Invariant Delta Functions in the Sense of Distributions*

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Invariant delta functions are most adequately interpreted in the sense of distributions of L. Schwartz. They are expressed as the sum of proper distributions and mass-dependent point functions. First terms are interpreted as the logarithmic or finite parts of the divergent integrals corresponding to the inverse square of the four-dimensional distance. Point function term of $A^{(1)}$ exhibits logarithmic singularities on the surface of the light cone, defining a finite value as a distribution.

§ 1. Introduction

To deal with the queer "functions" $A$, $A^{(1)}$, etc., and to clear up the ambiguities in the calculations beyond the usual conventional technics, we must treat them throughout in a mathematically justified way.

The theory of distributions will be powerful to this purpose. So we are to interpret the invariant delta functions not as mere point functions but as a sort of distributions.

At first a rough sketch about the concept of distribution will be given. For details the original books of L. Schwartz$^1$ should be refered to.

A distribution on an open space $V$ is a continuous linear functional $T(\varphi)$ defined for all infinitely differentiable complex valued function $\varphi(x)$, $x \in V$, whose carrier (the closure of the set of points $x \in V$ such that $\varphi(x) \neq 0$) is any compact subspace of $V$. Here continuity is said with respect to all the derivatives of $\varphi(x)$. The symbol $\varphi$ is used throughout this paper to denote the function with qualities mentioned above.

A partial derivative of a distribution $T$ on $R$ is defined by

$$(\partial/\partial x_i)T(\varphi) = -T(\partial \varphi/\partial x_i), \quad x=(x_1, \ldots, x_n) \in R^n,$$

Let $S$ and $T$ be two distributions. We define the convolution $S*T$ of $S$ and $T$ by

$$(S*T)(\varphi) = S(x)(T_y(\varphi(x+y))),$$

when the right hand side makes sense.

Examples:

(1) Any function $f(x)$ ($x \in R^n$), if summable on every compact set, defines a distribution $T_y$ on $R^n$:

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In this meaning a point function itself may be looked upon as a distribution and we write \( f(\varphi) \) instead of \( T_{ij}(\varphi) \).

(ii) Dirac's distribution \( \delta_{R^d} \), defined by

\[
\delta_{R^d}(\varphi) = \varphi(0).
\]

Now assume that the function \( F(t) \) defined for \( t > 0 \) (or \( t < 0 \)) may be written in the form

\[
F(t) = A + B \log |t| + \sum_{m \in \mathbb{N}} t^{-m} (A_m + B_m \log |t|) + E(t),
\]

where the exponent \( m \) appear finite times and \( \lim_{t \to 0} E(t) = 0 \).

Then we name \( A \) and \(-B\) as the finite part and the logarithmic part of \( F(t) \) respectively and write

\( A = P_f F(t) \) (or \( P_f F(0) \)),
\( -B = P_i F(t) \) (or \( P_i F(0) \)).

For instance we can easily prove the following relations:

\[
P_f \int_0^\infty \varphi(x) x^{-1} dx = \lim_{\epsilon \to 0} \left( \int_0^\infty \varphi(x) x^{-1} dx + \varphi(0) \log \epsilon \right),
\]
\[
P_i \int_{-\infty}^0 \varphi(x) x^{-1} dx = \lim_{\epsilon \to 0} \left( \int_{-\infty}^0 \varphi(x) x^{-1} dx - \varphi(0) \log \epsilon \right),
\]
\[
P_f \int_0^\infty \varphi(x) x^{-1} dx = \varphi(0),
\]
\[
P_i \int_{-\infty}^0 \varphi(x) x^{-1} dx = -\varphi(0).
\]

Eqs. (8)–(11) define distributions \( P_f^\pm x^{-1} \) and \( P_i^\pm x^{-1} \) on \( R^d \) by writing the left sides of these equations as

\( P_f^\pm x^{-1}(\varphi), \ P_i^\pm x^{-1}(\varphi), \ P_i^\pm x^{-1}(\varphi) \) and \( P_i^\pm x^{-1}(\varphi) \) respectively. (They are thought to be zero when the carrier of \( \varphi \) lies outside of the integral domain.)

We observe that the distribution \( P x^{-1} \) or \( \overline{P} x^{-1} \) defined by

\[
P x^{-1} = P_f^+ x^{-1} + P_f^- x^{-1}, \quad \overline{P} x^{-1} = P_i^+ x^{-1} + P_i^- x^{-1}
\]

coincides with the principal value of Cauchy or twice the Dirac's distribution \( \delta_{R^d} \).

