Non-Birkhoff Orbits with $2n$ Turning Points in the Standard Map

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One of the two fixed points of the standard map gives rise to a period-doubling bifurcation and becomes a saddle with reflection beyond a certain parameter value. In association with this bifurcation, symmetric non-Birkhoff periodic orbits (SNBOs) with $2n$ ($n \geq 2$) turning points appear and exhibit complicated behavior. We first analyze the structure of stable and unstable manifolds of this saddle and then derive dynamical order relations for these SNBOs and show that a period-3 SNBO implies the existence of SNBOs with all possible numbers of turning points.

§1. Introduction

Non-Birkhoff-type periodic orbits (NBOs) play an important role when we study the evolution of chaos in a one-parameter family of twist maps.$^{1-3}$ We have studied these periodic points in the standard map and in standard-like maps.$^{4-7}$ The NBOs have been used to estimate a lower bound on the topological entropy in one-dimensional circle maps$^8$ and in forced oscillator systems.$^9$

The standard map $T$ we consider is defined on a cylinder as

$$
y_{n+1} = y_n + a f(x_n),
$$

$$
x_{n+1} = x_n + y_{n+1} \pmod{2\pi},
$$

where $a$ is a positive parameter and $f(x) = \sin x$. There are two fixed points, $P = (0,0)$ and $Q = (\pi,0)$, where $P$ is a saddle and $Q$ is an elliptic point ($0 < a < 4$) or a saddle with reflection ($a > 4$).

We work on the universal cover $\mathbb{R}^2$ of the cylinder. We mainly study NBOs for which points of a period is almost confined to the fundamental domain, $0 \leq x < 2\pi$; i.e., the original cylinder region. We use the same notation for the lift map and lifted orbits as for the cylinder. For convenience, we call a point $(2\pi,0)$ a saddle $P'$.

It is well known$^{10}$ that the standard map $T$ can be expressed as a product of two involutions and, in addition, that there are two independent forms of the product.$^{11}$ Let $\{G_1, H_1\}$ and $\{G_2, H_2\}$ be two sets of involutions; i.e., $G_i^2 = H_i^2 = \text{Id}$ and $\det \nabla G_i = \det \nabla H_i = -1$ for $i = 1, 2$. With these involutions,

$$
T = H_1 \circ G_1 = H_2 \circ G_2,
$$

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where

\[ G_1 \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} -x \\ y + af(x) \end{array} \right) \quad \text{and} \quad H_1 \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} -x + y \\ y \end{array} \right), \] (4)

and

\[ G_2 \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x \\ -y - af(x) \end{array} \right) \quad \text{and} \quad H_2 \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x - y \\ -y \end{array} \right). \] (5)

The sets of fixed points of \( G_1 \) and \( H_i \) are the symmetry axes. In the universal cover, there exist infinitely many symmetry axes for \( G_1 \) and \( H_1 \). We use two axes, \( S_1 \) and \( S_2 \), of \( G_1 \), and two axes, \( S_3 \) and \( S_4 \), of \( H_1 \) in the domain close to the origin:

\[ S_1 : x = 0, \quad \text{and} \quad S_2 : x = \pi \quad \text{for} \quad G_1, \] (6)

\[ S_3 : y = 2x, \quad \text{and} \quad S_4 : y = 2(x - \pi) \quad \text{for} \quad H_1. \] (7)

To specify the branch of a symmetry axis for \( y > 0 \) (resp., \( y < 0 \)), we attach the superscript + (resp., −) to the expression of the axis. The symmetry axes of \( G_2 \) and \( H_2 \) are

\[ S_G : y = -\frac{a}{2} \sin x \quad \text{for} \quad G_2, \] (8)

\[ S_H : y = 0 \quad \text{for} \quad H_2. \] (9)

The orbit of a point \( z \in \mathbb{R}^2 \) is denoted \( o(T, z) = \{ \cdots , T^{-1}z, z, Tz, \cdots \} \). Following Hall,\(^1\) we define the extended orbit of a point \( z \in \mathbb{R}^2 \) by \( eo(T, z) = \{ T^kz + (2\pi l, 0) : k, l \in \mathbb{Z} \} \). We usually abbreviate \( o(T, z) \) and \( eo(T, z) \) as \( o(z) \) and \( eo(z) \). Let \( \pi_1(z) \) [resp., \( \pi_2(z) \)] be the projection onto the \( x \)-coordinate (resp., \( y \)-coordinate) of \( z \). A point \( z \in \mathbb{R}^2 \) is called a \( p/q \)-periodic point for the lifted standard map \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) if \( T^qz - (2\pi p, 0) = z \). A \( p/q \)-periodic point \( z \in \mathbb{R}^2 \) is called Birkhoff (see Hall\(^1\)) if \( \pi_1(r) < \pi_1(s) \Rightarrow \pi_1(Tr) < \pi_1(Ts) \) for any \( r, s \in eo(z) \). Otherwise, the point is said to be non-Birkhoff. A Birkhoff (resp., non-Birkhoff) periodic point is abbreviated BP (resp., NBP). Corresponding orbits are denoted BO and NBO. A periodic orbit is said to be symmetric if it has points on the symmetry axes.

Let \( z \) be a point with a positive rotation number. A point \( r \in eo(z) \) satisfying \( \pi_1(T^{-1}r) \leq \pi_1(r) \) and \( \pi_1(r) > \pi_1(Tr) \) is called a turning-back point, whereas a point \( q \in eo(z) \) satisfying \( \pi_1(T^{-1}q) \geq \pi_1(q) \) and \( \pi_1(q) < \pi_1(Tq) \) is called a turning-forward point. Both \( r \) and \( q \) are called turning points.

**Proposition 1.** A point \( p \) is a turning-back point if and only if \( \pi_2(p) \geq 0 \) and \( \pi_2(Tp) < 0 \). A point \( p \) is a turning-forward point if and only if \( \pi_2(p) \leq 0 \) and \( \pi_2(Tp) > 0 \).

