Time Correlations and Diffusion of a Conservative Forced Pendulum

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The time correlations and diffusion of chaotic orbits in a periodically forced pendulum without friction are studied. The pendulum exhibits a Poincaré section with period $T$ at times $t = jT$ ($j = 0, 1, 2, \cdots$). The time-correlation function $C(t) \equiv \langle p(t)p(0) \rangle$ of the angular velocity $p(t)$ oscillates with period $T$, even as $t \to \infty$, since the average quantities of the system have a periodicity with period $T$, due to the periodic external force. Studying the approach to asymptotic oscillation, we find that the time-correlation function $C(t)$ exhibits an inverse power decay $t^{-(\beta-1)}$ ($1 < \beta < 2$), where there exist islands of accelerator-mode tori. Then, it is also shown that the power spectrum $I_p(\omega)$ of $p(t)$ obeys an inverse power law $\omega^{-(2-\beta)}$ for small frequency $\omega \ll 2\pi/T$. We also calculate the mean square displacement $\sigma^2(n)$ of the angular variable $q_n \equiv q(nT)$ on the Poincaré section, and show that $\sigma^2(n) \propto n^\zeta$ ($\zeta = 3 - \beta$) for $n \to \infty$, leading to anomalous diffusion with $1 < \zeta < 2$.

§1. Introduction

The study of chaos has revealed various interesting features of dynamical systems.1–3 Chaotic motion is exhibited by most nonlinear systems extending from simple dynamical systems with a few degrees of freedom to complex fluid systems. Conversion of the Hamiltonian differential equations into Poincaré maps is a common device for studying the motion of nonlinear dynamical systems.4 Poincaré introduced the Poincaré section method in order to characterize the geometrical structures of chaotic regions in phase space. The main advantage of the Poincaré section method is that it reduces the number of degrees of freedom of the system by one, while keeping the topological nature of the system invariant. Many investigators have studied the chaotic dynamics of two-dimensional area preserving maps, which are the most simplified models of Hamiltonian dynamical systems. However, in general, the statistical properties of the chaotic dynamics depend on the initial phase of the Poincaré section. Little attention has been given to the difference between statistical properties of dynamical differential equations and those of their Poincaré maps.

In this paper, we would like to explore the relation between the statistical properties of dynamical differential equations and those of their Poincaré maps in Hamilto-
nian dynamical systems. The time-correlation functions and the diffusion of chaotic orbits are two of the most important statistical properties. Therefore we would like to study the time correlation and diffusion of chaotic orbits in a periodically forced pendulum without friction for which the angle \( q(t) \) and angular velocity \( p(t) \) evolve according to the equation

\[
\begin{pmatrix}
\dot{q} \\
\dot{p} \\
\dot{\phi}
\end{pmatrix} =
\begin{pmatrix}
p \\
-\sin q + b \cos \phi \\
\omega_0
\end{pmatrix},
\]

(1.1)

where \( b \) is the amplitude of the driving force with angular frequency \( \omega_0 \) and the phase is \( \phi(t) = \omega_0 t + \phi_0 (0 \leq \phi < 2\pi) \). Equation (1.1) is invariant under time reversal:

\[
t \to -t, \ q(t) \to q(-t), \ p(t) \to -p(-t), \ \phi_0 \to 2\pi - \phi_0.
\]

(1.2)

The system becomes chaotic for \( b \neq 0 \), as shown in Fig. 1. This system has a Poincaré section with period \( T = 2\pi/\omega_0 \). Let us consider a chaotic orbit \( X(t) \equiv \{q(t), p(t), \phi(t)\} \) in phase space. The phase \( \phi(t) \) takes the same value \( \phi_0 \) at times \( t = jT \ (j = 0, 1, 2, \cdots) \), so that \( \phi(t) = \phi_0 \) gives a Poincaré section \( \Sigma^{\phi_0} \) which the chaotic orbit \( X(t) \) intersects at times \( t = jT \). The intersection points on the Poincaré section \( \Sigma^{\phi_0} \) are denoted by

\[
p_j \equiv p(jT), \ q_j \equiv q(jT). \quad (j = 0, 1, 2, \cdots)
\]

(1.3)

The Poincaré section \( (p_j, q_j) \) with \( \phi = \phi_0 \) is mathematically defined by

\[
\Sigma^{\phi_0} \equiv \{(p_j, q_j) \in \mathbb{R} \times \mathbb{R} | \phi = \phi_0 \in (0, 2\pi]\}.
\]

(1.4)

Figure 2 shows an example of the Poincaré section in phase space. As \( b \) is increased, the chaotic regions on the Poincaré section become large, as shown in Fig. 3.

The trajectories on the Poincaré section are divided into chaotic seas and islands of tori for the Hamiltonian dynamical systems. Anomalous diffusion of the chaotic orbits occurs due to the intermittent sticking to the hierarchical structure of islands around islands, and the time-correlation functions decay slowly. As will be shown later, the time-correlation functions of chaotic orbits oscillate, even as \( t \to \infty \), since the average quantities of this system have a periodicity due to the periodic external force. This oscillation also corresponds to the periodicity of the Poincaré section with period \( T = 2\pi/\omega_0 \) shown in Fig. 4. Equation (1.1) is invariant under the transformation

\[
t \to t, \ q(t) \to -q(t), \ p(t) \to -p(t), \ \phi_0 \to \phi_0 + \pi,
\]

(1.5)
Fig. 2. A chaotic orbit in the phase space $(q, p, \phi)$ plotted over a time of $20T$ for $b = 0.7$ and $\omega_0 = 1$. The orbit intersects the Poincaré section $\Sigma_{\phi_0}$ with $\phi_0 = 0$ at $t = jT$ (for integer $j$).