They are obtained by differentiating the functions:

\[
P x^{-1} = (d/dx) \log |x|, \quad \overline{P} x^{-1} = (d/dx) \epsilon(x),
\]
where
\[ \varepsilon(x) = \text{sgn}(x). \]  

(16)

For the purpose of later use we give another expression which avails complex integral. Let \( C^+ \) (or \( C^- \)) be the contour along the real axis in the positive (or negative) direction, being replaced under (or above) the poles by semi-circles of radius \( r \), which will tend to zero.

Then we can observe
\[ \int_{C^+} \varphi(z) z^{-1} dz = P x^{-1}(\varphi) + (\pi i/2) \bar{P} x^{-1}(\varphi), \]
\[ \int_{C^-} \varphi(z) z^{-1} dz = -P x^{-1}(\varphi) + (\pi i/2) \bar{P} x^{-1}(\varphi), \]

left sides of which define distributions \( P^+ x^{-1}(\varphi) \) and \( P^- x^{-1}(\varphi) \) on the complex plane.

Thus we obtain
\[ P x^{-1} = (1/2) (P^+ x^{-1} - P^- x^{-1}), \]
\[ \bar{P} x^{-1} = (1/\pi i) (P^+ x^{-1} + P^- x^{-1}). \]

Analogous arguments are possible in the space of higher dimensions. We shall investigate the integral
\[ \int \varphi(x) (1-2\alpha^2) \text{d}^3x \]
in the interior and exterior of the light cone and show that its logarithmic or finite part corresponds to the proper distribution term of the invariant delta functions (§ 4). Explicit expressions of them, together with the point function terms, will be determined in a natural way from the classical solutions of the Cauchy's problems (§ 2, § 3). Because of this way of treating the problem, usual covariant expressions are avoided.

§ 2. Cauchy's problems and invariant delta functions

(i) Cauchy's problem of the homogeneous equation

Let \( L[u] \) be a linear differential form with respect to the three-dimensional space coordinates; consider following two problems of Cauchy

A) \[ \partial^2 u/\partial t^2 - L[u] = 0 \quad (t > 0), \]
\[ u_{t=0} = 0, \quad (\partial u/\partial t)_{t=0} = f(x). \]

(21)

(22)

B) \[ \partial^2 G^R/\partial t^2 - L[G^R] = 0 \quad (t > 0), \]
\[ G^R_{t=0} = 0, \quad (\partial G^R/\partial t)_{t=0} = \delta_\alpha^\beta . \]

(23)

(24)

As is easily verified by direct substitution, the solution of A) is expressed as the convolution on \( R^3 \) of the initial value of \( \partial u/\partial t \) and the solution of B).

\[ u(x, t) = f * G^R(x | t), \]

(25)

and defines a distribution.
where
\[ x = (x_1, x_2, x_3), \quad y = (y_1, y_2, y_3), \]
\[ dx = dx_1 dx_2 dx_3, \quad dy = dy_1 dy_2 dy_3. \]

We shall avail the eq. (26) to determine the distribution \( G(\varphi) \) from the knowledge of the solution \( u(x, t) \).

In case of the wave equation
\[ \partial^2 u / \partial t^2 - L[u] = -\Box u = \partial^2 u / \partial t^2 - (\partial^2 / \partial x_1^2 + \partial^2 / \partial x_2^2 + \partial^2 / \partial x_3^2) u, \quad (27) \]
the solution of A) is given by the famous formula of Poisson:
\[ u(x, t) = \frac{1}{(4\pi t)^{n/2}} \int f(x + ut) \, dw, \quad (28) \]
where
\[ u = (u_1, u_2, u_3), \quad u^2 = u_1^2 + u_2^2 + u_3^2, \]
and \( dw \) means the surface element of the unit sphere.

