Determination of a Seismic Wave Velocity from the Travel-Time Curve

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Summary

A solution of the following problem is presented: given a travel-time curve of a seismic wave, to determine the corresponding velocity distribution. This is a generalization of the Herglotz–Wiechert method to a medium with low-velocity zones.

The velocity depends only on depth and is a piecewise double-smooth function with a finite number of waveguides. A complete mathematical description of this solution is presented. In the presence of wave-guides the solution is ambiguous. Necessary and sufficient conditions for a velocity to be a solution are formulated and the set formed by plots of solutions is obtained.

The ambiguity arising from waveguides is reduced by a joint analysis of travel-time curves from surface and deep sources. In particular the following theorem is proved: If the travel times for a source between any adjacent waveguides as well as for a surface source are known, then the velocity between these waveguides can be determined uniquely.

1. Formulation of problem

1.1. We shall investigate the positive function \( u(y) \) on the half-axis \( y \geq 0 \). The following properties of \( u(y) \) are postulated: it can be discontinuous, and its first and second derivatives can be discontinuous or not exist but only in a finite number of points; everywhere except these points its second derivative is continuous; \( u(y) \) is limited on each finite interval, but is unlimited on the whole half-axis \( y \geq 0 \); \( u(0) = 1 \).

Suppose a heavy point on the plane \( x, y \) is in position \( 0 \) at zero moment (Fig. 1), and starts to move into lower \( (y > 0) \) halfplane.

We accept \( L \) for the path of the point; \( \alpha(y) \) for the angle between \( L \) and \( y \)-axis; \( p \) for \( \sin \alpha(0) \), \( 0 < \alpha(0) < \pi/2 \); \( X(p), Y(p) \) for the coordinates of the lowest point of \( L \); \( u(y) \) for the velocity of motion along \( L \); \( T(p) \) for the time of motion from 0 to \( (X(p), Y(p)) \). The trajectory of movement \( L \) is determined by the following properties:

(a) On its downward-going branch \( \sin \alpha(y) = pu(y) \).

(b) \( Y(p) = \inf \{ y, pu(y) \geq 1 \} \).

(c) If \( X(p) < \infty \), then the line \( x = X(p) \) is the axis of symmetry for \( L \). Hence, the ray \( L \) ends at the point \( x = 2X(p), y = 0 \). The time of motion along the ray is \( 2T(p) \).
The problem is to find $u(y)$ from the given travel-time curve
\[ \Gamma \{ x = 2X(p), t = 2T(p), p \in (0, 1) \} . \]

1.2. The seismological problem on finding the velocity–depth distribution from the travel-time curve can be reduced to the above-formulated problem, if the Earth is approximated to a sphere, with wave velocity depending on depth only, and if the wave is propagated along the rays, following the laws of geometrical optics. To show it we shall write $v(r)$ for velocity of wave propagation, $r \in [0, R]$ (Fig. 2). Time of propagation from the source $A$ to point $B$ is
\[ \int_{AB} ds = \sqrt{(dr^2 + r^2 d\theta^2)}. \]

By introducing the new variables:
\[ x = \frac{R \theta}{v(R)}, \quad y = \frac{R \ln R/r}{v(R)}, \quad u(y) = \frac{v(R)e^{-yv(R)/R}}{v(R) \cdot e^{-yv(R)/R}} \]
we arrive at the problem, formulated above.
We have assumed here that the distance between $A$ and $B$ is known exactly and not only by $\mod 2\pi$. The following relations explain the choice of $u(y)$:

$$
\int_{AB} \frac{ds}{v} = \int_{L} \frac{dl}{u}, \quad dl = \sqrt{(dx^2 + dy^2)}; \quad u(0) = 1.
$$

Here $L$ is the image of the seismic ray $AB$.

The rays $L$ mentioned in 1.1 are, of course, only a part of the images of seismic rays $AB$. The ray splits on discontinuities into reflected and refracted rays; in that case we consider only one refracted ray.

The geophysical problem 1.2 and the mechanical problem 1.1 are strictly equivalent, if the above remarks are taken into account.

1.3. We assume that $\Gamma$ is not an arbitrary curve but a travel-time curve which corresponds to a certain function $u(y)$. This function satisfies all conditions formulated in 1.1, with the following natural limitations: the set of points $y^0 > 0$ where

$$
u(y^0) < s(y^0) = \sup \{u(y), y \in [0, y^0]\}
$$

consists of a finite number of intervals. We assume also that the function $X(p)$, not known beforehand, of course, is not constant on any interval.

2. Some properties of $Y(p)$, $X(p)$ and $T(p)$

2.1. We introduce the function $f(y) = s^{-1}(y)$. It is evident that $f(y)$ does not increase with $y$, and that $0 < f(y) \leq 1$. Let us investigate $f(y)$ on the plane $y, p$ (Fig. 3).

