective energy-momentum tensor at linear order in the field amplitude. Although the radion has a negative mass squared of $-nH^2$ on the de Sitter brane, where $H$ is the Hubble parameter, we found that a corresponding instability is not manifested on the brane. We have noted, however, that the anisotropic stress induced by the radion is unusually large, though it still decays in time as $a^{-1}$ for a fixed value of $k$, where $a$ is the cosmic scale factor and $k$ is the comoving wavenumber, and the rms value of the effective energy density for a fixed value of $k$ decays as $a^{-3}$, irrespective of the spacetime dimensions $n$.

Focusing on the positive tension de Sitter brane with Hubble rate $H_+$, which models the braneworld inflation, we estimated the rms total energy density of the quantum radion by integrating over $k$ up to $aH_+$. We introduced the density parameter $\Omega_E$, which describes the magnitude of the radion energy density relative to the background energy density, and investigated the implication of the condition $\Omega_E \ll 1$ for the model parameters. For $H_+ \ell_{pl} \ll 1$ and $\ell_5 \lesssim \ell$, which are reasonable if the background is not in the quantum regime, we have found that almost any choice of the location of the negative tension brane is allowed. This implies the quantum radion does not seriously affect the braneworld inflation scenario, at least to linear order in the perturbative analysis.

The most intriguing question remaining is if the linear order analysis given here is sufficient. Naively, one may regard the effective action of the radion we have obtained as a piece to be added to the total effective action for an effective 4-dimensional theory that includes gravity. Then, the variation of the radion effective action with respect to the 4-metric would give the energy-momentum tensor of a scalar field with a negative mass squared, which would grow and diverge, in contrast to the decaying effective energy-momentum tensor we analyzed in this paper. However, such an energy-momentum tensor is quadratic in the radion, and therefore it is beyond the scope of our linear analysis. It is necessary to carry out the perturbative analysis to higher orders to definitely determine the effect of the quantum radion on the brane. This task is left to future study.
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Appendix A

Effective Action of Radion Derived and Compared

In this appendix, we give an outline of the derivation of the action given in Eq. (4.2) and compare it with the effective actions of radion derived in Refs. 20) and 13). The action is obtained by substituting the results given in Eq. (3.25) into the action of the system given in Eq. (4.1). Note that $\gamma_{\mu\nu}$ denotes the metric of the $n$-dimensional spacetime with constant curvature $K$, and $D_\mu$ is the covariant derivative with respect to $\gamma_{\mu\nu}$. We do not require $\tilde{\varphi}(x^\mu)$ to satisfy the field equation (3.26), and therefore the degree of freedom associated with the brane fluctuation remains.

We first substitute the results given in Eq. (3.25) into the bulk actions $I_{R-2A,1}$ and $I_{R-2A,2}$ of Eq. (4.1). Carrying out the calculation with the help of the commutation rule

$$[D_\mu D_\nu - D_\nu D_\mu] \omega_\rho = 2K\gamma_\mu[\rho \gamma_{\nu\sigma}] \sigma \omega^\sigma,$$  

(A.1)

results in the vanishing of all the second-order terms, except those terms that can be incorporated into the surface term on $\Sigma_{\pm}$. Thus, to second order in $\tilde{\varphi}$, the action (4.1) is found to have support only on $\Sigma_{\pm}$:

$$I^{(2)} = I^{(2)}_{\Sigma^+} + I^{(2)}_{\Sigma^-};$$

$$I^{(2)}_{\Sigma^\pm} = \frac{1}{\kappa^2} \int_{\Sigma^\pm} d^n x \sqrt{-q} \left[ -\frac{1}{2} \hat{k}^{\mu\nu} h_{\mu\nu} + \frac{1}{2} \hat{k}^{\sigma} h + h_{\mu\nu} \phi|^{\mu\nu} - h b^{-2} \Box_n \varphi 
- (n - 1) b^{-2} K h \varphi + \frac{1}{2} b^{-2} \sigma_{\pm} \kappa^2 \left( -\varphi \Box_n \varphi - n K \varphi^2 \right) \right];$$

$$\hat{k}_{\mu\nu} := \frac{1}{2} b^2 \partial_r \left( b^{-2} h_{\mu\nu} \right).$$  