Fig. 3. Poincaré section $\Sigma_{\phi_0}$ with $\phi_0 = 0$ and $\omega_0 = 1$.

so that the Poincaré section with $\phi = \phi_0$ and that with $\phi = \phi_0 + \pi$ are inversion-symmetric to each other. Therefore, the Poincaré sections with $\phi_0 = \pi$, $5\pi/4$, $6\pi/4$, $7\pi/4$ are inversion-symmetric to those with $\phi_0 = 0$, $\pi/4$, $2\pi/4$, $3\pi/4$, respectively. For this reason, we have omitted the case $\phi_0 \geq \pi$ in Fig. 4.

In this paper, we would like to explore the time correlation and diffusion of chaotic orbits in a widespread chaotic sea for the periodically forced pendulum (1.1) without friction. In §2, we consider the time-correlation function $C(t)$ of chaotic orbits in a widespread chaotic sea of the system. The time-correlation function oscillates even as $t \to \infty$, since the average quantities in the system have a periodicity,
due to the periodic external force. In §3, we introduce the time-correlation function $C_{\phi_0}(t)$ defined with the $\phi_0$-fixed average for the system. We would like to clarify the asymptotic behavior of $C_{\phi_0}(t)$ as $t \to \infty$ and the form in which $C_{\phi_0}(t)$ decays to this asymptotic behavior. In §4, we consider the rotational diffusion of chaotic orbits in a widespread chaotic sea on the Poincaré section $\Sigma^{\phi_0}$. The last section is devoted to a summary and remarks.

§2. Time-correlation functions and power spectrum

In this section, we present the numerical results for the time-correlation function of $p(t)$,

$$C(t) \equiv \langle p(t)p(0) \rangle \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau ds \ p(t+s)p(s), $$

(2.1)

in order to analyze the statistical properties of chaotic orbits in the system (1.1). This long-time average has a definite value, which does not depend on the initial state $X(0)$. To compute Eq. (2.1), we replace the time integral (2.1) by the sum over the discrete time series $t = i\Delta t \ (i = 0, 1, 2, \cdots)$,

$$C(t) \equiv \langle p(t)p(0) \rangle = \frac{1}{N} \sum_{i=0}^{N-1} p(t+i\Delta t)p(i\Delta t), $$

(2.2)

where we use a small time interval $\Delta t = T/2^7 = 2\pi/128$ with $N \geq 10^7$.

Figure 5 displays the numerical results for the time-correlation function $C(t)$ with $b = 0.7$. These numerical results indicate that the time-correlation function...
of the chaotic orbits oscillates with period $T$ even as $t \to \infty$. The oscillation of
the time-correlation function $C(t)$ should be observable for the various periodically
forced pendulums.

The oscillation of the time-correlation function can be explained theoretically by
transforming the evolution equation (1.1) into a linear stochastic equation with the
aid of the projection-operator method.\(^5\),\(^6\) In the following, we outline this theory
briefly.\(^6\) Let us denote the state variables in phase space by
\[ X(t) = \{q(t), p(t), \phi(t)\} \]
and denote their evolution operator by
\[ \Lambda \equiv \sum_{j=1}^{3} \dot{X}_j \frac{\partial}{\partial X_j} = p \frac{\partial}{\partial q} - \left\{ \sin q - b \cos \phi \right\} \frac{\partial}{\partial p} + \omega_0 \frac{\partial}{\partial \phi}, \quad (2.3) \]
so that $\dot{f}(X) = \Lambda f(X)$ and hence $f(X(t)) = \exp[t\Lambda]f(X)$. In order to obtain a
linear evolution equation for the time-correlation functions of macrovariables, $A(t) = \{A_1(t), A_2(t)\} = \{q(t), p(t)\}$, let us consider the projection of the nonlinear term
$\sin q$ of Eq. (1.1) onto the macrovariables $A$ with the aid of the projection operator
\[ \mathcal{P} f(X) = \sum_{m=1}^{2} \sum_{n=1}^{2} \left\langle f(X) A_m^\dagger \right\rangle \left\langle AA^\dagger \right\rangle^{-1} \{ A_n \}, \quad (2.4) \]
where
\[ \left\langle q \right\rangle = \left\langle p \right\rangle = \left\langle q \sin q \right\rangle = 0. \]
Since $\dot{p}(t) = e^{tA}p$, Eq. (1.1) leads to
\[ \dot{p}(t) = -e^{tA} \sin q + b \cos(\omega_0 t + \phi_0). \quad (2.6) \]
Using $Q \equiv 1 - \mathcal{P}$, $\sin q = \mathcal{P} \sin q + Q \sin q$, we have
\[ \dot{p}(t) = -\Omega_0^2 q(t) - e^{tA}Q \sin q + b \cos(\omega_0 t + \phi_0), \quad (2.7) \]
where $\Omega_0^2 \equiv \langle q \sin q \rangle / \langle q^2 \rangle$. Therefore, using the operator identity
\[ e^{tA} = e^{tQA} + \int_0^t ds \ e^{(t-s)A} \mathcal{P} \Lambda e^{sQA}, \quad (2.8) \]
we can rewrite Eq. (2.7) as
\[
\dot{p}(t) + \Omega_0^2 q(t) + \int_0^t ds \, \gamma(s) p(t - s) = b \cos(\omega_0 t + \phi_0) + r_2(t),
\]
where we have defined the memory function \( \gamma(t) \) and the fluctuating force \( r_2(t) \) by
\[
\gamma(t) \equiv -\langle \{ A r_2(t) \} p \rangle / \langle p^2 \rangle = \langle r_2(t) q \rangle / \langle p^2 \rangle,
\]
\[
r_2(t) \equiv e^{iQ^t} Q \{ - \sin q \} = e^{iQ^t} \{ - \sin q + \Omega_0^2 q \},
\]
with \( \tilde{r}_2 \equiv Q A p = r_2(0) + Q b \cos \phi_0 \). Since \( \mathcal{P} Q = 0 \), we have
\[
\langle r_2(t) p \rangle = \langle r_2(t) q \rangle = 0, \quad \langle \tilde{r}_2 p \rangle = \langle \tilde{r}_2 q \rangle = 0.
\]