Let us take $y^0$ for the points of discontinuity of $f(y)$; we shall include into the plot of $f(y)$ the direct line, connecting the points $(y^0, f(y^0 - 0))$ and $(y^0, f(y^0 + 0))$.

* Otherwise we might come to the following generalization of the problem: the curve $\tilde{\Gamma} = \{x = 2\tilde{X}(p), t = 2T(p), p \in (0, 1), 0 < X(p) \leq \pi R/\varpi(R), \tilde{X}(p) \equiv X(p)(\mod \pi R/\varpi(R))$ is given instead of the travel-time curve $\Gamma$. $\tilde{\Gamma}$ is the projection on $(x, t)$-plane of some curve $\Gamma^*$, determined by the equations $\{x = 2\tilde{X}(q), t = 2T(q), p = q; q \in (0, 1]\}$.

The travel time curve $\Gamma$ is determined by $\tilde{\Gamma}$ ambiguously. But only the finite number of different $\Gamma$ corresponds to $\tilde{\Gamma}$ in that practically important case, when $\Gamma^*$ will contain a finite number of components after identification of points $(x, t, p)$ and $(x + 2\pi R/\varpi(R), t, p)$. 
After an analogous addition to \( Y(p) \) the curves \( p = f(y) \) and \( y = Y(p) \) on the plane \( y, p \) will coincide. It follows from the relation:

\[
Y(p) = \inf \{ y, pu(y) \geq 1 \} = \inf \{ y, f(y) \leq p \}.
\]

On the intervals of monotony \( f(y) \) and \( Y(p) \) are the mutually inverse functions.

2.2. \( Y(p) \) does not increase with \( p \). If, for some \( p, Y(p-0) > Y(p+0) \), then \( f(y) = p \) on the interval \( (Y(p-0), Y(p+0)) \).

The last condition is necessary and sufficient for \( Y(p) \) to have a discontinuity at \( p \in (0, 1) \).

The assumption of the structure of a set \( y^0 \) formulated above leads to the conclusion, that the points \( p_1 > p_2 > ... > p_n > 0 \) with the following properties exist: if

\[
y_k = Y(p_k + 0), \quad \bar{y}_k = Y(p_k - 0)
\]

then each of the intervals \( j_k = (y_k, \bar{y}_k) \) has the points with \( f(y) < u^{-1}(y) \) and outside the intervals \( j_k f(y) = u^{-1}(y) \).

Assume for simplicity \( p_1 < 1 \); then for \( y \in [0, y_1] \) we have \( f(y) = u^{-1}(y) \). We shall assume also \( p_0 = 1, p_{n+1} = 0, y_0 = \bar{y}_0 = 0 \). We shall soon find that the points \( p_1, ..., p_n \) are very important.

2.3. Let us mark on the interval \( (0, 1) \) of \( p \)-axis a finite number of points: \( p = f(y-0) \) and \( f(y+0) \) if \( f(y-0) > f(y+0), p = f(y^0) \) if at \( y = y^0 f(y) \) is continuous, but \( f'(y) \) or \( f''(y) \) is discontinuous, or does not exist.

The rest of the points of \( (0, 1) \) we shall divide into two sets: \( \Pi \) and \( \Pi' \); the points for which \( f'(Y(p)) = 0 \), belong to \( \Pi' \).

Using the evident relations

\[
X(p) = \int_{0}^{y(p)} \tan x(y) \, dy = \int_{0}^{y(p)} \frac{p \, dy}{\sqrt{(u^{-2}(y)-p^2)}}, \quad T(p) = \int_{0}^{y(p)} \frac{dy}{u(y) \cos x(y)} = \int_{0}^{y(p)} \frac{dy}{u^2(y) \sqrt{(u^{-2}(y)-p^2)}},
\]

it is easy to show the following: for \( p \in \Pi \) the derivatives \( X'(p) \) and \( T'(p) \) exist and are continuous, and \( T'(p) = p X'(p) \); for \( p \in \Pi' \)

\[
X(p) = X(p+0) = T(p) = T(p+0) = +\infty.
\]

Let us take \( \Pi_0 \) for the subset of \( \Pi \), where \( X'(p) = 0; \Pi_0 \) is closed, and, according to 1.3, it is non-dense in \( \Pi \).