(A.2)

Note that $\varphi$ here is defined as the displacement of the corresponding brane measured from the side of smaller values of $r$; that is,

$$\varphi|_{\Sigma^\pm} = \pm L(z_{\pm}; \alpha) \tilde{\varphi}.$$  

(A.3)

Here, it is worth noting that the above action $I^{(2)}$ is different from the one obtained in Ref. 5) [see Eq. (A.19) there]. This is because the above action $I^{(2)}$ is obtained by substituting a particular form of the metric into the action of the system, while the action of Ref. 5) is constructed without any restriction on the metric. The particular form of the metric in Eq. (3.25) solves the equation of motion in the $z$-direction, and thus it already extremizes the action in the $z$-direction. Therefore,
$I^{(2)}$ is an action to be maximized in the $x^\mu$-directions only, to give the equation of motion for $h_{\mu\nu}$ as a function of $\varphi(x^\mu)$. It should be noted that the action obtained in Ref. 5) is an action to be maximized in both the $z$-direction and $x^\mu$-directions independently, to give the equations of motion for both $h_{\mu\nu}$ and $\varphi$.19)

Let us define $L_h$ and $L_\varphi$ as

$$
L_h := -\hat{k}^{\mu\nu} h_{\mu\nu} + \hat{k}_\sigma h + h_{\mu\nu} \varphi \big|^{\mu\nu} - hb^{-2} \Box \varphi - (n - 1)b^{-2} K h \varphi,
$$

$$
L_\varphi := \frac{1}{2} \left( h_{\mu\nu} \varphi \big|^{\mu\nu} - hb^{-2} \Box \varphi - (n - 1)b^{-2} K h \varphi \right) + \frac{1}{2} b^{-2} \sigma_\pm \kappa^2 \left( - \varphi \Box \varphi - n K \varphi^2 \right).
$$

(A.4)

Then $I^{(2)}_{\Sigma_\pm}$ is expressed as

$$
I^{(2)}_{\Sigma_\pm} = \frac{1}{\kappa^2} \int_{\Sigma_\pm} d^n x \sqrt{-g} \left[ \frac{1}{2} L_h + L_\varphi \right].
$$

We find that the variation of $L_h$ with respect to $h_{\mu\nu}$ gives the boundary condition of $h_{\mu\nu}$ on the brane, while that of $L_\varphi$ with respect to $\varphi$ gives the equation of motion for $\varphi$.

Substituting Eq. (3.25) into $I^{(2)}_{\Sigma_\pm}$, $L_h$ vanishes and only $L_\varphi$ remains. Using the commutation rule (A.1), $L_\varphi$ is simplified to give

$$
I^{(2)} = I^{(2)}_{\Sigma_+} + I^{(2)}_{\Sigma_-}
$$

$$
= \frac{(n - 1) \ell^{n-2}}{\kappa^2} \int d^n x \sqrt{-\gamma} \left( b_- \partial_z u^{(2)}(z_-) - b_+ \partial_z u^{(2)}(z_+) \right) \varphi \left[ (\Box + n K) \right] \varphi,
$$

which is Eq. (4.2).

The effective action for the radion was derived previously by Chiba20) and by Chacko and Fox13) in the longitudinal gauge. However, in our opinion, there seems to exist some subtleties in their derivations that may be worth clarifying, in addition to the difference in the choice of gauge. For this purpose, we now compare their derivations with ours.