We multiply Eq. (2.9) by \( p(0) \) and take its long-time average. Then, since \( \langle r_2(t) p(0) \rangle = 0 \), we get the following evolution equation for \( Q(t) \equiv \langle q(t) p(0) \rangle \):
\[
\ddot{Q}(t) + \Omega_0^2 Q(t) + \int_0^t ds \, \gamma(s) \dot{Q}(t - s) = b \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau ds \, p(s) \cos(\omega_0 t + \omega_0 s + \phi_0).
\]
Let us rewrite the cosine of Eq. (2.13) in terms of \( \cos(\omega_0 s + \phi_0) \) and \( \sin(\omega_0 s + \phi_0) \) and define
\[
\zeta_2 \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau ds \, p(s) \cos(\omega_0 s + \phi_0) = \sqrt{\zeta_2^2 + \eta_2^2} \cos \phi_2,
\]
\[
\eta_2 \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau ds \, p(s) \sin(\omega_0 s + \phi_0) = \sqrt{\zeta_2^2 + \eta_2^2} \sin \phi_2.
\]
Then Eq. (2.13) becomes
\[
\ddot{Q}(t) + \Omega_0^2 Q(t) + \int_0^t ds \, \gamma(s) \dot{Q}(t - s) = B_2 \cos(\omega_0 t + \phi_2),
\]
where \( B_2 \equiv b \sqrt{\zeta_2^2 + \eta_2^2} \), \( \phi_2 \equiv \tan^{-1}(\eta_2/\zeta_2) \).

We may assume that the memory function \( \gamma(t) \) decays to zero in a short time \( \tau_r \), representing the loss of memory of the initial states. Then, for \( t \gg \tau_r \), we have
\[
\int_0^t ds \, \gamma(s) \dot{Q}(t - s) \to \gamma \dot{Q}(t),
\]
where \( \gamma \equiv \int_0^\infty ds \, \gamma(s) \). Then Eq. (2.16) will converge to the linear Markovian equation
\[
\ddot{Q}(t) + \gamma \dot{Q}(t) + \Omega_0^2 Q(t) = B_2 \cos(\omega_0 t + \phi_2)
\]
for \( t \gg \tau_r \). The general solution of Eq. (2.18) is given by
\[
Q(t) = A \exp[-\gamma t/2] \cos(\omega_1 t + \alpha) + \frac{B_2 \cos(\omega_0 t + \phi_2 + \delta)}{\sqrt{(\omega_0^2 - \Omega_0^2)^2 + (\gamma \omega_0)^2}}.
\]
where \( \omega_1 \equiv \sqrt{\Omega_0^2 - (\gamma/2)^2} \) with \( \Omega_0^2 > \gamma^2/4 \), \( \delta \equiv \tan^{-1} [\gamma \omega_0/(\omega_0^2 - \Omega_0^2)] \), and \( A \) and \( \alpha \) are constants. Hence, for \( t \gg 1/\gamma \), \( \tau_r, C(t) = \dot{Q}(t) \) becomes a periodic oscillation

\[
\xi(t) = -\frac{B_2 \omega_0 \sin(\omega_0 t + \phi_2 + \delta)}{\sqrt{\omega_0^2 - \Omega_0^2} + (\gamma \omega_0)^2} \tag{2.20}
\]
as the forced oscillation driven by the periodic external force \( B_2 \cos(\omega_0 t + \phi_2) \). Therefore the oscillation in Fig. 5 can be explained by Eq. (2.20).

The power spectrum of \( p(t) \) is given by

\[
I_p(\omega) \equiv \lim_{\tau \to \infty} \frac{\tau}{2\pi} \left\langle \left| \frac{1}{\tau} \int_0^\tau dt \, p(t) e^{-i\omega t} \right|^2 \right\rangle, \tag{2.21}
\]

\[
= \lim_{\tau \to \infty} \frac{1}{\pi} \int_0^\tau dt \left( 1 - \frac{t}{\tau} \right) C(t) \cos(\omega t). \tag{2.22}
\]
The time-correlation function \( C(t) \) becomes the forced oscillation function \( \xi(t) \) given by (2.20) for \( t \gg \tau_r \), and, as will be shown later, exhibits the inverse power decay

\[
\hat{C}(t) \equiv C(t) - \xi(t) \propto t^{-(\beta - 1)} \tag{2.23}
\]
of this oscillation \( \xi(t) \), where \( 1 < \beta < 2 \). Therefore, the power spectrum (2.22) has a line spectrum at \( \omega = \omega_0 \) and becomes an inverse power-law spectrum

\[
I_p(\omega) \propto \omega^{-(2 - \beta)} \tag{2.24}
\]
for \( \omega \ll \omega_0 \). Indeed, the numerical experiment for the power spectrum (2.21) whose results are plotted in Fig. 6 verifies the inverse power-law spectrum (2.24), indicating that \( \beta = 1.48 \). In computing Eq. (2.21), we replaced the time integral in (2.21) by the sum over the discrete time series \( t = m\Delta t \) (\( m = 0, 1, 2, \cdots \)) and set \( \tau = N' \Delta t \):

\[
I_p(\omega) = \frac{\tau}{2\pi} \frac{1}{N} \sum_{l=0}^{N-1} \left| \frac{1}{N'} \sum_{m=0}^{N'-1} p(l\Delta t + m\Delta t) e^{-i\omega m\Delta t} \right|^2, \tag{2.25}
\]
where \( \Delta t = T/2^7 \) and \( N' = 2^{15} \).