**Proof.** We only prove the first statement. The second statement is proved in a similar manner. (Sufficiency) By assumption, we have \( \pi_1(T^{-1}p) \leq \pi_1(p) \) and \( \pi_1(p) > \pi_1(Tp) \). In view of Eq. (2), we have \( \pi_2(p) \geq 0 \) from the first inequality, and \( \pi_2(Tp) < 0 \) from the second inequality. (Necessity) In view of Eq. (2), \( \pi_2(Tp) < 0 \) implies \( \pi_1(p) > \pi_1(Tp) \), and \( \pi_2(p) \geq 0 \) implies \( \pi_1(T^{-1}p) \leq \pi_1(p) \). (Q.E.D.)
In this paper, we restrict our attention to symmetric non-Birkhoff periodic orbits (SNBOs) starting on $S^+_1$ with $2n$ ($n \geq 1$) turning points and rotation number $1/q$ ($q \geq 3$). For SNBOs with two turning points, we already proved Theorem 1 for the standard map and standard-like maps, which determines the dynamical ordering for SNBOs. The periodic orbits with $2n$ ($n \geq 1$) turning points are the objects standing between the Birkhoff periodic orbits which monotonously move along the $x$-axis and the periodic orbits bifurcated from the fixed point $Q$ which forever revolve around $Q$. The periodic orbits with $2n$ ($n \geq 1$) turning points are direct evidence for the existence of chaos, because they represent folding and stretching motion in the phase space. The authors believe that the study of these periodic points contribute the understanding chaos.

**Theorem 1** [Refs. 6) and 7)]. The following dynamical order relations hold for symmetric non-Birkhoff periodic orbits with two turning points starting on the symmetry axis $S^+_1$.

\[
\begin{align*}
I_1: & \quad (1/3) \to (1/4) \to (1/5) \to (1/6) \to \cdots \\
I_2: & \quad (1/5) \to (1/6) \to (1/7) \to (1/8) \to \cdots \\
I_3: & \quad (1/7) \to (1/8) \to (1/9) \to (1/10) \to \cdots \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{align*}
\]

First, we give a brief explanation of the notation used in Theorem 1. Here $I_i$ is an interval on the symmetry axis $S^+_1$ [see Eq. (6)], and $(1/q)_2$ represents a $1/q$-SNBO with two turning points. The expression $(1/3)_2 \to (1/4)_2$ means that the existence of a $(1/3)_2$ implies the existence of a $(1/4)_2$. The symbol $\to$ represents a forcing relation. We use the same notation for the forcing relation between intervals. The symbols $\downarrow$ and $\to$ have the same meaning.

At $a = 4$, the period-doubling bifurcation of $Q$ occurs. New SNBOs with $2n$ ($n \geq 2$) turning points appear after this bifurcation. Our main aim is to prove that there are infinitely many SNBOs between the first and the second columns in Theorem 1 and to derive the following dynamical ordering for them (Theorem 4 proved in §4).

**Theorem 4.** For SNBOs starting on $S^+_1$, the following dynamical order relations hold:

\[
\begin{align*}
I_1: & \quad (1/3) \to (1/5) \to (1/7) \to \cdots \to (1/8) \to (1/6) \to (1/4) \\
I_2: & \quad (1/5) \to (1/7) \to (1/9) \to \cdots \to (1/10) \to (1/8) \to (1/6) \\
I_3: & \quad (1/7) \to (1/9) \to (1/11) \to \cdots \to (1/12) \to (1/10) \to (1/8) \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{align*}
\]
Here, the periodic orbits of the first and last columns are the SNBOs listed in the first and second columns of the table given in Theorem 1, and periodic orbits sandwiched between these two are new SNBOs that appear after the period-doubling bifurcation of $Q$.

In §2, the structure of stable and unstable manifolds of $P$ is summarized and that of $Q$ is analyzed. Fundamental properties are proved in §3. The proof of Theorem 4 is given in §4.

§2. Stable and unstable manifolds of $P$ and $Q$

2.1. Stable and unstable manifolds of $P$

Here, we summarize the structure of stable and unstable manifolds of $P$ (see Ref. 6) for a detailed analysis). A saddle $P$ and its copy $P'$ have stable and unstable manifolds. Let $W^u_P$ be a branch of the unstable manifold starting at $P$ in the upper-right direction, and let $W^s_{P'}$ be a branch of the stable manifold going to $P'$ from the upper-left in the universal cover (see Fig. 1). Let $u$ be the first intersection point of $W^u_P$ and $S^+_2$. $W^s_{P'}$ passes through $u$. This comes from the facts that $S_2$ is invariant under $G_1$ and $W^u_P$ is mapped to $W^s_{P'}$ by $G_1$. The transverse intersection of $W^u_P$ and $W^s_{P'}$ at $u$ has been proved. Let $v$ be the first intersection point of $W^u_P$ (or $W^s_{P'}$) and $S^+_4$. The points $u$ and $v$, along with their forward and backward iterates are the primary homoclinic points. Let us denote by $(a, b)_A$ an open arc of a one-dimensional manifold $A$, where $a, b \in A$. Closed and semi-closed arcs are defined in an obvious manner. Let $\gamma_u = [u, v]_{W^u_P}$ and $\gamma_s = [u, v]_{W^s_{P'}}$. Then, let $V$ be an open region bounded by $\gamma_u$ and $\gamma_s$, and let $U$ be an open region bounded by $[T^{-1}v, u]_{W^u_P}$ and $[T^{-1}v, u]_{W^s_{P'}}$. These are primary homoclinic lobes. Due to reversibility, two lobes $U$ and $V$ are related by

$$U = G_1 V. \quad (10)$$

Fig. 1. Stable and unstable manifolds of $P$, two primary homoclinic points $u$ and $v$, and two primary homoclinic lobes $U$ and $V$. 

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We define a sequence of open intervals in \( S^+_4 \) as
\[
I_i = T^{-i}V \cap S^+_4. \quad (i \geq 1)
\] (11)

It is easy to show that each \( I_i \) has only one component if it exists. The order of appearance of \( I_i \) is determined by the lambda lemma \(^{13}\) as
\[
I_1 \to I_2 \to I_3 \to \cdots.
\] (12)

2.2. Stable and unstable manifolds of \( Q \)

We have not yet analyzed the behavior of stable and unstable manifolds of the saddle \( Q \) in our series of papers. We here rather precisely analyze their behavior and geometrical structure.