First, we need to transform the metric perturbation in Eq. (3.25), which is given in the RS gauge, into that in the longitudinal gauge, in which the locations of the branes are unperturbed, $h_{r\mu} = 0$ and $h_{\mu\nu} \propto \gamma_{\mu\nu}$. With a coordinate transformation of the form

$$
r \to r - L(z; \alpha) \tilde{\varphi}, \quad x^\mu \to x^\mu - \{ \alpha \ell J + \psi_{\mu}^{(2)} \} \tilde{\varphi} \big|^{\mu},
$$

we obtain the metric perturbation in the longitudinal gauge:

$$
h_{rr} = -2(n - 2)(\ell/b)^{2-n} \tilde{\varphi}, \quad h_{r\mu} = 0, \quad h_{\mu\nu} = 2b^2(\ell/b)^{2-n} \tilde{\varphi} \gamma_{\mu\nu}.
$$

(A.6)

Let us first consider Chiba’s effective action for the radion given in Eq. (14) of Ref. 20). The metric ansatz Chiba uses to derive the action is

$$
ds^2 = h_{\gamma}^2 dr^2 + e^{-2kh} \bar{g}_{\mu\nu} dx^\mu dx^\nu; \quad h = r + f(x^\mu)e^{2kr},
$$

(A.7)
in which $g_{\mu \nu}$ is a general 4-dimensional metric that depends solely on $x^\mu$. Here, the coordinate $z$ in his paper is replaced with the coordinate $r$ to avoid confusion. Linearizing the metric (A.7) with respect to $f(x^\mu)$, it reduces to a metric perturbation of the form (A.6) with the following correspondence between his notation and ours:

$$x^\mu \leftrightarrow \ell x^\mu, \quad k \leftrightarrow \ell^{-1}, \quad f(x^\mu) \leftrightarrow -\ell \tilde{\varphi}(x^\mu), \quad g_{\mu \nu} \leftrightarrow \gamma_{\mu \nu}, \quad (A.8)$$

where we have fixed $n = 4$. [We note that Eq. (A.7) can be extended to the general case of $n$ dimensions simply by replacing $h = r + f(x^\mu)e^{2kr}$ with $r + f(x^\mu)e^{(n-2)kr}$ in the case of $K = 0$.] With the above correspondence, we find that his result, when linearized, does not agree with ours.

We now identify the source of this difference by following his derivation. Substituting the decomposition of the Ricci scalar of the metric (A.7),

$$R[g] = -20k^2 + e^{2kh}R[\bar{g}] + 6k^2e^{4kr+2kh}h_{\nu\tau}^{-1}(-1 + 2kf e^{2kr})\nabla^\nu f \nabla^\tau f + (\text{total divergence term in } x^\mu), \quad (A.9)$$

into the original action of the system that he uses [Eq. (1) of Ref. 20)], and integrating it over $r$, we obtain

$$S_{C} = 2 \int d^4x \sqrt{-g(\pm)} \left\{ 5kM^3(\alpha^{-2}e^{-4(\alpha-1)kf} - 1) 
+ \frac{M^3}{2k} \left( 1 - \frac{1}{\alpha} e^{-2(\alpha-1)kf} \right) e^{2kf}R[\bar{g}] 
- 3kM^3(\alpha - 1)(\nabla(\pm)f)^2 - 3kM^3(\alpha^{-2}e^{-4(\alpha-1)kf} - 1) \right\} 
- \sigma(\pm) \int d^4x \sqrt{-g(\pm)} - \sigma(-) \int d^4x \sqrt{-g(-)}, \quad (A.10)$$

where the following correspondence is understood:

$$g(\pm)_{\mu \nu} \leftrightarrow q_{\pm \mu \nu}, \quad M^3 \leftrightarrow \kappa^{-2}, \quad \alpha \leftrightarrow (\ell/b)_-^2, \quad \sigma(\pm) \leftrightarrow \sigma_\pm, \quad \Lambda \leftrightarrow \Lambda_{n+1}/(2\kappa^2). \quad (A.11)$$