Then we get
\[ u(\varphi) = \frac{1}{(4\pi)^{n/2}} \int f(x + ut) \varphi(x) \, dw \, dx 
= \frac{1}{(4\pi)^{n/2}} \int f(y) \, dy \int \varphi(y - ut) \, dw, \]
from which we obtain, on comparing with the right side of (26) and writing \( D^R \) for \( G^R \),
\[ D^R(\varphi(x + y)) = \frac{1}{(4\pi)^{n/2}} \int \varphi(y - ut) \, dw. \]

On putting \( y = 0 \) we can determine the expression of \( D^R(\varphi) \)
\[ D^R(\varphi) = \frac{1}{(4\pi)^{n/2}} \int \varphi(ut) \, dw = \frac{1}{(4\pi)^{n/2}} \int \varphi(x) \, dw, \quad (r = \sqrt{x^2}). \quad (29) \]

Thus the distribution \( D^R \) means to make spherical mean multiplied by the radius of the sphere.

Next we consider the following case
\[ \partial^2 u / \partial t^2 - L[u] = - (\Box + x^2) u, \quad G^R = \Delta^R. \quad (30) \]

The solution of A) is given as follows:
\[ u(x, t) = \frac{t}{4\pi} \int f(x + ut) \, dw + \frac{x}{4\pi} \int f(x - y) \frac{f_0'(x \sqrt{t^2 - y^2})}{\sqrt{t^2 - y^2}} \, dy. \quad (31) \]

The second term of (31) being the convolution between point functions, we get immediately
where
\[ S^+ = \begin{cases} \sqrt{t^2 - x^2} & (t^2 \geq x^2, \ t \geq 0), \\ 0 & \text{(otherwise)} \end{cases} \] (33)

Take care that so far $D^R$ and $\Delta^R$ are considered on $\mathbb{R}^3$; the time variable $t$ appears here as a continuous parameter.

(ii) Inhomogeneous problem and elementary solutions

C) \[ \frac{\partial^2 v}{\partial t^2} - L[v] = f(x, t), \] (34)
\[ v(x, 0) = (\partial v / \partial t)(x, 0) = 0. \] (35)

The solution of C) is expressed by the solution $u$ of the homogeneous equation (i) A), as follows:
\[ v(x, t) = \int_0^t u(x, \tau; t - \tau) \, d\tau, \] (36)
where $\tau$ means the parameter introduced in the initial value of $\partial u / \partial t$; we write $f(x, \tau)$ instead of $f(x)$.

Assume that $f(x, t)$ and $G(\varphi)$ are both zero on any subspace of $t < 0$, on substituting (25) we can interpret (36) defining four-dimensional convolution:
\[ v(x, t) = f * G^R(x, t). \] (37)

Putting (37) in (34), we obtain
\[ \frac{\partial^2 v}{\partial t^2} - L[v] = f*(\frac{\partial^2 G^R}{\partial t^2} - L[G^R]) = f \]
from which follows
\[ \frac{\partial^2 G^R}{\partial t^2} - L[G^R] = \delta_R. \] (38)

Thus the distribution $G^R$ (on $\mathbb{R}^4$) proves to be the elementary solution of the operator $\left( \frac{\partial^2}{\partial t^2} - L \right)$.

§ 3. Extension to the whole space and the distribution $\Delta^{(1)}$

In the usual manner we shall extend our theory in the domain $t < 0$.

Consider the elementary solution $G^A$ with carrier in the domain $t \leq 0$;
\[ \frac{\partial^2 G^A}{\partial t^2} - L[G^A] = \delta_R, \] (39)
\[ (G^A)_{t=0} = 0, \quad (\partial G^A / \partial t)_{t=0} = -\delta_R. \] (40)

If we set
\[ G = G_A - G_R, \quad \bar{G} = (G_A + G_R) / 2, \] (41)
we observe that $G$ and $\bar{G}$ satisfy following relations:
\[ \partial^2 G / \partial t^2 - L[G] = 0 \text{,} \quad (G)_{t=0} = 0 \text{,} \quad (\partial G / \partial t)_{t=0} = -\delta_k^i, \quad (42) \]
\[ \partial^2 G / \partial t^2 - L[G] = \delta_k^i \text{,} \quad (G)_{t=0} = (\partial G / \partial t)_{t=0} = 0. \quad (43) \]

Explicit expressions of \( \Delta^A \), \( \bar{\Delta} \), \( \bar{\Delta} \) etc. may be easily obtained.