Due to the period-doubling bifurcation of \( Q \) for \( a > 4 \), new period-2 points \( r^+ \in S^+_4 \) and \( r^- \in S^-_4 \) with \( T^2r^+ = Tr^- = r^+ \) appear. Let \( W^u_Q \) be the branch of the unstable manifold of \( Q \) going upward from \( Q \), and let \( W^s_Q \) be that going downward. Similarly, let \( W^s_Q \) be the branch of the stable manifold of \( Q \) coming from above to \( Q \), and let \( W^s_Q \) be that coming from below. These are shown in Fig. 2. Let \( w^\pm \) be the intersection points of \( S^+_4 \) and \( S_G \). The initial slopes of \( W^u_Q \) and \( W^s_Q \) are \((a \pm \sqrt{a^2 - 4a})/2\) at \( Q \). We have the inequalities
\[
\frac{a - \sqrt{a^2 - 4a}}{2} < 2 < \frac{a}{2} < \frac{a + \sqrt{a^2 - 4a}}{2} < a
\] (13)

![Fig. 2. Structure of the stable and unstable manifolds of \( Q \) at \( a = 5.25 \).](https://academic.oup.com/ptp/article-abstract/109/2/187/1903488)
for the slopes, at \( Q \), of the curves \( W^q_Q \), \( S_4 \), \( S_G \), \( W^s_Q \) and \( y = a \sin \hat{x} \).

The following result will be used in later sections, but it is also interesting in its own right.

**Theorem 2.**
(a) \( W^s_Q \) and \( S^+_4 \) intersect transversely for the first time other than at \( Q \) at a point \( z^+ \). By symmetry, \( z^- = H_2 z^+ \) is the first intersection point of \( W^s_Q \), \( W^u_Q \) and \( S^-_4 \).
(b) \( W^s_Q \) and \( W^u_Q \) have transverse intersections except possibly at discrete parameter values for which the slope of \( W^s_Q \) and \( W^u_Q \) at \( z^+ \) is zero and the slope of \( W^s_Q \) and \( W^u_Q \) diverges at the point where they cross \( S^+_4 \).

**Remark.** Numerical calculations indicate that \( W^s_Q \) and \( W^u_Q \) always intersect transversely at \( z^+ \). In particular, the slope of \( W^s_Q \) at \( z^+ \) is negative, and hence \( W^s_Q \) always has a unique peak to the left of \( S_4 \). However we have not been able to prove this.

We now discuss notation. Let \( D^+ \) be an open region surrounded by \( \gamma_s \), \([Q, u]_{S^+_4}\), and \( D^- = H_2 D^+ \). Using linear analysis, it is easy to show that the local stable manifold of \( Q \) is located in \( D^+ \cup D^- \). Let \( o(p_0) = \{ \cdots , p_{-2}, p_{-1}, p_0, p_1, p_2, \cdots \} \) with \( p_i = (x_i, y_i) \). We use \( \hat{x} = x - \pi \). Correspondingly, let \( \hat{\gamma}_1(p) = \hat{x} \) if \( \gamma_1(p) = x \).

**Lemma 1.** \( W^s_Q \) cannot be completely contained in \( \text{clos}(D^+) \) and \( W^s_Q \) cannot be completely contained in \( \text{clos}(D^-) \).

**Proof.** Let us first show that \( x_0 > x_{-2} \) if \( p_0 \in D^- \) and \( p_{-1} \in D^+ \). Indeed, the lower boundary of \( D^+ \) is \( y = 2(x - \pi) \), and the upper boundary of \( D^- \) is \( y = 2(x - \pi) \). From this we have \( y_0 < 2(x_0 - \pi) \) and \( y_{-1} > 2(x_{-1} - \pi) \). Then, \( x_0 - x_{-2} = (x_0 + x_{-1}) - (x_{-1} + x_{-2}) = (2x_0 - y_0) - (2x_{-1} - y_{-1}) > 0 \). Similarly, if \( p_{-1} \in D^+ \) and \( p_{-2} \in D^- \), then \( x_{-1} < x_{-3} \). Then, using the above relation repeatedly, we have \( x_0 > x_{-2} > x_{-4} > \cdots > x_{-2n} \) and \( x_{-1} < x_{-3} < x_{-5} < \cdots < x_{-2n-1} \) for \( p_{-2i} \in D^- \) \( (i = 0, 1, \cdots , n) \) and \( p_{-2i-1} \in D^+ \) \( (i = 0, 1, \cdots , n-1) \).

Now we appeal to reductio ad absurdum. We assume that \( W^s_Q \) is contained in \( D^- \), and derive a contradiction. If \( p_0 \in W^s_Q \cap D^- \), then by hypothesis

\[
x_0 > x_{-2} > x_{-4} > \cdots , \quad x_{-1} < x_{-3} < x_{-5} < \cdots
\]

Consequently, \( x_i \) with odd and even \( i \) converge separately. Correspondingly, \( y_{-2i} \) and \( y_{-2i-1} \) converge. Hence \( p_{-2i} \) and \( p_{-2i-1} \) converge in \( \text{clos}(D^-) \) and \( \text{clos}(D^+) \), respectively. The limit points are period-2 points. The points \( r^\pm \) are the only possible choices. However, if \( r^\pm \) is stable, the surrounding points rotate around it, whereas if it is unstable, it is a saddle with reflection. In any case, points approaching \( r^\pm \) under iterates cannot approach monotonically from the left of \( r^+ \) nor monotonically from the right of \( r^- \). We thus arrive at a contradiction. (Q.E.D.)