The inverse power decay (2.23) of \( \hat{C}(t) \) indicates that the exponential decay of the first term of Eq. (2.19) is not valid for the present system. Indeed, as will be shown in §4, there exist accelerator-mode islands in the present system, which produce the inverse power decay (2.23).

The transformation of Eq. (1.1) into Eq. (2.18) indicates that the evolution equation, which is reversible under time reversal, is replaced by an irreversible stochastic equation on long time scales. We get \( \Omega_0^2 = 0.32 \) from numerical results, with \( N = 10^7 \) for \( b = 0.7 \) and \( \omega_0 = 1 \). However, it is difficult to get numerical values of \( B_2, \phi_2 \) and \( \delta \) by an analytic method. In the next section, we consider how to treat the oscillation (2.20) by a numerical method.
§3. Time-correlation functions defined with the $\phi_0$-fixed average

We consider the time-correlation function

$$C_{\phi_0}(t) \equiv \langle p(t)p(0) \rangle_{\phi_0} \equiv \lim_{M \to \infty} \frac{1}{M} \sum_{j=0}^{M-1} p(t+jT)p(jT), \quad \left( T = \frac{2\pi}{\omega_0} \right)$$

(3.1)

with a long-time average over the Poincaré section $\Sigma^{\phi_0}$, in order to understand the oscillation of the time-correlation function $C(t)$. This average becomes a periodic function of the initial phase $\phi_0$ with period $2\pi$ and is called the $\phi_0$-fixed average. The average of this $C_{\phi_0}(t)$ over the initial phase $\phi_0$ gives the time-correlation function (2.1).

It should be noted that the long-time average (3.1) has been employed also in studies of chaos using the Poincaré maps and their models.\(^3\)

Figure 7 shows numerical results for the following $\phi_0$-fixed average of $p(t)$:

$$\langle p(t) \rangle_{\phi_0} \equiv \lim_{M \to \infty} \frac{1}{M} \sum_{j=0}^{M-1} p(t+jT).$$

(3.2)

From this, $\langle p(t) \rangle_{\phi_0}$ turns out to have the form

$$\langle p(t) \rangle_{\phi_0} = B_{\phi_0} \sin(\omega_0 t + \phi_0 + \delta_1).$$

(3.3)

Figure 8 shows numerical results for the time-correlation function with the $\phi_0$-fixed average $C_{\phi_0}(t)$. The amplitudes and phases of the oscillation of $C_{\phi_0}(t)$ depend on $\phi_0$.

In order to derive a linear evolution equation for $C_{\phi_0}(t)$, instead of the projection
operator (2.5), let us introduce the projection operator

\[
\mathcal{P}'f(X) = \sum_{m=1}^{2} \sum_{n=1}^{2} \left( f(X)A_m^\dagger \right)_{\phi_0} \left[ \left( AA^\dagger \right)^{-1} \right]_{mn} A_n,
\]

(3.4)

\[
= \frac{1}{D} \left\{ \langle f(X)q \rangle_{\phi_0} \langle p^2 \rangle_{\phi_0} - \langle f(X)p \rangle_{\phi_0} \langle qp \rangle_{\phi_0} \right\} q
+ \frac{1}{D} \left\{ - \langle f(X)q \rangle_{\phi_0} \langle pq \rangle_{\phi_0} + \langle f(X)p \rangle_{\phi_0} \langle q^2 \rangle_{\phi_0} \right\} p,
\]

(3.5)

where \( D \equiv \langle q^2 \rangle_{\phi_0} \langle p^2 \rangle_{\phi_0} - \left( \langle qp \rangle_{\phi_0} \right)^2 \). Using this projection, we obtain, instead of (2.7),

\[
\dot{p}(t) = -\Omega^2_{0\phi_0} q(t) - \sigma p(t) - e^{tA} Q' \sin q + b \cos(\omega_0 t + \phi_0),
\]

(3.6)

where \( Q' \equiv 1 - \mathcal{P}' \) and

\[
\Omega^2_{0\phi_0} \equiv \left\{ \langle \sin q \rangle_{\phi_0} \langle p^2 \rangle_{\phi_0} - \langle \sin p \rangle_{\phi_0} \langle qp \rangle_{\phi_0} \right\}/D,
\]

(3.7a)

\[
\sigma \equiv \left\{ - \langle \sin q \rangle_{\phi_0} \langle pq \rangle_{\phi_0} + \langle \sin p \rangle_{\phi_0} \langle q^2 \rangle_{\phi_0} \right\}/D.
\]

(3.7b)

The frequency \( \Omega^2_{0\phi_0} \) is a periodic function of \( \phi_0 \) with period \( \pi \) shown in Fig. 9. Therefore, using the operator identity (2.8), we can rewrite Eq. (3.6) as

\[
\dot{p}(t) + \Omega^2_{0\phi_0} q(t) + \sigma p(t) + \int_0^t ds \left\{ \gamma_q(s) q(t-s) + \gamma_p(s) p(t-s) \right\} = b \cos(\omega_0 t + \phi_0) + r'_2(t),
\]