Comparing the action (A.10) with Eq. (5) in Ref. 20), it seems that the first term, $5kM^3(\alpha^{-2}e^{-4(\alpha-1)kf} - 1)$, is missing there. This term results from the integration of the Ricci scalar of the background RS bulk spacetime, $-20k^2$ in Eq. (A.9). Furthermore, if the correspondence (A.11) that we deduced is correct, the values of the brane tensions $\sigma(\pm) = \pm 3M^3k$, as given in Eq. (3) of Ref. 20) are wrong by a factor of 2; i.e., the correct values are $\sigma(\pm) = \pm 6kM^3$. In addition, in his original action, where the integral over the extra dimension is given by twice the integral over $r$ in the range $(0, r_c)$ (that is, from the positive tension brane to the negative tension brane), the geometrical boundary terms of the bulk spacetime [$I_{K,1}$ and $I_{K,2}$ in our action given in Eq. (4.1)] should be added.
With these corrections, we find
\[ S_C(\sigma_{\pm}) \rightarrow \pm 6kM^3 + I_{K,1} - I_{K,2} \]
\[ = 2 \int d^4x \sqrt{-g(+)} \left\{ \frac{M^3}{2k} \left( 1 - \frac{1}{\alpha} e^{-2(\alpha-1)kf} \right) e^{2kf R[\bar{g}]} - 3kM^3(\alpha - 1)(\nabla_+(f))^2 \right\}, \]
where we have used the fact that \( K_{\mu\nu} = -ke^{-kh}\bar{g}_{\mu\nu} \). Since our effective action for the radion is obtained by fixing \( \bar{g}_{\mu\nu} = \eta_{\mu\nu} \), the action to be compared is
\[ S_C(\sigma_{\pm}) \rightarrow \pm 6kM^3, \bar{g}_{\mu\nu} \rightarrow \eta_{\mu\nu} \]
\[ + \int d^4x \sqrt{-g(+)} \left\{ -3kM^3(\alpha - 1)(\nabla_+(f))^2 \right\}. \quad \text{(A.12)} \]

Using Eqs. (A.8) and (A.11) and the correspondence \( N_{2\mathcal{K}} \leftrightarrow (\alpha-1)/2 \), which follows from them, we find that the above action agrees with our result.

Let us next consider the derivation by Chacko and Fox and their result. The correspondence between their notation and ours is
\[ G_{MN} \leftrightarrow g_{ab}, \quad \bar{G}_{\mu\nu} \leftrightarrow q_{\mu\nu}, \quad M^3 \leftrightarrow (4\kappa^2)^{-1}, \quad \Lambda_B \leftrightarrow \Lambda_5/\kappa^2, \quad \bar{\Lambda}_0 \leftrightarrow \sigma_+, \]
\[ \bar{\Lambda}_A \leftrightarrow \sigma_-, \quad \alpha \leftrightarrow 2\ell^{-1}, \quad H \leftrightarrow b^2, \quad \psi \leftrightarrow 2\ell^2\bar{\phi}, \quad g_{\mu\nu} \leftrightarrow \gamma_{\mu\nu}, \]
from which we deduce
\[ \int_0^a dr' f^{-1} \leftrightarrow \ell^{-1}N_{2\mathcal{K}}. \quad \text{(A.14)} \]

Using the above correspondence, we find that their result given in Eq. (34) of Ref. 13) agrees exactly with ours.

There is, however, a slight subtlety that we now point out. Chacko and Fox state that they work in a compact extra dimension with \( \mathbb{Z}_2 \) symmetry. Therefore there is no need to introduce geometrical boundary terms in the action. In fact, the form of the linearized action given by Eq. (33) of Ref. 13) is clearly free from such terms. However, if we perform the integral over \( r \) of the linearized action, there appear contributions from the branes due to the curvature singularities. The interesting fact is that these contributions cancel with those from the tension terms in the longitudinal gauge. Perhaps this is related to the fact that the coordinated locations of the branes are unperturbed in the longitudinal gauge. This implies that the correct answer might be obtained by simply simultaneously ignoring the singular contributions from the branes and the tension terms, and then integrating the linearized action only over the bulk between the branes.