Both initial arcs of \( W^u_Q \) and \( W^s_Q \) starting from \( Q \) are graphs. We express these
graphs by

\[ y = F_s(\hat{x}) \quad \text{and} \quad y = F_u(\hat{x}). \]  

We denote by the derivative of a function with respect to its argument. Then, the functional equations for \( F(\hat{x}), F'(\hat{x}), \) and \( F''(\hat{x}) \), where \( F(x) \) represents either \( F_s(x) \) or \( F_u(x) \), are

\[
F(\hat{x}_{n+1}) = F(\hat{x}_n) - a \sin \hat{x}_n, \\
F'(\hat{x}_{n+1}) = F'(\hat{x}_n) - a \cos \hat{x}_n, \\
\text{and} \\
F''(\hat{x}_{n+1}) = \frac{F''(\hat{x}_n) + a \sin \hat{x}_n}{(F'(\hat{x}_n) - a \cos \hat{x}_n + 1)^2}.
\]

These equations are easily derived from the mapping equations (1) and (2) taking into account that points are on a graph.

Let us give the transformation between the first and second derivatives of \( F_u \) and \( F_s \). We note that if \( p \in W^u_Q \), then \( H_1p \in W^u_Q \), and that if \( p \in W^s_Q \), then \( H_1p \in W^s_Q \), and that similar relations can be obtained by replacing \( H_1 \) by \( G_1, G_2, \) and \( G_2 \). Denoting \( \xi(p) = F'(\hat{x}_1(p)) \) and \( \eta(p) = F''(\hat{x}_1(p)) \), we have

\[
\eta(H_1p) = \frac{\xi(p)}{\xi(p) - 1}, \quad \eta(G_1p) = -\xi(p) + a \cos \hat{x}_1, \quad \eta(G_2p) = -\xi(p) - a \sin \hat{x}_1.
\]

By (13), the denominator of Eq. (16) for \( W^u_Q \) [i.e., \( F'_u(\hat{x}_n) - a \cos \hat{x}_n + 1 \)] is negative if \( \hat{x} \) is close to zero. We know that the sign of \( \hat{x}_n \) alternates, because iterated points oscillate to the left and right of \( Q \). Let us increase the value of \( \hat{x}_n \) from zero. As long as the denominator of Eq. (16) is negative, the slope \( F'_u(\hat{x}_n+1) \) is positive and \( W^u_q \) continues to be a graph for the interval \( \hat{x}_n+1 \) of \( \hat{x} \), and correspondingly, \( W^u_q \) continues to be a graph for the interval \( [0, \hat{x}_n+1] \) of \( \hat{x} \). On the other hand, the graph \( y = F_u(\hat{x}) \) does not extend to \( \hat{x} \rightarrow \infty \). In fact, letting \( p_c \) be the point where \( W^u_q \) ceases to be a graph, and where the slope diverges, we have \( \hat{x}_1(T^{1-p_c}) < \pi/2 \) because the denominator in Eq. (16) is positive for \( \hat{x} \geq \pi/2 \).

It can be shown that \( y = F''_u(\hat{x}) \) is zero at \( Q \) and positive on \( (Q,p_c)_{W^u_q} \). Correspondingly, arc \( (Q,H_1p_c)_{W^s_q} \) is a graph, by Eq. (18), and the arc \( (Q,G_2p_c)_{W^s_q} \) is a graph, by Eq. (21). The curvature of \( y = F_s(\hat{x}) \) is negative for \( (Q,H_1p_c)_{W^s_q} \), by Eq. (18), and it is negative for \( (Q,G_2p_c)_{W^s_q} \), by Eq. (21).
Let $z^\pm$ be the first intersection points of $W_{Q}^{s\pm}$ with $S_4$. The existence of $z^\pm$ is insured by Lemma 1. $W_{Q}^{u+}$ passes through $z^+$ by the symmetry $H_1$. Let us first show that $z^+$ is above $w^+$ along $S^+_4$. If $z^+$ is below $w^+$, then $W_{Q}^{s+}$ first intersects $S^+_G$ at some point $q \in (Q, z^+)_W^{s+}$. The arc $H_1(Q, q)_W^{s+}$ has no intersection with $S^+_4$, whereas the arc $G_2(Q, q)_W^{s+}$ does have intersection with $S^+_4$. This means that the latter is a longer arc of $W_{Q}^{s+}$ with one common end $Q$. Then note that the $x$-coordinates of the points of $H_1(Q, q)_W^{s+}$ are larger than those of $H_1(Q, q)_W^{s-}$. This means that $H_1(Q, q)_W^{s+}$ is a longer arc of $W_{Q}^{u+}$. We thus arrive at a contradiction. Similarly, it can be proved that $z^+ = w^+$ is not realized.

We claim that $p_c \in (Q, z^+)_W^{s+}$. Let us assume the contrary and derive a contradiction. Assume that $p_c$ is above the line $y = 2\hat{x}$ or $p_c = z^+$. The first intersection of $W_{Q}^{u+}$ with $y = 2\hat{x}$ is $z^+$. Let the first intersection of $W_{Q}^{u+}$ with $S^+_G$ be $q$. One easily observes that $G_2(Q, q)_W^{s+}$, an arc of $W_{Q}^{s+}$, first intersects $y = 2\hat{x}$ at a point whose $y$-coordinate is smaller than that of $z^+$. This contradicts the fact that $z^+$ is the first intersection point of $W_{Q}^{s+}$ and $S^+_4$.

Now, $p_c$ is below $S_4$. Then $G_2p_c$ cannot be above $S_4$, because, if it were, $(Q, G_2p_c)_W^{s+}$ would not intersect $S_4$, and the arc $(Q, H_1G_2p_c)_W^{s+}$ would have a peak to the right of $p_c$, and hence it would have a point of infinite slope to the right of $p_c$, which is a contradiction. The arc $(Q, G_2p_c)_W^{s+}$ intersects $S_4$ transversely, because it has a negative curvature. Thus Theorem 2(a) is proved.

If the slope of $W_{Q}^{s+}$ at $z^+ \in (Q, G_2p_c)_W^{s+}$ is negative, then $W_{Q}^{s+}$ and $W_{Q}^{u+}$ intersect transversely at $z$ and the proof of Theorem 2(b) completes. As is easily shown, the arc $(Q, z)_W^{s+}$ has a peak at least for $a \geq 2\pi$. Hence Theorem 2(b) is true in this range of parameter. We decrease the value of $a$ from $2\pi$ to $4$. We only consider the behavior of the intersection angle of $W_{Q}^{s+}$ and $S^+_4$ at $z$. This angle may increase and may pass through zero. However, the angle never stay zero for a finite interval of parameter values though it may stay exponentially small. Thus Theorem 2(b) is proved. Therefore the intersection angle can be zero at discrete values of parameter. This completes the proof of Theorem 2.

