the limits in equation (5) should be written \( \int_{d}^{\infty} \) rather than \( \int_{d}^{\infty} \).

**Authors’ Closure**

The authors agree with Professor Schapery that there is an error in equation (3) of the main paper. The equality

\[
G(c) = G(c_o) \quad \text{for} \quad \tau = t - d
\]

is valid only when \( G \) is a linear functional or for some special forms of nonlinear functionals. This error, which was unfortunately overlooked, does not invalidate, however, the concept of duration of the memory as expressed in equations (6) and (15) of the main text.

Hence, for the case of a Fréchet expansion of the nonlinear functional \( G \), the inequality (5) of the main paper becomes

\[
\|\sigma - \sigma_d\| \leq M \sum_{n=1}^{\infty} \left[ \int_{d}^{\infty} \cdots \int_{d}^{\infty} \frac{\partial}{\partial u_1} \cdots \frac{\partial}{\partial u_n} K_n^{(\infty)}(u_1, \ldots, u_n) du_1 \cdots du_n \right] \]

This expression (2) permits the evaluation of \( d \) as a function of \( M \) and \( \delta \), although the presence of the multiple integrals with unequal limits yields more complicated computations. For instance, it is possible to use the mean value theorem to find a bound for (2), since \( \{K_n^{(\infty)}, \ldots\} \) does not change signs

\[
\int_{d}^{\infty} \cdots \int_{d}^{\infty} K_n^{(\infty)}(u_1, \ldots, u_n) du_1 \cdots du_n
\]

where the \( u_i^* \) designates a particular value of \( u_i \) over the interval \( 0 \leq u_i \leq d \), for which \( K_n^{(\infty)} \) is maximum.

In the case of a two-term series, equation (2) becomes

\[
\|\sigma - \sigma_d\| \leq M \int_{d}^{\infty} \left| K^{(\infty)}(u_1) \right| du_1
\]

The same procedures apply to bound the missing terms in the section “Duration of Creep.”

### Theory of Laminated Plates

J. M. Whitney. The author presents a plate theory which is applicable to laminates consisting of a large number of alternating plane, parallel isotropic layers referred to as reinforcing and matrix layers. Gross displacements which are linear with respect to the thickness coordinate are assumed. Thus the theory accounts for gross thickness-stretch deformation as well as gross shear deformation. The theory is then developed by following essentially the same procedure used by Mindlin (author’s reference [4]) for isotropic homogeneous plates. Piste equations of motion are obtained, however, by integrating the three-dimensional stress equations of motion from the continuum theory developed by Sun, Achenbach, and Herrmann (author’s references [1, 2]) rather than integrating the equations of linear theory of elasticity as Mindlin did. As a result, the author’s plate theory also includes the effect of local thickness-stretch deformation and local shear deformation which makes this theory differ from any other existing laminated plate theory.

It should be pointed out, however, that this approach to lamination problems is very restrictive. In particular the theory cannot be easily extended to the general case of anisotropic laminated plates. The continuum theory, and thus the plate theory as well, could be readily extended to alternating plies of anisotropic materials. This would allow the theory to include cross-ply composites (plies with the fibers alternately oriented at 0° and 90° deg to the \( x_1 \)-axis) and angle-ply composites (plies with the fibers alternately oriented at +90° and −90° to the \( x_1 \)-axis). Extending this theory to more general composites which are of practical interest would require developing the continuum theory for three or more layers in the repeating unit, and the theory would very rapidly become out of hand.

It should also be noted that dispersion curves for flexural motion can be adequately described by an effective stiffness theory without microstructure which includes gross shear deformation (see reference [1]). Since the title of the author’s paper implies a theory which is applicable to laminated plate problems of a general nature, discussion of existing effective stiffness theories without microstructure such as those in references [2-4] is in order.

Although not appearing explicitly in the governing equations of motion, resultant moments on the individual layers are present in the classical effective stiffness theory of laminated plates [2-4]. Thus the existence of moments within each layer (or couple stresses if you prefer) is not unique to the present theory.

### References


3. Air Force Materials Laboratory, Wright Patterson Air Force Base, Ohio.

4. Numbers in brackets designate References at end of Discussion.
Author's Closure

The author would like to thank Dr. Whitney for his comments. The author agrees that for more general laminates the plate theory may become laborious and the suppression of the microstructure may become desirable. Such simplicity is achieved at the expense of accuracy, however.

It has been shown in [3] that the effect of the microstructure in a composite beam is very substantial and that the model without microstructure is adequate only in the range of very long wavelengths. The same can be said about the laminated plate. The examples given by the discussers' reference [1] deal with the laminates in which the difference in shear rigidities of the layers is small, and, consequently, the shear deformation induced by the heterogeneity is negligible. A more detailed discussion on this point can be found in [6].

With regard to the last comment, we should notice that the couple stresses in the microstructure plate theory are originated by the introduction of some independent kinematic variables employed to describe the local deformations in the layers which are different from the gross deformation of the plate. In contrast, the couple stresses in the discussers' references [1, 2, 4] are not distinguishable from the gross moments of the plate.

References


The discussers applied the authors' equations to determine the response of a large prestressed concrete reactor vessel (PCRV) subjected to the El Centro (May, 1940, Calif.) earthquake input, an increase in the peak base acceleration over the free-field input maximum was also observed for several cases. The authors attributed such an increase mainly to the lack of radiation of energy into the soil. While this is an important consideration, the discussers wish to point out that aside from the factor of the radiation damping, another important reason for the response to be significantly amplified is that the author's formulation cannot consider damping in the structure. However, the Volterra type of interaction equation as formulated by the authors can be modified to account for the existence of structural modal damping. Using the author's notation and considering the effects of a large base mass, the modified interaction equation may be put in the form:

\[
\ddot{u}(t) = \frac{uA}{bM_f} \int_0^t \left[ \ddot{u}_p(\tau) - \bar{u}(\tau) \right] d\tau - \frac{b}{2\pi cM_0} \sum \int_0^t \ddot{u}(t-\tau) \times \text{Im} \left[ g \left( \frac{b^2}{c} \right) \right] d\tau - \frac{b}{2\pi cM_0} \sum M_0 \omega_j K_j(t) - \frac{b}{2\pi cM_0} \sum M_0 \omega_j \int_0^t \bar{u}(t-\tau) \text{Im} \left[ g \left( \frac{b^2}{c} \right) \right] d\tau \ (1)
\]

where the function \( K_j \) appears twice with different arguments and is defined, for argument \( s \), as

\[
K_j(s) = \frac{1 - 2F_j^2}{\sqrt{1 - \xi_j^2}} \int_0^s \ddot{u}(\eta)e^{-i\omega_j(s-\eta)} \sin \omega_j \sqrt{1 - \xi_j^2} \times \cos \omega_j \sqrt{1 - \xi_j^2} (s-\eta) d\eta \ (2)
\]

The new symbols \( \xi_j \) and \( M_0 \) denote, respectively, the \( j \)th modal damping factor and the base mass. In writing the foregoing equation, it has been assumed that the system was initially at rest at \( t = 0 \) and the inclusion of a large base mass was handled in the same way as the authors did in [1]. Fig. 1 shows the amplification factors of PCRV subjected to El Centro type earthquake.

**Fig. 1** Amplification factors of PCRV subjected to El Centro type earthquake.