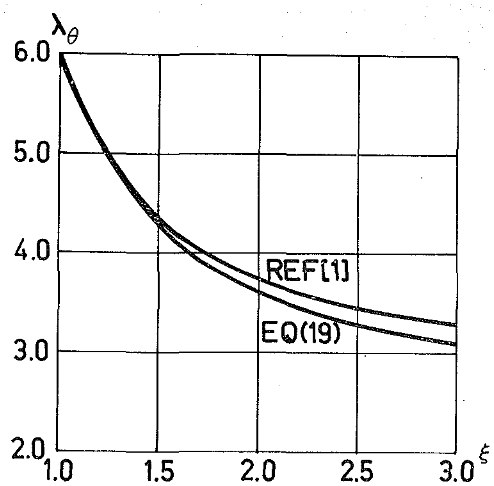

Fig. 1 Variation of circumferential extension ratio λ_θ

$$\lambda_\xi = \lambda = (\lambda_\theta)^{-1/2} = 0.408 \tag{26}$$

The coefficients a and b are obtained using (10), (18), and (26)

$$a = 1.204, \qquad b = 0.145$$

Using (19c) and (26), we get the parameter t corresponding to $\xi = 1$ as

$$t = 0.795i$$

Substitution of t into (19a), and (19b) gives the constants C_1 and C_2

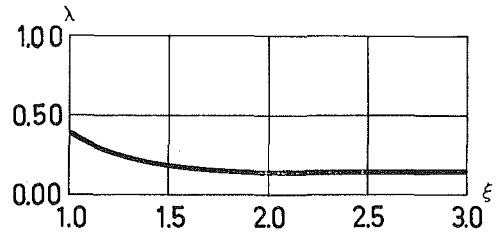

Fig. 2 Variation of deformed thickness

$$C_1 = -0.915i, \qquad C_2 = 2.679$$

The solution is then obtained by successively changing t and computing the corresponding values of ξ, λ_θ, and λ_ξ. The variation of $\lambda_\theta(\xi)$ is plotted in Fig. 1, together with the numerical solution taken from [1]. The variation of the deformed thickness is calculated through (4) and plotted in Fig. 2.

References

1 Rivlin, R. S., and Thomas, A. G., "Large elastic deformation of Isotropic Materials VIII. Strain Distribution Around a Hole in a Sheet," *Philosophical Transactions*, Series A, Vol. 243, 1951, pp. 289–298.

2 Green, A. E., and Adkins, J. E., *Large Elastic Deformations and Nonlinear Continuum Mechanics*, Oxford, 1960.

3 Yang, W. H., "Stress Concentration in a Rubber Sheet Under Axially Symmetric Stretching," Journal of Applied Mechanics, Vol. 34, Trans. ASME, Vol. 89, Series E, 1967, pp. 942–946.

4 Biricikoglu, V., and Kalnins, A., "Finite Stretching of a Circular Plate," to be published.

5 Appell, M. P., "On the Invariants of Certain Differential Equations," (in French), *Journal de Mathematiques Pures et Appliquées*, 4th series, Vol. 5, 1889, pp. 361–423.

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journal of applied mechanics discussion

Determination of the Duration of Memory for Viscoelastic Materials[1]

R. A. SCHAPERY.[2] The authors have written an interesting paper on a subject of considerable practical significance. There is, however, an error in limits in equations (3) and (12) which appears to invalidate the results for the *nonlinear* range of behavior.

That there is an error can be demonstrated by retaining, for simplicity, only the first and second-order terms in a multiple integral representation for stress:

$$\sigma(t) = f[e(t)] + \int_{-\infty}^{t} K_1^{(1)}(t-\tau_1)e(\tau_1)d\tau_1 + \int_{-\infty}^{t}\int_{-\infty}^{t} K_2^{(2)}(t-\tau_1, t-\tau_2)e(\tau_1)e(\tau_2)d\tau_1 d\tau_2 \tag{a}$$

where the authors' notation is followed. The norm of the difference between stress σ and the stress σ_c is found as

$$||\sigma - \sigma_c|| \leq M\int_d^\infty |K_1^{(1)}(u_1)|du_1 + M^2 \times \int_d^\infty\int_d^\infty |K_2^{(2)}(u_1,u_2)|du_1du_2 + 2M^2\int_d^\infty\int_0^d |K_2^{(2)}(u_1,u_2)|du_1du_2 \tag{b}$$

The authors omitted all multiple integrals having unequal limits, such as the third term in the foregoing equation (b). Further,

[1] By M. H. Gradowczyk, J. Soussou, and F. Moavenzadeh, published in the June, 1970, issue of the Journal of Applied Mechanics, Vol. 92, Series E, pp. 449–453.

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the limits in equation (5) should be written $\int_d^\infty$ rather than $\int_\infty^d$.

Authors' Closure

The authors agree with Professor Schapery that there is an error in equation (3) of the main paper. The equality

$$\left\| \begin{matrix} \tau = t \\ G(e) \\ \tau = -\infty \end{matrix} \quad - \begin{matrix} \tau = t \\ G(e_c) \\ \tau = -\infty \end{matrix} \right\| = \left\| \begin{matrix} \tau = t - d \\ G(e) \\ \tau = -\infty \end{matrix} \right\| \qquad (1)$$

is valid only when G is a linear functional or for some special forms of nonlinear functionals. This error which was unfortunately overlooked, does not invalidate, however, the concept of duration of the memory as expressed in equations (6) and (15) of the main text.