Proposition 2. The point $r^+$ is in the region bounded by $\Gamma^+ = [Q, z^+]_W^{s+} \cup [Q, z^+]_W^{u+}$. The point $r^-$ is in the region bounded by $\Gamma^- = [Q, z^-]_W^{s-} \cup [Q, z^-]_W^{u-}$.

Proof. The proof is almost trivial, because of the conservation of the Poincaré index computed along the boundary of $\Gamma^+$ or $\Gamma^-$. (Q.E.D.)

The arc $(Q, z^+)_W^{s+}$ is a graph and has a negative curvature, as in the proof of Theorem 2. It may or may not have a peak. The $y$-coordinate of the highest point in $(Q, z^+)_W^{s+}$ tends to infinity with $a$. Increasing the value of $a$, the arc $(Q, z^+)_W^{s+}$ touches $y = x$ at some value of $a$, and then there is a unique chord of $y = x$ cut by $(Q, z^+)_W^{s+}$ for larger values of $a$. If we operate with $T^{-1}$ on this chord, the image will be in $I_1$. We denote this subarc of $I_1$ by $I_1^*$ (see Fig. 3). In a similar manner,
we define $I^*_n \subset I_n$ as the image under $T^{-n}$ of the arc $T^nS^+_1 \in V$ cut by $(Q, z^+)_{W^+_Q}$.

§3. Properties of SNBOs for $a > 4$

We use matrix notation to specify SNBOs in Theorem 1. We denote by $(i, j)$ the periodic orbit of the $i$-th row and the $j$-th column of the table in Theorem 1. The $(2, 4)$ element, for example, represents $(1/8)_2$. Correspondingly, define $a_c(i, j)$ to be the critical parameter value at which the SNBO of the $(i, j)$ element appears due to the saddle-node bifurcation (or, the so-called tangent bifurcation).

Below, we summarize the properties of periodic orbits in the first and second columns in Theorem 1.\(^7\)

Properties of $(i, 1)$ elements.
(F1) $p_0 \in I_i$ has the period $q = 2i + 1$.
(F2) $p_i \in V$ is located above $S^+_4$ and to the right of $(u, P')_{W^+_P}$ (with respect to the natural direction of motion on $W^+_P$).
(F3) $p_{i+1}$ is located on $S^-_4$.
(F4) $p_{i+2} (= H_1 p_i)$ is located below $S^+_4$ and to the left of $(u, P')_{W^+_P}$.

Properties of $(i, 2)$ elements.
(S1) $p_0 \in I_i$ has the period $q = 2i + 2$.
(S2) $p_i \in V$ is located above $S^+_4$ and to the right of $(u, P')_{W^+_P}$.
(S3) $p_{i+1}$ is located on $S^-_4$.
(S4) $p_{i+2} \in U$ is located on the half line $y = x - \pi < 0$. 

Fig. 3. Appearance of $I^*_1$ in $I_1$ due to the tangency of $S^+_1$ and $W^\pm_Q$ ($a = 6.273$). Note that $W^s_Q$ (resp., $W^s_{Q}$) touches $S^+_1$ if the suffix $i$ is odd (resp., even).
(S5) \( p_{i+3} (= H_1 p_i) \) is located below \( S_4^+ \) and to the left of \((u, P')_{W_p'}\).

**Proposition 3.**

\[
\lim_{i \to \infty} a_c(i, 1) > \lim_{i \to \infty} a_c(i, 2) > 4. \tag{22}
\]

**Proof.** The first inequality is obvious from Theorem 1, and thus we prove only the second one. We will show that there is no SNBO of the \((i, 2)\)-element \((i = 1, 2 \cdots)\) of Theorem 1 at \(a = 4\). We assume the contrary, that an SNBO, \((1/(2i+2))_2\), exists for some \(i \geq 1\). Let \(O\{p_0, p_1, \cdots, p_{2i+1}\} \) be one cycle of this SNBO.

Recall that there exists for \(a > 3\) a symmetric period-3 orbit revolving around \(Q\) such that \(s_0, s_1 = Ts_0, s_2 = T^2s_0\) with \(s_0 \in S_4^+, s_1 \in S_4^−, s_2 \in \{y = x−\pi, y < 0\}\). The point \(s_0\) is that at which \(T^{−1}S_2^−\) and \(S_4^+\) cross (Fig. 4). The open arc \(L = (Q, s_0)_{T^{−1}S_2^−}\) can be shown to be above \(S_4^+\), as seen in Fig. 5(c). We have \(p_i \in T^{−1}S_2^−\), due to item (S3). Then, \(p_i \in L\), due to item (S2). The point \(p_{i+3}\) is in \(H_1 L\), due to item (S5). \(H_1 L\) is to the right of \(S_4\), because \(L\) is to the left of \(S_4\). We claim that \(H_1 L\) at \(a = 4\) is located below \(T^{−1}S_2^−\) (Fig. 4). This implies that \(p_{i+3}\) is located to the right of \((u, P')_{W_p'}\). This contradicts item (S5), and the proof is complete. The remaining thing is to prove our claim. \(T^{−1}S_2^−\) has the expression \(y = \pi − x − 4\sin x\), whereas \(H_1 L\) has the expression \(2y − x + 4\sin(y − x) + \pi = 0\). At \(s_0\), the slope of \(L\) is positive, and hence that of \(H_1 L\) is negative. We only need to show that the rightmost point of \(H_1 L\), say \(p^* = (x^*, y^*)\), is below the curve \(y = \pi − x − 4\sin x\). The slope is infinite at this point. Then we have \(\cos(y^* − x^*) = −1/2\), which implies that \(p^*\) is below \(T^{−1}S_2^−\). (Q.E.D.)

Numerically we obtain \(\lim_{i \to \infty} a_c(i, 1) = 6.64 \cdots\) and \(\lim_{i \to \infty} a_c(i, 2) = 5.25 \cdots\).