(3.8)

where \( r'_2(t) \equiv e^{tQ'\lambda} Q' \{ - \sin q \} \) and

\[
\gamma_p(t) \equiv \langle r'_2(t) r'_2 \rangle_{\phi_0} \langle q^2 \rangle_{\phi_0} /D,
\]

(3.9a)

\[
\gamma_q(t) \equiv - \langle r'_2(t) r'_2 \rangle_{\phi_0} \langle qp \rangle_{\phi_0} /D,
\]

(3.9b)
Fig. 9. $\phi_0$-dependence of $\langle p^2 \rangle_{\phi_0}$, $\langle qp \rangle_{\phi_0}$, $\langle q^2 \rangle_{\phi_0}$, and $\Omega^2_{\phi_0}$ with $M = 10^7$ for $b = 0.7$ and $\omega_0 = 1$.

with $\bar{r}_2' \equiv Q' Ap = r_2'(0) + Q' b \cos \phi_0$. Since $P' Q' = 0$, we have

$$\langle r_2'(t)p \rangle_{\phi_0} = \langle r_2'(t)q \rangle_{\phi_0} = 0. \quad (3.10)$$

If $\langle qp \rangle_{\phi_0} = 0$, then $\Omega^2_{0\phi_0}$ and $\gamma_p(t)$ agree with $\Omega^2_{0\phi_0}$ and $\gamma(t)$ of Eq. (2.9).

Now, let us assume that the memory functions $\gamma_q(t)$ and $\gamma_p(t)$ decay to zero in a short time $\tau_r$, representing the loss of memory of the initial states. Then, the memory terms become $-\gamma_q q(t)$ and $-\gamma_p p(t)$, so that we have the Markovian stochastic equation

$$\dot{p}(t) + \gamma_1 p(t) + \Omega^2 I q(t) = b \cos(\omega_0 t + \phi_0) + r_2(t) \quad (3.11)$$

on the long-time scale $t \gg \tau_r$, where $\langle r_2(t)p(0) \rangle_{\phi_0} = 0$ and we have put

$$\gamma_1 \equiv \gamma_p + \sigma, \quad \Omega^2_I \equiv \Omega^2_{0\phi_0} + \gamma_q, \quad \gamma_p \equiv \int_0^\infty ds \gamma_p(s), \quad \gamma_q \equiv \int_0^\infty ds \gamma_q(s). \quad (3.12)$$

In the following, we assume that $\gamma_1 > 0$ and $\Omega^2_I > 0$.

Let us introduce

$$Q_{\phi_0}(t) \equiv \langle q(t)p(0) \rangle_{\phi_0}, \quad (3.13)$$

in analogy to Eq. (3.1). Then, since $\langle p(0) \cos(\omega_0 t + \phi_0) \rangle_{\phi_0}$ becomes

$$\lim_{M \to \infty} \frac{1}{M} \sum_{j=0}^{M-1} p(jT) \cos(\omega_0 \{t + jT\} + \phi_0) = \langle p(0) \rangle_{\phi_0} \cos(\omega_0 t + \phi_0), \quad (3.14)$$
Eq. (3-11) leads to
\[ \ddot{Q}_{\phi_0}(t) + \gamma_1 \dot{Q}_{\phi_0}(t) + \Omega_1^2 Q_{\phi_0}(t) = \langle p(0) \rangle_{\phi_0} b \cos(\omega_0 t + \phi_0) \] (3-15)
on the long-time scale \( t \gg \tau_r \). In many cases, we may have \( \Omega_1^2 > \gamma_1^2 / 4 \). Then Eq. (3-15) can be integrated similarly to (2-19), leading to the forced oscillation
\[ C_{\phi_0}(t) = - \frac{\langle p(0) \rangle_{\phi_0} b \omega_0 \sin(\omega_0 t + \phi_0 + \delta_1)}{\sqrt{\omega_0^2 - \Omega_1^2})^2 + (\gamma_1 \omega_0)^2} \] (3-16)
for \( t \gg 2/\gamma_1, \tau_r \), where \( \delta_1 \equiv \tan^{-1}[\gamma_1 \omega_0 / (\omega_0^2 - \Omega_1^2)] \). Thus, \( C_{\phi_0}(t) \) turns out to be forced oscillation driven by the periodic external force \( b \cos(\omega_0 t + \phi_0) \). Equation (3-16) verifies the oscillation of \( C_{\phi_0}(t) \) shown in Fig. 8. Figure 10 shows the numerical results for \( \langle p(0) \rangle_{\phi_0} \).

Let us assume for Eq. (3-11) that \( \langle r(2) \rangle_{\phi_0} = 0 \), in accordance with the phenomenological theory of stochastic equations. Then we have
\[ \langle \dot{p}(t) \rangle_{\phi_0} + \gamma_1 \langle p(t) \rangle_{\phi_0} + \Omega_1^2 \langle q(t) \rangle_{\phi_0} = b \cos(\omega_0 t + \phi_0) \] (3-17)
for \( t \gg \tau_r \). This is integrated similarly to (3-16), leading to the forced oscillation
\[ \langle p(t) \rangle_{\phi_0} = - \frac{b \omega_0 \sin(\omega_0 t + \phi_0 + \delta_1)}{\sqrt{\omega_0^2 - \Omega_1^2})^2 + (\gamma_1 \omega_0)^2} \] (3-18)
for \( t \gg 2/\gamma_1, \tau_r \), which is in agreement with the numerical result (3-3). This oscillation differs from Eq. (3-16) only by the factor \( \langle p(0) \rangle_{\phi_0} \). Therefore we have
\[ \langle p(t)p(0) \rangle_{\phi_0} = \langle p(t) \rangle_{\phi_0} \langle p(0) \rangle_{\phi_0} \] (3-19)
for \( t \to \infty \), so that the mixing condition does hold for this time-correlation function.