Hence, for the case of a Fréchet expansion of the nonlinear functional G, the inequality (5) of the main paper becomes

$$\|\sigma - \sigma_c\| \leq \sum_{n=1}^{\infty} M^n \left| \int_\infty^d \cdots \int_\infty^d K_n^{(n)}(u_1, \ldots, u_n) du_1 \ldots du_n \right| + \sum_{n=2}^{\infty} nM^n \left[\sum_{m=1}^{n-1} \left| \underbrace{\int_\infty^d \cdots \int_\infty^d}_{n-m-1} \underbrace{\int_d^0 \cdots \int_d^0}_{m} \times K_n^{(n)}(u_1, \ldots u_n) du_1 \ldots du_n \right| \right] \leq \delta \qquad (2)$$

This expression (2) permits the evaluation of d as a function of M and δ, although the presence of the multiple integrals with unequal limits yields more complicated computations. For instance, it is possible to use the mean value theorem to find a bound for (2), since $|Kn^{(n)}(\ldots)|$ does not change signs

$$r(M, d) \leq \sum_{n=1}^{\infty} M^n \left| \int_\infty^d \cdots \int_\infty^d K_n^{(n)}(u_1, \ldots, u_n) \right| du_1 \ldots du_n + \sum_{n=2}^{\infty} nM^n \left[\sum_{m=1}^{n-1} \left| \int_\infty^d \cdots \int_\infty^d K_n^{(n)} \times (u_1^*, u_2^*, \ldots u_m^*, u_{m+1}, \ldots u_n) du_{m+1} \ldots du_n \right| \right] \qquad (3)$$

where the u_i^* designates a particular value of u_i over the interval $0 \leq u_i \leq d$, for which $K_n^{(n)}$ is maximum.

In the case of a two-term series, equation (2) becomes

$$\|\sigma - \sigma_c\| \leq M \int_d^\infty |K^{(1)}(u_1)| du_1 + M^2 \int_d^\infty \int_d^\infty |K_2^{(2)}(u_1, u_2)| du_1 du_2 + 2M^2 \int_d^\infty \int_0^d |K_2^{(2)}(u_1, u_2)| du_1 du_2 \qquad (4)$$

By applying the mean value theorem, the following expression is obtained:

$$\|\sigma - \sigma_c\| \leq M \int_d^\infty |K^{(1)}(u_1)| du_1 + M^2 \int_d^\infty \int_d^\infty |K_2^{(2)}(u_1, u_2)| du_1 du_2 + 2M^2 d \int_d^\infty |K_2^{(2)}(u_1^*, u_2)| du_2 \qquad (5)$$

A different bound can be derived if $K_i^{(i)}$ does not change the sign

$$\|\sigma - \sigma_c\| \leq M|K_1(d) - K_1(\infty)| + M^2|K_2(d, d) - 2K_2(d, \infty) + K_2(\infty, \infty)| + 2M^2|K_2(d, 0) - K_2(d, d) - K_2(0, \infty) + K_2(d, \infty)|$$

The same procedures apply to bound the missing terms in the section "Duration of Creep."

Theory of Laminated Plates[1]

J. M. WHITNEY.[2] The author presents a plate theory which is applicable to laminates consisting of a large number of alternating plane, parallel isotropic layers referred to as reinforcing and matrix layers. Gross displacements which are linear with respect to the thickness coordinate are assumed. Thus the theory accounts for gross thickness-stretch deformation as well as gross shear deformation. The theory is then developed by following essentially the same procedure used by Mindlin (author's reference [4]) for isotropic homogeneous plates. Plate equations of motion are obtained, however, by integrating the three-dimensional stress equations of motion from the continuum theory developed by Sun, Achenbach, and Herrmann (author's references [1, 2]) rather than integrating the equations of linear theory of elasticity as Mindlin did. As a result, the author's plate theory also includes the effect of local thickness-stretch deformation and local shear deformation which makes this theory differ from any other existing laminated plate theory.

It should be pointed out, however, that this approach to lamination problems is very restrictive. In particular the theory cannot be easily extended to the general case of anisotropic laminated plates. The continuum theory, and thus the plate theory as well, could be readily extended to alternating plies of anisotropic materials. This would allow the theory to include cross-ply composites (plies with the fibers alternately oriented at, 0 and 90 deg to the x_1-axis) and angle-ply composites (plies with the fibers alternately oriented at $+\theta$ and $-\theta$ to the x_1-axis). Extending this theory to more general composites which are of practical interest would require developing the continuum theory for three or more layers in the repeating unit, and the theory would very rapidly become out of hand.

It should also be noted that dispersion curves for flexural motion can be adequately described by an effective stiffness theory without microstructure which includes gross shear deformation (see reference [1][3]). Since the title of the author's paper implies a theory which is applicable to laminated plate problems of a general nature, discussion of existing effective stiffness theories without microstructure such as those in references [2–4] is in order.

Although not appearing explicitly in the governing equations of motion, resultant moments on the individual layers are present in the classical effective stiffness theory of laminated plates [2–4]. Thus the existence of moments within each layer (or couple stresses if you prefer) is not unique to the present theory.

References

1 Yang, P. C., Norris, C. H., and Stavsky, Y., "Elastic Wave Propagation in Heterogenous Plates," *International Journal of Solids and Structures*, Vol. 2, 1966, pp. 665–684.

[1] By C. T. Sun, published in the March, 1971, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 38, Series E, pp. 231–238.

[2] Air Force Materials Laboratory, Wright Patterson Air Force Base, Ohio.

[3] Numbers in brackets designate References at end of Discussion.