For \(a > 3\), there exists another period-3 orbit, \(\{s'_0 \in S_4^−, s'_1 = Ts'_0 \in S_4^+, s'_2 = Ts'_1\}\), where \(s_i\) and \(s'_i\) \((i = 0, 1, 2)\) are located in symmetric positions with respect to \(Q\).

Fig. 4. At \(a = 4\), \(H_1 L\) stays below \(T^{−1}S_2^−\), and the period-3 orbit \(\{s_0, s_1, s_2\}\) revolving around \(Q\) exists.
Proposition 4. (a) $T^{-1}S_4$ is a graph. In particular, its part $T^{-1}S_4^-$ has at most one peak for $\pi < x < 2\pi$ and has for $a > 4$ a unique intersection $r^+$ with $S_4^+$. (b) For $a > 4$, $T(Q, r^-)_{S_4^-}$ is located to the right of $S_4^+$, whereas $T^{-1}(Q, r^-)_{S_4^-}$ is located to the left of $S_4^+$. 

![Graphs of T^{-1}S_4, T^{-1}S_4^+, T^{-1}S_4^- at a = 5.](https://academic.oup.com/ptp/article-abstract/109/2/187/1903488)

Fig. 5. (a) $TS_4$, (b) $T^{-1}S_4$, and (c) $T^{-1}S_2$ at $a = 5$. 


Proof. Direct calculation of the inverse map gives (a), and linear analysis gives (b). (Q.E.D.)

In Figs. 5(a)–(c), the situations described in Proposition 4 are displayed.

Proposition 5. The SNBOs of the \((i,1)\)-element in Theorem 1 have a point in \(I_i^*\), whereas the SNBOs of the \((i,2)\)-element have a point in \(I_i \setminus I_i^*\) for \(i = 1, 2, \ldots\).

Proof. The existence of \((i,1)\)-elements implies \(T^{i+1}I_i \cap S_4^- \neq \emptyset\). The intersection points are on \((r^-, Q)_{S_4^-}\), because \(T^{-1}(S_4^- \setminus r^-, Q)_{S_4^-}\) is to the right of \(S_4^+\) by Proposition 4 and has no common points with \(V \supset T^iI_i\). On the other hand, the arc \((r^-, Q)_{S_4^-}\) is contained in the region bounded by arcs \([Q, z^-]_{W_4^*}\) and \([Q, z^-]_{W_2^*}\). Consequently, if \(T^{i+1}I_i\) intersects \(S_4^-\), then it also intersects \((Q, z^-)_{W_4^*}\), because \(T^{i+1}I_i\) is contained in \(TV\) and the boundary of \(TV\) never intersects \((Q, z^-)_{W_4^*}\). Now, there are exactly two points \(q_1, q_2 \in T^{i+1}I_i \cap (Q, z^-)_{W_4^*}\). (The proof that the number of points is two is complicated but it is elementary.) The arc of \(T^{i+1}I_i\) between \(q_1\) and \(q_2\) is \(T^{i+1}I_i^*\) by definition. Obviously \(T^{i+1}I_i \cap S_4^-\) is contained in \(T^{i+1}I_i^*\). Thus the first statement is proved.

The points of \(T^{i+1}I_i \cap S_2^-\) belong to an SNBO of the \((i,2)\)-element in Theorem 1. These points are not contained in \(T^{i+1}I_i^*\). (Q.E.D.)

§4. Main results

If \(T^{i+k}I_i\) intersects \(S_4^-\) for \(k \geq 1\), whereas \(T^{i+k'}I_i\) does not intersect \(S_4^-\) for any \(0 \leq k' < k\), then we call these intersections the first intersections and these intersection points the first intersection points of \(T^{i+k}I_i\) in \(S_4^-\). We say that the first intersections take place at \(i + k\). If \(T^{i+k}I_i\) intersects both \(S_4^-\) and \(S_4^+\) but \(T^{i+k'}I_i\) (\(0 \leq k' < k\)) does not intersect \(S_4^+\), then we call the intersections and intersection points \(T^{i+k}I_i \cap S_4^+\) the second intersections and second intersection points in \(S_4^+\). We say that the second intersections take place at \(i + k\). Repeating this, we can define the \((2n)\)-th intersections and \((2n)\)-th intersection points of \(T^{i+k}I_i\) in \(S_4^+\), and the \((2n+1)\)-st intersections and \((2n+1)\)-st intersection points of \(T^{i+k}I_i\) in \(S_4^-\). Similar definitions can be made for the intersections of \(T^{i+k}I_i\) and \(S_2^\pm\). For later convenience, we introduce the zero-th intersections. This is the state expressed by \(T^iI_i\).

Theorem 3.
(a) If \(T^{i+k}I_i\) \((k \geq 1)\) has \(n\)-th \((\text{with } n \geq 1)\) intersection points in \(S_2\), these are points of SNBOs of \((1/(2i + 2k))_{2n}\). If \(n\) is odd (resp., even), the intersection points are located in \(S_2^-\) (resp., \(S_2^+\)).
(b) If \(T^{i+k+1}I_i\) \((k \geq 1)\) has the \(n\)-th \((\text{with } n \geq 1)\) intersection points in \(S_4\), these are points of SNBOs of \((1/(2i + 2k + 1))_{2n}\). If \(n\) is odd (resp., even), the intersection points are located in \(S_4^-\) (resp., \(S_4^+\)).

Proof. (a) Intersection points are obviously periodic and their period is \(2i + 2k\).
Due to the reversibility $G_1$ with respect to $S_2$, orbits proceed $2\pi$ in one period. Thus, the rotation number is $1/(2i + 2k)$. Let $O(p_0) = \{p_0(\in I_i), p_1, \cdots, p_{2i+2k-1}\}$ be the one cycle of the orbit of one of the intersection points. For the proof, it is enough to show that the number of turning points is $2n$; i.e., that there are $n$ turning-back points and $n$ turning-forward points. Then, in view of Proposition 1, it is enough to show that there are $n$ pair of points $q_j, r_j, j = 1, \cdots, n \in \{p_i\}_{i=0, \cdots, 2i+2k-1}$ such that $\pi_2(q_j) \geq 0$ and $\pi_2(T(q_j)) < 0$, and $\pi_2(r_j) \leq 0$ and $\pi_2(T(r_j)) > 0$.