We examined the mixing property of chaotic orbits in a widespread chaotic sea by numerical simulations. If the chaotic orbits in a widespread chaotic sea possess the mixing property, then \( C_{\phi_0}(t) \) tends toward
\[ \frac{\langle p(t)p(0) \rangle_{\phi_0}}{\langle p(0) \rangle_{\phi_0}} \to \langle p(t) \rangle_{\phi_0} \] (3-20)
for \( t \gg \tau_r \).

Figure 11 shows numerical results for the quantity \( C_{\phi_0}(t)/\langle p(0) \rangle_{\phi_0} \). Here we have omitted the case of \( \phi_0 = 0 \), because \( \langle p(0) \rangle_{\phi_0} = 0 \) for \( \phi_0 = 0, \pi \), as shown in Fig. 10. The oscillation of \( C_{\phi_0}(t)/\langle p(0) \rangle_{\phi_0} \) has an identical form for different \( \phi_0 \) for \( t \gg \tau_r \). However, the phases of the oscillation of \( C_{\phi_0}(t) \) are different.
Fig. 11. Time-correlation function $C_{\phi_0}(t) / \langle p(0) \rangle_{\phi_0}$ vs $t/T$ with $M = 10^7$ for $b = 0.7$ and $\omega_0 = 1$.

Fig. 12. Time-correlation function $\langle p(t - \phi_0/\omega_0)p(0) \rangle_{\phi_0} / \langle p(0) \rangle_{\phi_0}$ vs $t/T$ with $M = 10^7$ for $b = 0.7$ and $\omega_0 = 1$.

for different $\phi_0$. In order to make the oscillation of $C_{\phi_0}(t)$ identical, we calculate $\langle p(t - \phi_0/\omega_0)p(0) \rangle_{\phi_0} / \langle p(0) \rangle_{\phi_0}$. Then for $t \gg \tau_r$, we have

$$\frac{\langle p(t - \phi_0/\omega_0)p(0) \rangle_{\phi_0}}{\langle p(0) \rangle_{\phi_0}} = \frac{C_{\phi_0}(t - \phi_0/\omega_0)}{\langle p(0) \rangle_{\phi_0}} \rightarrow \langle p(t - \phi_0/\omega_0) \rangle_{\phi_0}. \quad (3.21)$$

Figure 12 shows numerical results for $C_{\phi_0}(t - \phi_0/\omega_0) / \langle p(0) \rangle_{\phi_0}$. The amplitudes and phases of the oscillation of $C_{\phi_0}(t - \phi_0/\omega_0) / \langle p(0) \rangle_{\phi_0}$ have an identical form for different $\phi_0$ for $t \gg \tau_r$. The asymptotic forms of $C_{\phi_0}(t)$ given in (3.20) and (3.21) were first clarified numerically by Tominaga et al., for the Duffing equation.

These results give convincing evidence of the mixing of chaotic orbits in the widespread chaotic sea of this system. From these numerical results, it turns out
that the time-correlation function

\[
\hat{C}_{\phi_0}(t) \equiv \left\langle \left( p(t) - \langle p(t) \rangle_{\phi_0} \right) \left( p(0) - \langle p(0) \rangle_{\phi_0} \right) \right\rangle_{\phi_0} = \langle p(t)p(0) \rangle_{\phi_0} - \langle p(t) \rangle_{\phi_0} \langle p(0) \rangle_{\phi_0}
\]  

(3.22)

covers to 0 for \( t \gg 1/\gamma_1, \tau_r \), so that the mixing condition (3.19) is numerically verified. It also turns out that the time-correlation function \( \hat{C}_{\phi_0}(t) \) exhibits inverse power decay \( t^{-(\beta-1)} \) for \( t > 30T \) with \( \beta = 1.47 \), which is independent of the initial phase \( \phi_0 \), as shown in Fig. 13.

From these numerical results, we find that the time-correlation function

\[
\hat{C}(t) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\phi_0 \hat{C}_{\phi_0}(t) = \frac{1}{2\pi} \int_0^{2\pi} d\phi_0 \langle p(t)p(0) \rangle_{\phi_0} - \frac{1}{2\pi} \int_0^{2\pi} d\phi_0 \langle p(t) \rangle_{\phi_0} \langle p(0) \rangle_{\phi_0}
\]
converges to 0 for \( t \gg 1/\gamma_1, \tau_r \). The time-correlation function \( \hat{C}(t) \) exhibits the inverse power decay \( t^{-(\beta-1)} \) with \( \beta = 1.46 \), as shown in Fig. 14. This time-correlation function agrees with the time-correlation function (2.23). It should be noted that the value of \( \beta \) obtained from the numerical results for \( \hat{C}(t) \) in Fig. 14, 1.46, is about the same as that obtained from the numerical results for \( \hat{C}_\phi(0) \) with \( b = 0.7 \) in Fig. 13, 1.47. Furthermore, these results for \( \beta \) are consistent with the result \( \beta = 1.48 \) obtained for the power spectrum (2.24) shown in Fig. 6.