Now let \( P \tau^{-1} \) be the distribution defined on any straight line with time-like direction, then we define \( \pi A^{(1)} \) as the convolution of \( \Delta \) and \( P \tau^{-1} \):

\[ A^{(1)} = (1/\pi) \Delta * P \tau^{-1}. \quad (44) \]

On substituting the expression of \( \Delta \):

\[ \Delta = D + (1/4\pi) W, \]
\[ W = -\varepsilon(t) x S^{-1} J_0'(x S), \quad (46) \]

where

\[ S = \begin{cases} \sqrt{t^2 - x^2} & (t^2 > x^2), \\ 0 & (t^2 \leq x^2), \end{cases} \quad (47) \]

and taking the direction of \( \tau \) on the time axis, we obtain

\[ A^{(1)} = D^{(1)} + (1/4\pi) \int_{-\infty}^{\infty} W(\tau) (t-\tau)^{-1} d\tau, \quad (48) \]

where \( D^{(1)} \) is defined by

\[ \pi D^{(1)}(\varphi) = (D * P \tau^{-1})(\varphi) = D\left(\int_{-\infty}^{\infty} \varphi(x, \tau) (\tau-t)^{-1} d\tau \right) \quad (49) \]

and will be treated in the next section.

The second term of \( A^{(1)} \) can be calculated as follows:

\[ P \int_{-\infty}^{\infty} \frac{W(\tau)}{t-\tau} d\tau = -P \int_{-\infty}^{\infty} \frac{x}{s} J_0'(x s) d\tau = \int_{-\infty}^{\infty} \frac{x}{s} J_0'(x s) d\tau = P \int_{-\infty}^{\infty} \frac{x}{s} J_0'(x s) \left\{ \frac{1}{\tau - t} + \frac{1}{\tau + t} \right\} d\tau \]
\[ = P \int_{-\infty}^{\infty} \frac{x}{s} J_0'(x s) \frac{2\tau d\tau}{\tau^2 - t^2}. \quad (50) \]

Set

\[ \sigma = (\tau^2 - x^2)^{1/2}, \quad \sigma_0 = (\tau^2 - x^2)^{1/2}, \quad (51) \]

then (50) becomes

\[ P \int_{-\infty}^{\infty} \frac{W(\tau)}{t-\tau} d\tau = P \int_{0}^{\infty} \frac{x}{a} J_0'(xa) \frac{d\sigma^2}{\sigma^2 - \sigma_0^2}. \quad (52) \]

To calculate this integral we set

\[ xa = \xi, \quad x^2 \sigma_0^2 = \begin{cases} a^2 & (\sigma_0^2 > 0), \\ -a^2 & (\sigma_0^2 < 0). \end{cases} \quad (53) \]
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Then we have to consider the integral $I_+$ or $I_-$ defined by

\[ I_+ = P \int_0^{\infty} f_0(\xi) (\xi^2 - a^2)^{-1} d\xi, \quad \text{(54)} \]

\[ I_- = \int_0^{\infty} f'_0(\xi) (\xi^2 + a^2)^{-1} d\xi, \quad \text{(55)} \]

according as the point in question lies in the interior or exterior of the light cone.

The integrals $I_{\pm}$ can be evaluated by the aid of the following contour integrals

\[ \int_{C} \gamma^2 (\gamma^2 a^2)^{-1} H_1^{(i)}(\gamma) d\gamma, \quad \text{(56)} \]

where $C$ is the contour in the half plane $\Im(\gamma) \geq 0$, consisting of semicircles $|\gamma|=R$, $|\gamma \pm a|=r$ and the parts of the real axis; second circles are not necessary in case of $I_-$. On letting $R \to \infty$, $r \to 0$, the integral along the real axis becomes

\[ 2(\mp a^2 I_{\pm} + 1). \]

Then we obtain, after calculating the residues,

\[ I_+ = (\pi/2a) N_1(a) + (1/a^2), \quad \text{(57)} \]

\[ I_- = (1/a) K_1(a) - (1/a^2), \quad \text{(58)} \]

where $N_1$ is the Neumann's function and $(-2/\pi) K_1$ is the Hankel's function with imaginary argument.

Thus we have completed the calculations of the second term of $A^{(3)}$, and we get

\[ A^{(3)} = D^{(3)} + \begin{cases} (x/4\pi a) N_1(xa) + (1/2\pi^2 \sigma^2) & (\sigma^2 > 0), \\ (x/2\pi^2(-\sigma^2)^{1/2}) K_1(x(-\sigma^2)^{1/2}) - (1/2\pi^2 \sigma^2) & (\sigma^2 < 0). \end{cases} \quad \text{(59)} \]

Notice that the second term, being a point function, exhibits logarithmic singularities on the surface of the light cone.