In the present paper, we took a different approach. We divided the covering space of the bulk into two patches bounded by the branes. This division requires additional geometrical boundary terms for each patch in the action of the system. If we apply our method to their original action employed by Chacko and Fox, Eq. (1) of Ref. 13), it can be rewritten as
\[ S_{CF} = 2 \int_{r=0}^{a} d^4x \sqrt{-G}(2M^3R - \Lambda_B) - \int_{r=0}^{a} d^4x \sqrt{-G}\bar{\Lambda}_0 - \int_{r=0}^{a} d^4x \sqrt{-G}\bar{\Lambda}_A \]
\[ + 2 \int_{r=0}^{a} d^4x \sqrt{-G}(4M^3K) - 2 \int_{r=0}^{a} d^4x \sqrt{-G}(4M^3K), \quad \text{(A.15)} \]
where $K_{\mu\nu}$ is the extrinsic curvature of the $r$-constant hypersurface defined with the normal vector pointing toward the direction of increasing $r$. Apart from the difference in notation, as expressed above, this form of the action is the one we used as the starting point of our calculation. [See Eq. (4.1).] Substituting the perturbation given in Eq. (A-6) into the above action, we find that the terms arising from the first integral in Eq. (A-15) that can be incorporated into the total divergence form cancel out with the perturbation of the last four terms of Eq. (A-15), i.e., the tension terms and the geometrical boundary terms. This cancellation gives an action with the same form as that given in Eq. (33) of Ref. 13) by restricting the integral to be only over the bulk. Consequently, it leads to their final result, Eq. (34) of Ref. 13).

Appendix B

Effective Brans-Dicke Parameter

In this appendix, we derive the effective Brans-Dicke parameter on the $n$-dimensional positive tension brane with curvatures $\mathcal{K} = 0, 1$.

The action for the Brans-Dicke theory in $n$-dimensional spacetime is

$$I = \frac{1}{16\pi G_n} \int \sqrt{-q} \,\, d^n x \left[ \Psi^{(n)} R - 2\Lambda - \Psi^{-1} \omega_{\text{BD}} q^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi \right] + \int \sqrt{-q} \,\, d^n x \mathcal{L}_m,$$  \hspace{1cm} (B.1)

where $q_{\mu\nu}$ is the metric of the $n$-dimensional spacetime, $\omega_{\text{BD}}$ is the Brans-Dicke parameter, and $\mathcal{L}_m$ is the Lagrangian density of matter fields. Let $\Psi(x^\mu) = f_0 \exp[W(x^\mu)]$.

Then, the equations of motion linearized in $W(x^\mu)$ are

$$(n)G_{\mu\nu} + \Lambda q_{\mu\nu} = \frac{8\pi G_n}{f_0} \,\, \tau_{\mu\nu} + \left[ D_\mu D_\nu - q_{\mu\nu} q \right] W(x^\mu),$$  \hspace{1cm} (B.2)

$$f_0 \Box_q W(x^\mu) = \frac{1}{n - 1 + (n - 2)\omega_{\text{BD}}} \left( 8\pi G_n \tau - 2\Lambda f_0 \right),$$  \hspace{1cm} (B.3)

where $\tau_{\mu\nu}$ is the energy-momentum tensor that arises from $\mathcal{L}_m$. Using Eq. (B.3), Eq. (B.2) can be put into the form

$$(n)G_{\mu\nu} + A q_{\mu\nu} = \frac{8\pi G_n}{f_0} \,\, \tau_{\mu\nu} + \left[ D_\mu D_\nu - q_{\mu\nu} q \right] W(x^\mu),$$  \hspace{1cm} (B.4)

where $\tilde{\tau}_{\mu\nu} := \tau_{\mu\nu} - (1/n)q_{\mu\nu} \tau$.