\section*{4. Anomalous diffusion of chaotic orbits}

In this section, we analyze the rotational diffusion of chaotic orbits in terms of the long-time correlation (2.23). The amount by which angle \( q(t) \) changes during a time \( \tau \) is given by

\[
S(\tau) \equiv q(\tau) - q(0) = \int_0^\tau dt \dot{q}(t) = \int_0^\tau dt \ p(t).
\] (4.1)

The mean square displacement of the angle \( q(t) \) can be written as

\[
\langle S^2(\tau) \rangle = \langle \left\{ \int_0^\tau dt \ p(t) \right\}^2 \rangle = 2 \int_0^\tau dt \ (\tau - t) C(t)
\] (4.2)

in terms of the time-correlation function (2.1).

Equation (2.23) leads to \( C(t) = \xi(t) + \hat{C}(t) \) with \( \hat{C}(t) \propto t^{-(\beta-1)} \) for \( t \geq 30T \). Inserting this into (4.2) leads to

\[
\left\langle \{S(\tau) - \langle S(\tau) \rangle \}^2 \right\rangle \propto \tau^{3-\beta}
\] (4.3)

for large \( \tau \), where \( 2 > \beta > 1 \). Therefore, the inverse power decay of \( C(t) \) produces anomalous diffusion with \( \zeta = 3 - \beta > 1 \).

Next, let us consider the rotational diffusion of chaotic orbits on the Poincaré section \( \Sigma_\phi \) shown in Fig. 4. In terms of \( u_j \equiv q_{j+1} - q_j \ (q_j \equiv q(jT)) \), with integer \( j \), the amount by which \( q_j \) changes during \( n \) iterations is given by

\[
S_n \equiv q_n - q_0 = \sum_{j=0}^{n-1} u_j.
\] (4.4)

Equation (4.4) leads to the variance

\[
\sigma^2(n) \equiv \langle (S_n - \langle S_n \rangle_\phi_0)^2 \rangle_\phi_0 = n\hat{C}_0 + 2 \sum_{j=1}^{n-1} (n - j)\hat{C}_j,
\] (4.5)

\[
\hat{C}_n \equiv \langle (u_n - \langle u_n \rangle_\phi_0)(u_0 - \langle u_0 \rangle_\phi_0) \rangle_\phi_0,
\] (4.6)
where \( \langle \cdots \rangle_{\phi_0} \) denotes the long-time average
\[
\langle u_n u_0 \rangle_{\phi_0} \equiv \lim_{M \to \infty} \frac{1}{M} \sum_{j=0}^{M-1} u_{n+j} u_j.
\]
(4.7)

Here \( n \ll M \) must be satisfied if \( M \) is finite. The average of \( S_n \) is
\[
\langle S_n \rangle_{\phi_0} = \bar{u} n,
\]
(4.8)
where \( \bar{u} \equiv \langle u_j \rangle_{\phi_0} \) gives the average drift velocity. In general, we have
\[
\sigma^2(n) \propto n^{\zeta},
\]
(4.9)
for large \( n \). This is referred to as ‘anomalous diffusion’ when the variance \( \sigma^2(n) \) does not depend linearly on \( n \), with \( 1 < \zeta < 2 \).

If there are no accelerator-mode islands, then a diffusion constant \( D \) would exist, leading to the variance
\[
\sigma^2(n) = 2Dn
\]
(4.10)
for \( n \gg n_c \), \( n_c \) being the correlation time of the time series \( \{u_j\} \).

Islands of tori are created around a stable periodic orbit \( \{p_j^* (= p^*(jT)), q_j^*(= q^*(jT))\} \) \( (j=1,2,\cdots,Q) \) of period \( Q \), which satisfies
\[
p_{j+Q}^* = p_j^*, \quad q_{j+Q}^* - q_j^* = \pm 2\pi l,
\]
(4.11)
where \( l \) is an integer. Then \( q_j \) shifts by \( \pm 2\pi l/Q \) every period \( Q \) with mean speed \( v_s \equiv \pm 2\pi l/Q \). If \( l \neq 0 \), then this is called the accelerator-mode periodic orbit of period \( Q \) and step \( l \) \( (8,9) \). Figure 15 shows accelerator-mode islands around an accelerator-mode periodic orbit with \( (b) \) \( Q=1, \ l=2, \ v_s = 4\pi \), \( (d) \) \( Q=3, \ l=7, \ v_s = 14\pi/3 \) and \( Q=5, \ l=12, \ v_s = 24\pi/5 \) for \( b = 0.7 \) and \( \omega_0 = 1 \). There is a symmetry line \( p(t)=0 \) on the Poincaré section \( \Sigma^{\phi_0} \) with \( \phi_0 = 0 \). From Eq (1.2), it is seen that there are accelerator-mode islands around an accelerator-mode periodic orbit with \( v_s = -4\pi \) \( (Q=1, \ l=2) \), \( v_s = -24\pi/5 \) \( (Q=5, \ l=12) \) and \( v_s = -14\pi/3 \) \( (Q=3, \ l=7) \) for \( b = 0.7 \) and \( \omega_0 = 1 \). It should be noted that there always exist such accelerator-mode islands in the boundary areas between the chaotic sea and KAM tori extending over \( q = -\infty - \infty \), as shown in Fig. 15(d) for \( b \neq 0 \).

We now give the results of numerical experiments on the variance \( \sigma^2(n) \). The parameter values used are \( b = 0.7, \ \omega_0 = 1, \ \phi_0 = 0 \). Figure 16 displays the results of numerical experiments on the variance \( \sigma^2(n) \propto n^{\zeta} \) for \( b = 0.7 \). These results show anomalous diffusion, with \( \zeta = 1.51 \).