§ 4. Invariant delta functions as the logarithmic or finite part of $\sigma^{-2}$

We shall investigate the divergent integral

\[ \int \varphi(x, t) \sigma^{-2} dR^4 \]

with singularities on the surface of the light cone.

Let $V_{\epsilon}^+$ be the domain defined by

\[ (1-\epsilon)^2 \xi^2 - r^2 \geq 0, \quad 1 > \epsilon > 0. \quad \text{(60)} \]

Then the integral for a fixed $\epsilon$

\[ \int_{V_{\epsilon}^+} \varphi(x, t) \sigma^{-2} dR^4, \quad \text{(61)} \]

exists in the limit; first take $|t| \geq \delta > 0$ then let $\delta \to 0$. 

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The logarithmic part of the integral (61) defines a distribution $P_1 s^{-2}$ and $\tilde{P}_1 s^{-2}$, as
\begin{equation}
P_1 s^{-2} = (P_1 s^{-2})_{t > 0} + (P_1 s^{-2})_{t < 0},
\end{equation}
\begin{equation}
\tilde{P}_1 s^{-2} = (P_1 s^{-2})_{t > 0} - (P_1 s^{-2})_{t < 0},
\end{equation}
\begin{equation}
(P_1 s^{-2})_{t \geq 0}(\varphi) = P_1 \int_{V_\varepsilon^+} \varphi(x, t) \sigma^{-2} dR^4,
\end{equation}
\begin{equation}(P_1 s^{-2})_{t \leq 0}(\varphi) = -P_1 \int_{V_\varepsilon^-} \varphi(x, t) \sigma^{-2} dR^4,
\end{equation}
\begin{equation}(P_1 s^{-2})_{t \equiv 0}(\varphi) = \varphi(0, t) \sigma^{-2} dR^4,
\end{equation}
\begin{equation}\varphi(x, t) = \varphi(ru, t)
\end{equation}
when we observe that the integral remains finite when $\varepsilon$ tends to zero.

From eq. (65) we observe that
\begin{equation}(P_1 s^{-2})_{t > 0}(\varphi(x, t)) = (P_1 s^{-2})_{t \leq 0}(\varphi(x, -t)),
\end{equation}
and following relations hold:
\begin{equation}D^a = (1/2\pi) (P_1 s^{-2})_{t > 0},
\end{equation}
\begin{equation}D^a = (1/2\pi) (P_1 s^{-2})_{t < 0},
\end{equation}
\begin{equation}D^a = (1/4\pi) P_1 s^{-2}.
\end{equation}

Eq. (65) shows that the integral (61) has only logarithmic divergence, hence we can define the finite part of (61), by
\begin{equation}(P_1^+ s^{-2})(\varphi) = \lim_{\varepsilon \to 0} \left( \int_{V_\varepsilon} \varphi(x, t) \sigma^{-2} dR^4 + (P_1 s^{-2})(\varphi) \log \varepsilon \right).
\end{equation}

On the domain $V_\varepsilon$:
\begin{equation}t^2 - (1 - \varepsilon)^2 r^2 \leq 0,
\end{equation}
we can evaluate the logarithmic part as follows:
\begin{equation}P_1 \int_{V_\varepsilon} \varphi \cdot \sigma^{-2} dR^4 = P_1 \int_{0}^{r(1-\varepsilon)} dr \int_{0}^{\varepsilon} \varphi(ru, r) \sigma^{-2} dR^4.
\end{equation}
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\begin{align*}
&+ \int_{\mathbb{R}^3} \varphi(ru, -r) \, d\omega \int_{-r/(1-\varepsilon)}^0 (t^2 - r^2)^{-2} \, dt \\
&= -P_1 s^{-2}(\varphi)
\end{align*}
on employing the relation
\[
\int_0^{r/(1-\varepsilon)} r^2 (t^2 - r^2)^{-1} \, dt = (-r/2) (\log \varepsilon - \log (2 - \varepsilon) + 2 - 2\varepsilon).
\]

As before we can define the finite part
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} \varphi(r, t) \, d\omega = (P_1 s^{-2}) (\varphi) \log \varepsilon.
\]