To obtain the effective Einstein equations on the positive tension brane for an energy density small compared with the tension of the brane, we substitute Eq. (3.8) of Ref. 5) into Eq. (2.2) of this paper. The result is

$$(n)G_{\mu\nu} + A_n q_{\mu\nu} = \kappa_n^2 \tau_{\mu\nu} + \frac{(\ell/b_+)^{n-2}\kappa^2}{2\ell N(n-2)\mathcal{K}} \tilde{\tau}_{\mu\nu}.$$
Comparing Eq. (B.4) with Eq. (B.5), we find that the effective gravity on the brane takes the Brans-Dicke form with the following identifications:

\[
\kappa_n^2 = \frac{8\pi G_n}{f_0} \frac{(n-2)\omega_{BD}}{n-1 + (n-2)\omega_{BD}},
\]

and

\[
\frac{(\ell/b_+)^{n-2}}{\ell N_{(n-2)\kappa}} \phi = W(x^\mu).
\]

Eliminating \(f_0\) from Eq. (B.6), we obtain

\[
\omega_{BD} = (n-1) \frac{\cosh \sqrt{\kappa z_+}}{(\sinh \sqrt{\kappa z_+}/\sqrt{\kappa})^{n-2}} N_{(n-2)\kappa},
\]

while eliminating \(\omega_{BD}\) from Eq. (B.6), we obtain

\[
8\pi G_n = \frac{\kappa^2 \ell^{-1}}{2} \left( (n-2) \cosh \sqrt{\kappa z_+} + \frac{1}{N_{(n-2)\kappa}} \left( \sinh \sqrt{\kappa z_+}/\sqrt{\kappa} \right)^{n-2} \right).
\]

For \(\kappa = 1\) and \(n = 4\), we have \(N_{(n-2)\kappa} = \cosh z_+ - \cosh z_-\) and

\[
\omega_{BD} = 3 \frac{\cosh z_+}{\sinh^2 z_+} 2\kappa, \quad 8\pi G_4 = \frac{\kappa^2 \ell^{-1}}{2} \left[ 2 \cosh z_+ + \frac{\sinh^2 z_+}{N_{2\kappa}} \right].
\]

For \(\kappa = 0\) and \(n = 4\), we have \(N_{(n-2)\kappa} = (z_+^2 - z_-^2)/2\), and Eqs. (B.9) and (B.10) recover the result of Garriga and Tanaka.\(^4\)

**Appendix C**

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**E**\(_{\mu\nu}\) à la Cosmological Perturbation Theory

In this appendix, we express the effect of the radion on the brane in the language of cosmological perturbation theory. We use the notation of Ref. 21). To begin with, we list a few important facts to be kept in mind:

(a) The background is de Sitter spacetime.
(b) \(\tau_{\mu\nu} = -E_{\mu\nu}/\kappa^2_n\) is gauge-invariant.
(c) \(\tau_{\mu\nu}^{E}\) is traceless.

The facts (a) and (b) are somewhat related. If the background bulk spacetime is anti-de Sitter, (b) is always true. Independent of this fact, from the brane point of view,
matter perturbation variables are automatically gauge-invariant on a pure de Sitter space. For example, if we consider a density perturbation, the gauge transformation law for $\delta \rho$ is given by

$$\delta \rho \rightarrow \bar{\delta \rho} = \delta \rho + (n - 1)aH(\rho + p)T,$$

where $T$ is a gauge function describing the shift in the time coordinate $\eta \rightarrow \bar{\eta} = \eta + T$. Thus a density perturbation on an exactly de Sitter space, where $\rho + p = 0$, is always gauge-invariant, and the same is true for all the matter perturbation variables. It should be noted, however, that this special property of the de Sitter background can lead to an erroneous result if the perturbation is considered in a gauge specified by conditions involving the matter variables. We return to this point below. For the moment, what we can claim is that the gauge-invariance is guaranteed provided that the only perturbations considered are those in a class of gauges whose conditions involve only geometrical variables such as the metric. Let us call this class of gauges the ‘geometrical gauges’.