The variance can be analyzed in the following manner. A chaotic orbit in the chaotic sea sticks to the accelerator-mode islands repeatedly, with an inverse-power distribution function \( f(\tau) \) of sticking times \( \tau \): \( f(\tau) \propto \tau^{-1-\beta}, \ (1 < \beta < 2) \) for \( \tau \gg 1 \), where there exists a finite mean sticking time \( \bar{\tau} \). The probability for the chaotic orbit to stick longer than \( n \) is given by \( (9,10) \)
\[
W(n) \equiv \sum_{\tau=n}^{\infty} \tau f(\tau) \propto n^{-(\beta-1)}, \quad (1 < \beta < 2)
\]
(4.12)
Fig. 15. (a) Poincaré section \( \Sigma^{\phi_0} \) with \( \phi_0 = 0 \) for \( b = 0.7 \) and \( \omega_0 = 1 \). (b) Accelerator-mode islands around an accelerator-mode periodic orbit with \( Q = 1, l = 2, v_s = 4\pi \). (c) Normal islands around a fixed point with \( Q = 1, l = 0, v_s = 0 \). (d) Visible islands inside the box \( d1 \) are accelerator-mode islands around an accelerator-mode periodic orbit with \( Q = 5, l = 12, v_s = 24\pi/5 \). Visible islands inside the box \( d2 \) are accelerator-mode islands around an accelerator-mode periodic orbit with \( Q = 3, l = 7, v_s = 14\pi/3 \).

Figures 15 and 16 show the sticking behavior between accelerator-mode islands and normal islands. For \( n \gg 1 \) during the sticking to the accelerator-mode islands, we have \( q_n - q_0 \approx n v_s \). Then, using the sticking probability, we have

\[
\sigma^2(n) \simeq (nv_s)^2 W(n) \propto n^{3-\beta}
\]

(4.13)

for \( n \gg \bar{n}_c \), where \( \bar{n}_c \) is the mean duration of those segments of a chaotic orbit that lie inside the chaotic sea.\(^{10} \) This is compared with (4.9) to give \( \zeta = 3 - \beta \), so that Fig. 16 leads to \( \beta = 1.49 \).

Islands of tori form a self-similar hierarchical structure of islands around accelerator-mode islands. This hierarchical structure must be independent of the initial phase \( \phi_0 \), so that the value of \( \beta \) is independent of the initial phase \( \phi_0 \).

Since the long-time correlation function (4.6) has a long-time correlation of the form \( \tilde{C}_n \simeq (v_s - \bar{w})^2 W(n) \propto n^{-(\beta-1)} \), with \( 1 < \beta < 2 \) for \( n \gg \bar{n}_c \), the second term of Eq. (4.5) leads to \( \sigma^2(n) \propto n^\zeta \), with \( \zeta = 3 - \beta > 1 \), so that Eq. (4.10) breaks down.

Figure 17 displays the numerical results for the time-correlation function \( \tilde{C}_n \) at \( b = 0.7 \). This leads to \( \beta = 1.48 \), which is about the same as \( \beta = 1.49 \), obtained from the numerical results, \( \zeta = 1.51 \), given in Fig. 16. It should be noted that these values of \( \beta \) are also consistent with the value \( \beta = 1.48 \) of the power spectrum (2.24) obtained from Fig. 6.
§5. Summary

In the present paper, we have explored the time correlation and diffusion of chaotic orbits in a widespread chaotic sea for the periodically forced pendulum (1.1) without friction.

The time-correlation function $C(t)$ defined by (2.1) oscillates even as $t \to \infty$, because the average quantities in the system have a periodicity due to the external periodic force, as shown in Fig. 5. The decay of $C(t)$ to this oscillation obeys an inverse power law (2.23), so that the power spectrum obeys an inverse power law (2.24), as shown in Fig. 6.

In order to treat the periodicity, we introduced the time-correlation function $C_{\phi_0}(t)$ by (3.1). As a result, we obtained convincing evidence of the mixing (3.19) of chaotic orbits in the widespread chaotic sea of the system, as shown in Figs. 11 and 12. The time-correlation function $\hat{C}_{\phi_0}(t)$ defined by (3.22), where the periodicity is removed from the time-correlation function (3.1), exhibits the inverse power decay form $t^{-(\beta-1)}$, as shown in Fig. 13.

This long-time correlation of $\hat{C}_{\phi_0}(t)$ leads to the result that the rotational diffusion of chaotic orbits becomes anomalous in the widespread chaotic sea on the Poincaré section $\Sigma_{\phi_0}$, as shown in Figs. 16 and 17. This anomaly is brought about by the intermittent sticking of chaotic orbits to the accelerator-mode islands, as shown in Fig. 15. The value $\beta = 1.49$ for the numerical results in Fig. 16 is consistent with the value $\beta = 1.47$ for the numerical results in Fig. 13 for the time-correlation function $\hat{C}_{\phi_0}(t)$ and the value $\beta = 1.48$ for the power spectrum (2.24) in Fig. 6.

Islands of tori form a self-similar hierarchical structure of islands around accelerator-mode islands. This hierarchical structure must be independent of the initial phase $\phi_0$, so that the value of $\beta$ is independent of the initial phase $\phi_0$.

These anomalous statistical properties are expected to be possessed by other types of periodically forced pendulums without friction.

Finally it should be noted that, since the decay of the time-correlation function $C(t)$ to the periodic forced oscillation $\xi(t)$ obeys the inverse power law $t^{-(\beta-1)}$,
we have to explore the forced oscillation $\xi(t)$ and the decay of $C(t)$ to $\xi(t)$ from the non-Markovian equations (2.16) and (3.8), instead of the Markovian equations (2.18) and (3.15). This may be carried out by taking the Laplace transform of the non-Markovian equations and introducing the continued-fraction expansion of the memory functions, as will be reported in a subsequent paper.

References