From eqs. (70) and (72) the distribution \( P_1 s^{-2} \) is introduced by
\[
(P_1 s^{-2}) (\varphi) = \lim_{\varepsilon \to 0} \left( \int_{\mathbb{R}^3} \varphi \cdot \sigma^{-2} \, dR^4 - P_1 s^{-2}(\varphi) \log \varepsilon \right).
\]

or explicitly
\[
(P_1 s^{-2}) (\varphi) = \lim_{\varepsilon \to 0} \int_0^\infty \left[ \int_{-\infty}^{\infty} \varphi \cdot \sigma^{-2} \, d\omega \right] dt \int_{\mathbb{R}^3} \varphi \cdot \sigma^{-2} \, d\omega
\]
\[
= \frac{1}{2} \int_0^\infty \left[ \int_{-\infty}^{\infty} \varphi \cdot (r-t)^{-1} \, d\omega \right] dt \int_{\mathbb{R}^3} \varphi \cdot \sigma^{-2} \, d\omega,
\]
which means, on writing \( t \) or \( \tau \) for \( r \) or \( t \),
\[
(P_1 s^{-2}) (\varphi) = \left\{ (P_1 s^{-2})_{\tau<0} - (P_1 s^{-2})_{\tau>0} \right\} \left( \int_{-\infty}^{\infty} \varphi(x, \tau) (\tau-t)^{-1} \, d\tau \right)
\]
\[
= -\left\{ [(P_1 s^{-2}) \ast P_1 t^{-1}] (\varphi) \right\}.
\]

Then we can deduce the relation
\[
D^{(1)} = (-1/2\pi^2) P_1 s^{-2}
\]
from eqs. (49), (68) and (75).

Next we shall show the relation
\[
P_1 s^{-2} = -\nabla \log |\sigma|,
\]
the right side of which defining the distribution
\[
- \int_{\mathbb{R}^4} \log |\sigma| \cdot \nabla \varphi \, dR^4.
\]

On making use of the four-dimensional Green's formula on \( V_\varepsilon \pm \) and the relation
\[
\nabla \log |\sigma| = -\sigma^{-2} \quad (\sigma^2 \approx 0),
\]
the integral (78) is expressed as the sum of \( P_1 s^{-2}(\varphi) \) (eq. (73)) and the surface integral
containing transversal derivatives. The latter cancels and we have the result.

Lastly we give the expressions corresponding to eqs. (19) and (20) of § 1; they are

\[ P\sigma^{-2} = \left(\frac{1}{2}\right) \left( P_\varphi \sigma^{-2} - P_\psi \sigma^{-2} \right), \]

\[ \overline{P}_\psi \overline{s}^{-2} = \left(\frac{1}{2}\right) \left( P_\varphi \sigma^{-2} + P_\psi \sigma^{-2} \right), \]

where by \( P_\varphi \sigma^{-2}(\varphi) \) we mean the expression (74) whose time integral part is replaced by the contour integral on the complex \( t \)-plane along the contour \( C^\pm \) defined in § 1.

These relations show that \( \left(\frac{i}{4\pi^2}\right) P_\varphi \sigma^{-2} \) are the distribution \( D^\pm \) defined by

\[ D^\pm = \left(\frac{1}{2}\right) (D \mp iD^{(3)}), \]

If the contour \( C^+ \) is altered by replacing the semicircle under the pole \( t = -r \) by the semicircle above it, we obtain the contour \( C_\varphi \) and \( \left(\frac{-1}{2\pi^2}\right) P_\varphi \sigma^{-2} \) defined as before coincides with \( \overline{D}^{(3)} = D^{(3)} - 2iD \).

\[ \frac{}{5. Conclusion} \]

The problem of Fourier transforms\(^3\) is not treated in this paper, but there is no difficulty to describe them as we have already succeeded to clear up the ambiguities in the expressions of delta functions.

At the present stage of quantum field theory we are confronted with many difficulties by treating proper distributions as point functions. Some of them may be got rid of if we can define delta functions first on any space-like surface then extend continuously to the whole space. Still it is meaningless, for instance, to argue about the values which delta functions will take at one space time point.

An interesting trial was made by W. Güttinger\(^4\) to examine the calculations of the quantum field theory with distribution analysis, utilizing the arbitrary constants which appear by division process or by transforming the variable, but it seems to us that the whole field theory should be rewritten to fit the treatment so far made—in the sense of distributions.

\[ \text{References} \]