First, we introduce the matter perturbation variables as

$$\delta T^{00}_0 = -\rho \delta(\eta) Y(x^i), \quad \delta T^{0j}_0 = \rho q(\eta) Y_j(x^i),$$

$$\delta T_{ij}^0 = P \pi_L(\eta) \delta_j^i Y(x^i) + P \pi_T(\eta) Y_{ij}(x^i),$$

where $Y(x^i), Y_i(x^i)$ and $Y_{ij}(x^i)$ are the spatial harmonics satisfying

$$(D^k D_k + k^2) Y(x^i) = 0, \quad Y_i(x^i) := -k^{-1} D_i Y(x^i),$$

$$Y_{ij}(x^i) := k^{-2} \left[ D_i D_j - \frac{1}{3} \sigma_{ij} D^k D_k \right] Y(x^i),$$

and $\delta T^{\mu\nu}$ is the effective radion energy momentum tensor,

$$\delta T_{\mu\nu} = \tau^{E}_{\mu\nu} = -\frac{1}{\kappa^2_n} E_{\mu\nu}.$$ 

Note that the usual velocity perturbation, $v = \rho q / (\rho + P)$, cannot be defined, because of $\rho + P = 0$ on the background. Also note that the traceless nature of $\tau^{E}_{\mu\nu}$ implies $\pi_L = -\delta / (n - 1)$. For the metric perturbation, we introduce the variables

$$h_{\mu\nu} dx^\mu dx^\nu = -2a^2 A Y d\eta^2 - 2a^2 B Y_j d\eta dx^j + a^2 [2H_L \delta_{ij} + 2H_T Y_{ij}] dx^i dx^j,$$

where we have chosen the spatially flat chart of the de Sitter space.

As noted above, the choice of gauge is irrelevant, as long as we choose a geometrical gauge. The perturbation equations can then be derived by simply writing down the energy momentum conservation law, $\delta T^{\mu\nu} = 0$, on the unperturbed background. Therefore, for definiteness, let us choose the Newton (or longitudinal) gauge, in which the shear of the constant time hypersurfaces vanishes (i.e., $k^{-1} H'_T - B = 0$), and denote the matter variables in this gauge as

$$\delta = -(n - 1) \pi_L = \Delta_s, \quad q = Q_s, \quad \pi_T = \Pi.$$
Adopting the scale factor $a$ as the time variable, we find

$$a \frac{d}{da} \Delta_s + n \Delta_s = -\frac{k}{aH} Q_s, \quad (C.7)$$

$$a \frac{d}{da} Q_s + nQ_s = \frac{k}{aH} \left( \frac{1}{n-1} \Delta_s + \frac{n-2}{n-1} II \right). \quad (C.8)$$

These equations can be combined to obtain a second order equation for $\Delta_s$:

$$a^2 \frac{d^2}{da^2} \Delta_s + 2(n+1)a \frac{d}{da} \Delta_s + \left( \frac{1}{n-1} \frac{k^2}{a^2 H^2} + n(n+1) \right) \Delta_s \nonumber\nonumber$$

$$= -\frac{n-2}{n-1} \frac{k^2}{a^2 H^2} II. \quad (C.9)$$

We see from the above equation that if there were no anisotropic stress $II$, then $\Delta_s \propto a^{-n}$ or $a^{-(n+1)}$ after horizon-crossing. However, in the present case, we have $II \propto k^2 a^{-2} \psi \propto ka^{-1}$. This slow decay rate of the anisotropic stress acts as a source for the energy density $\Delta_s$. With $II \propto ka^{-1}$, the right-hand side of Eq. (C.9) behaves as $k^3 a^{-5}$, which implies $\Delta_s \propto k^3 a^{-3}$ (for $n > 3$). The behavior of $Q_s$ is found to be $Q_s \propto k^2 a^{-2}$. To be a bit more precise, the leading order behaviors of $\Delta_s$, $Q_s$ and $II$ are given by