We are going to show that there is exactly one turning point in $O(p_0)$ between the $j$-th and $(j+1)$-st intersections. For definiteness, let us assume that $j$ is even and that the $j$-th and $(j+1)$-st intersections take place at $i + k_j$ and $i + k_{j+1}$, respectively. We have $k_j < k_{j+1}$. We can treat the problem in a similar manner when $j$ is odd. By definition, at the $(j+1)$-st intersections, $T^{i+k_{j+1}}I_i$ intersects $S^-_2$, whereas $T^{i+k_{j+1}-1}I_i$ does not, apart from $j$ consecutive intersections with $S^+_2$ and $S^-_2$. Similarly, $T^{i+k}_{j}I_i$ intersects $S^+_2$, whereas $T^{i+k_{j}-1}I_i$ does not, apart from $j - 1$ consecutive intersections with $S^+_2$ and $S^-_2$.

Let $t_1$ and $t_2$ be two intersection points of $T^{i+k_{j+1}}I_i$ with $S^-_2$ at the $(j+1)$-st intersections. Obviously, these two points are those of periodic orbits with rotation number $1/(2i + 2k_{j+1})$. Let $\gamma_t$ be the open arc of $T^{i+k_{j+1}}I_i$ between $t_1$ and $t_2$. This arc is to the left of $S^-_2$ but to the right of $y = \hat{x}$. The points $T^{-1}t_1$ and $T^{-1}t_2$ are both on $L$ defined in the proof of Proposition 3, and $T^{-1}\gamma_t$ is to the right of $L$. If there is no part of $T^{-1}\gamma_t$ to the right of $y = \hat{x}$, then we have $k_{j+1} = k_j + 1$. If $T^{-1}\gamma_t$ has common points with $y = \hat{x}$, then we consider $T^{-2}\gamma_t$. This arc is necessarily to the left of $L$ because the half line $y = \hat{x}$, $y < 0$ becomes $T^{-1}S^-_2$ upon operation with $T^{-2}$. Therefore, we have $k_j = k_{j+1} - 1$ or $k_j = k_{j+1} - 2$.

Now, as is easily confirmed, a point $p_{i+k_{j+1}}$ of $O(p_0)$ is in $\text{clos}(\gamma_t)$. Because $p_{i+k_{j+1}-1}$ or $p_{i+k_{j+1}-2}$ is above the $x$-axis, the point $p_{i+k_{j+1}-1}$ or $p_{i+k_{j+1}-2}$ is a turning-back point by Proposition 1.

(b) The proof here is similar to that of (a). (Q.E.D.)

**Lemma 2.** If $T^{i+1}I_i$ has first intersection points in $S^-_4$, then $T^{i+2k}I_i$ has $(2k)$-th intersections in $S^+_4$, and $T^{i+2k+1}I_i$ has $(2k+1)$-st intersections in $S^-_4$ for any $k \geq 1$.

**Proof.** The intersection points of $T^{i+1}I_i$ and $S^-_4$ are in $(Q, r^-)_{S^+_4}$, and $T(Q, r^-)_{S^-_4}$ sits to the right of $S^+_4$ [see Fig. 5(a)]. The two endpoints of $T^{i+2}I_i$ are in $(v, P^r)_{W^s_{P^r}}$, and $T^{i+2}I_i$ winds clockwise around $Q$. Therefore $T^{i+2}I_i$ has first intersection points in $S^-_4$ and second ones in $S^+_4$ by Proposition 4. Thus, the first statement for $k = 1$ is proved. Repeating this, the proofs for the remaining statements are obtained. (Q.E.D.)

**Lemma 3.** The following forcing relations hold:

\[
\left(\frac{1}{2i + 2k + 1}\right)_{2k+2} \in I^*_i \rightarrow \left(\frac{1}{2i + 2k + 2}\right)_{2k+2} \in I^*_i; \\
\left(\frac{1}{2i + 2k + 1}\right)_{2k+2} \in I^*_i \rightarrow \left(\frac{1}{2i + 2k + 3}\right)_{2k+4} \in I^*_i,
\]

(23)-(24)
\[
\left(\frac{1}{2i + 2k + 1}\right)_{2k+2} \in I^*_i \rightarrow \left(\frac{1}{2i + 2k + 1}\right)_{2k} \in I^*_i \rightarrow \left(\frac{1}{2i + 2k + 1}\right)_{2k} \in I^*_i \rightarrow \cdots \rightarrow \left(\frac{1}{2i + 2k + 1}\right)_{2k} \in I^*_i ,
\]

(25)

Here \(i \geq 1\) and \(k \geq 0\).

**Proof.** Let us derive the forcing relation (23). The remaining relations can be derived in a similar manner. By Theorem 3, the existence of \((1/(2i + 2k + 1))_{2k+2} \in I^*_i\) means that \(T^{i+k}I_i\) has \((k+1)\)-st intersections with \(S_4^+\) (resp., \(S_4^-\)) if \(k\) is even (resp., odd), and in turn it implies that \(T^{i+k}I_i\) has \((k+1)\)-st intersections with \(S_2^+\) (resp., \(S_2^-\)) if \(k\) is even (resp., odd). Intersection points are those of \((1/(2i + 2k + 2))_{2k+2} \in I^*_i\) by Theorem 3. These points are obviously in \(I^*_i\). Lemma 2 implies Eq. (24), and Proposition 4(b) implies Eq. (25).

(Q.E.D.)

**Lemma 4.** The following forcing relations hold:

\[
\left(\frac{1}{2i + 2k + 1}\right)_{2k+2} \in I^*_i \rightarrow \left(\frac{1}{2i + 2k + 3}\right)_{2k+2} \in I^*_i+1 ,
\]

(26)

\[
\left(\frac{1}{2i + 2k + 2}\right)_{2k+2} \in I^*_i \rightarrow \left(\frac{1}{2i + 2k + 4}\right)_{2k+2} \in I^*_i \rightarrow \left(\frac{1}{2i + 2k + 4}\right)_{2k+2} \in I^*_i \rightarrow \cdots \rightarrow \left(\frac{1}{2i + 2k + 4}\right)_{2k+2} \in I^*_i .
\]

(27)

Here \(i \geq 1\) and \(k \geq 0\).