$$\Delta_s = -\frac{C}{(n-1)(n-3)} \left( \frac{k}{aH} \right)^3, \quad Q_s = \frac{C}{n-1} \left( \frac{k}{aH} \right)^2, \quad II = C \left( \frac{k}{aH} \right), \quad (C.10)$$

where $C$ is a constant. These results are, of course, consistent with those obtained in the main text, given in Eq. (4.15). Thus, it can be said that this unusual behavior of $\Delta_s$ is caused by the large (though decaying) anisotropic stress.

It is instructive to re-express the above result in the so-called comoving gauge, in which the $\eta = \text{const}$ hypersurfaces are chosen in such a way that

$$\delta T^0_j = \rho q Y_j = 0. \quad (C.11)$$

Note that this condition involves the matter variable $q$. A peculiarity of the comoving gauge is that the density perturbation in this gauge is not equal to $\Delta_s$, although the matter perturbation variables should be gauge-invariant on the pure de Sitter spacetime, as noted above. The density perturbation on the comoving hypersurface, which we denote by $\Delta$, is related to $\Delta_s$ as

$$\Delta = \Delta_s + (n-1) \frac{aH}{k} Q_s. \quad (C.12)$$

It is immediately seen from Eq. (C.10) that the $Q_s$ term dominates, and the leading order behavior of $\Delta$ is given by

$$\Delta = C \frac{k}{aH}, \quad (C.13)$$
which decays much more slowly than $\Delta_s$. The cause of this seemingly inconsistent result is the comoving gauge condition (C.11) which forces $q$ to be zero. Because $q$ is a gauge-invariant quantity, this condition can never be satisfied, and thus the comoving slice does not exist. In fact, it is straightforward to show that all the metric perturbation variables are ill-defined in the comoving gauge. Therefore, although the variable $\Delta$ is well-defined, as in Eq. (C.12), this $\Delta$ does not represent a density perturbation in any gauge.

Let us analyze the behaviors of the metric perturbation variables in the Newton gauge. Although it is not essential, for simplicity, we set $H_T = 0$, which implies $B = 0$. We denote the metric variables in the Newton gauge by

$$ A = \Psi, \quad H_L = \Phi. $$

(C.14)

Then, the $(0, \mu)$-components and the traceless part of the $(i, j)$-components of the Einstein equations give

$$ (n - 2) \frac{k^2}{a^2} \Phi = \kappa_n^2 \rho \Delta = 3H^2 \Delta, $$

$$ (n - 3) \Phi + \Psi = -\kappa_3^2 \frac{a^2}{k^2} P \Pi = 3 \left( \frac{aH}{k} \right)^2 \Pi. $$

(C.15)

Inserting the leading-order behaviors of $\Delta$ and $\Pi$ given above into these equations, we find

$$ \Psi = \Phi = \frac{3}{n - 2} \left( \frac{aH}{k} \right). $$

(C.16)

It may be a bit surprising that these variables are proportional to $a$, seemingly indicating an instability. The conclusion is that the Newton gauge is not really a good gauge. It is necessary to examine if there exists a gauge in which the metric perturbations behave regularly, and indeed there exists such a gauge. Under the shift of the time slice $\eta \to \bar{\eta} = \eta + T$, $A$ and $H_L$ transform as

$$ A \to \bar{A} = A - H \frac{d}{d\eta} (aT), \quad H_L \to \bar{H}_L = H_L - HaT. $$

(C.17)

Then, it is easy to see that the leading terms of $\Phi$ and $\Psi$ are simultaneously eliminated by setting $T = \Phi/(aH) = \text{const}$. Thus the above apparent instability is just a reflection of a bad choice of gauge.

References

19) Private communication with R. M. Wald.