**Proof.** From the definition of \(I_i\), the two endpoints \(A\) and \(B\) with \(\pi_2(A) > \pi_2(B)\) of \(T^{i+k}I_i \supset T^{i+k}I^*_i\) and the two endpoints \(C\) and \(D\) with \(\pi_2(C) > \pi_2(D)\) of \(T^{i+k+1}I_{i+1} \supset T^{i+k+1}I^*_{i+1}\) are located in \((v, P')_{W^*_{P'}}\). Then, \(C\) is in \((v, A)_{W^*_{P'}}\) and \(D\) is in \((B, P')_{W^*_{P'}}\). This means that \(T^{i+k+1}I_{i+1}\) is located outside the area bounded by \(T^{i+k}I_i\) and \([A, B]_{W^*_{P'}}\). This implies that if \(T^{i+k}I^*_i\) intersects \(S_4^-\), then \(T^{i+k+1}I^*_{i+1}\) also intersects \(S_4^-\), because their endpoints are on \((Q, z^-)_{W^*_{Q^-}}\). Using this, Eq. (26) is derived. The derivation of Eq. (27) is similar.

(Q.E.D.)

**Lemma 5.** The following two order relations hold:

\[
\left(\frac{1}{2i + 2k + 1}\right)_{2k+2} \in I^*_i \rightarrow \left(\frac{1}{2i + 2k + 1}\right)_{2k} \in I^*_i \rightarrow \cdots \rightarrow \left(\frac{1}{2i + 2k + 1}\right)_{2k} \in I^*_i ,
\]

(28)

\[
\left(\frac{1}{2i + 2k + 2}\right)_{2k+2} \in I^*_i \rightarrow \left(\frac{1}{2i + 2k + 2}\right)_{2k} \in I^*_i \rightarrow \cdots \rightarrow \left(\frac{1}{2i + 2k + 2}\right)_{2k} \in I^*_i .
\]

(29)

**Proof.** As in the proof of Lemma 3, the existence of \((1/(2i + 2k + 1))_{2k+2} \in I^*_i\) means that \(T^{i+k}I_i\) has \((k+1)\)-st intersections with \(S_4^+\) (resp., \(S_4^-\)) if \(k\) is even (resp., odd). Let us decrease the value of \(a\). Then \(T^{i+k}I_i\) shrinks and it has no longer \((k+1)\)-st intersections below some value of \(a\). Then, we deal with \(k\)-th intersections of \(T^{i+k}I_i\)
with $S^-_4$ (resp., $S^+_4$) if $k$ is even (resp., odd). Obviously, intersection points are those of $(1/(2i + 2k + 1))_{2k} \in I^*_i$. Further decreasing the value of $a$ and repeating similar arguments, we obtain the order relations (28). The order relations (29) can be obtained in a similar manner.

Lemma 3 implies the following order relation for $i = 1$:

$$(1/3)_2 \rightarrow (1/5)_4 \rightarrow (1/7)_6 \rightarrow \cdots$$

Rearranging this order, we have

$$(1/3)_2 \rightarrow (1/5)_4 \rightarrow (1/7)_6 \rightarrow \cdots \rightarrow (1/8)_6 \rightarrow (1/6)_4 \rightarrow (1/4)_2.$$  (30)

We can derive similar relations for $i \geq 2$. Combing these, we have Theorem 4.

**Theorem 4.** For SNBOs starting in $S^+_i$, the following dynamical order relations hold:

<table>
<thead>
<tr>
<th>$I_1$</th>
<th>$(1/3)_2 \rightarrow (1/5)_4 \rightarrow (1/7)_6 \rightarrow \cdots \rightarrow (1/8)_6 \rightarrow (1/6)_4 \rightarrow (1/4)_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2$</td>
<td>$(1/5)_2 \rightarrow (1/7)_4 \rightarrow (1/9)_6 \rightarrow \cdots \rightarrow (1/10)_6 \rightarrow (1/8)_4 \rightarrow (1/6)_2$</td>
</tr>
<tr>
<td>$I_3$</td>
<td>$(1/7)_2 \rightarrow (1/9)_4 \rightarrow (1/11)_6 \rightarrow \cdots \rightarrow (1/12)_6 \rightarrow (1/10)_4 \rightarrow (1/8)_2$</td>
</tr>
</tbody>
</table>
| \vdots | \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots \vdots |}

Note that Lemma 4 is used to obtain the relations indicated by $\downarrow$.

Due to Lemma 5, we can regard $(1/7)_6$ in Theorem 4 as $(1/7)_6 \rightarrow (1/7)_4 \rightarrow (1/7)_2$ in $I^*_7$. We can confirm the existence of all types of SNBOs for a given period. For period-7, we have $(1/7)_6, (1/7)_4, (1/7)_2 \in I^*_1, (1/7)_4, (1/7)_2 \in I^*_2$, and $(1/7)_2 \in I^*_3$. As a result, $(1/3)_2$ implies the existence of SNBOs with all possible numbers of turning points for periods larger than 3.

Let $a_c(i)$ be the value of $a$ at which $T^{i+1}I_i$ is tangent to $W^{-}_{Q}$ for the first time. This is the critical value at which a new interval $I^*_i$ appears in $I_i$. The existence of the $(i, 1)$ element means $T^{i+1}I_i \cap (Q, r^-)_{S^-_4} \neq \emptyset$. In view of the relative positions of $(Q, r^-)_{S^-_4}, W^{-}_{Q}$ and $S^-_2$, the condition $T^{i+1}I_i \cap (Q, r^-)_{S^-_4} \neq \emptyset$ implies $T^{i+1}I_i \cap W^{-}_{Q} \neq \emptyset$, and the condition $T^{i+1}I_i \cap W^{-}_{Q} \neq \emptyset$ implies $T^{i+1}I_i \cap S^-_2 \neq \emptyset$. Combining these facts and Proposition 3, we have the relation

$$a_c(i, 1) > a_c(i) > a_c(i, 2) > 4.$$  (31)

**References**