Now the following property of the travel-time curve \( \Gamma \) can be proved. Suppose \((x_0, t_0)\) is a point on the smooth arc \( \gamma \) of \( \Gamma \). Let us write \( s \) for the angle between \( \gamma \) and \( x \)-axis at \((x_0, t_0)\). Then, \( 2X(p) = x_0, 2T(p) = t_0 \), if \( p = \tan s \in \Pi + \Pi' \). Consequently, one can find \( X(p), T(p) \) from a given travel-time curve, at first for all \( p \in \Pi - \Pi_0 \) and then, from continuity, for \( p \in \Pi_0 + \Pi' \). Thus, \( X(p), T(p) \) can be determined unambiguously for all \( p \in (0, 1) \) except, maybe, the finite number of points.
3. The conditions required for \( u(y) \) being a solution

3.1. Let us introduce the following notations:

\[
\phi(q) = \frac{2}{\pi} \int_0^1 \frac{X(p) \, dp}{\sqrt{(p^2-q^2)^2}}, \quad \tau(q) = \int_0^{y(q)} \sqrt{(u^{-2}(y)-q^2)} \, dy, \quad q \in (0, 1);
\]

\[
\psi(q) = \sum_{i=1}^{k} \frac{2}{\pi} \int_{y_i}^{y_i} \arctan \left( \frac{u^{-2}(y)-p_i^{-2}}{p_l^{-2}-q^2} \right) \, dy,
\]

\[
\sigma_k = \int_{y_k}^{y_{k+1}} \sqrt{(u^{-2}(y)-p_k^2)} \, dy, \quad q \in (p_{k+1}, p_k), \quad 1 \leq k \leq n.
\]

Addition of the symbol* to these notations means that \( u(y), Y(q), y_k \) and \( y_k \) are replaced by \( u^*(y), Y^*(q), y_k^* \) and \( y_k^* \) respectively.

3.2. It is clear that \( \tau(p) \) has discontinuities of magnitude \( \sigma_k \) in the points \( p_k \),

\( 1 \leq k \leq n \), and is continuous in all other points. \( p_k \) and \( \sigma_k \) are known, because \( \tau(p) = T(p) - pX(p) \). And the data on \( \tau(p) \) are mutually equivalent, because \( \tau'(p) = -X(p) \).

3.3. Let us show now, that \( Y(p) = \phi(p) + \psi(p) \), \( p \in (0, 1) \).

Let us write \( D_\phi, D_\psi, D^k \) for the following sets on \((y, p)\)-plane respectively (Fig. 3):

\[
\{0 < y < Y(q), \quad q < p < u^{-1}(y)\},
\]

\[
\{0 < y < Y(q), \quad q < p < f(y)\},
\]

\[
\{y \in j_k, \quad p_k \leq p < u^{-1}(y)\}, \quad k = 0, 1, \ldots, n.
\]

Introduce

\[
F(p, y, q) = (2/\pi)p[(u^{-2}(y)-p^2)(p^2-q^2)]^{-\frac{1}{2}}.
\]

If \( q \in (p_{k+1}, p_k), 0 \leq k \leq n \), then

\[
Y(q) = \int_0^{y(q)} \left[ \int_q^{\tau(q)} F(p, y, q) \, dp \right] \, dy = \int_{p_{k+1}}^{p_k} F \, dS + \sum_{i=0}^{k} \int_{p_{i+1}}^{\tau(q)} F \, dS = \phi(q) + \psi(q).
\]

In particular, \( Y(q) = \phi(q) \) if \( q \in (p_1, 1) \).

3.4. Suppose, we succeed in choosing the intervals \( j_k^* = (y_k^*, \tilde{y}_k^*) \), \( 1 \leq k \leq n \), and can choose and define \( u^*(y) \) on them in such a way, that:

(a) \( \sigma_k^* = \sigma_k \),

(b) \( Y^*(q) = \phi(q) + \psi^*(q) \) does not increase with \( q \),

(c) \( Y^*(p_k+0) = y_k^*, \quad Y^*(p_k-0) = \tilde{y}_k^* \) for any \( k = 1, \ldots, n \).

Let us determine \( u^*(y) \) outside \( j_k^* \) from \( Y^*(q) \) in such a way, that

\[ Y^*(p) = \inf \{ y, pu^*(y) \geq 1 \}. \]

One can prove then that \( \tau^*(y) = \tau(y) \), i.e. \( u^*(y) \) is the solution to our problem. It is clear that any solution can be obtained by this procedure.
4. The set of solutions

4.1. $\psi^*(q)$ depends on $h_i$ only, and does not depend on $y^*_i$, $\bar{y}^*_i$ under the following conditions:

\[ y^*_i < y^*_i + h_i \leq \bar{y}^*_i, \]
\[ u^*(y) = u_i = \text{const} \]

when

\[ y \in (y^*_i, y^*_i + h_i) \]

and

\[ u^*(y) = p_i^{-1} \]

when

\[ y \in (y^*_i + h_i, \bar{y}^*_i), \quad 1 \leq i \leq n. \]

Suppose, in addition, that $h_i$ are small, and $h_i \sqrt{(u_i^{-2} - p_i^2)} = \sigma_i$; then we have:

\[ \psi^*(q) < \psi'(q) \]

and $Y^*(q) = \phi(q) + \psi^*(q)$ does not increase on $(0, 1)$. We assume $y^*_k = Y^*(p_k + 0)$, $\bar{y}^*_k = Y^*(p_k - 0)$ and determine $u^*(y)$ outside $y^*_k$ from $Y^*(q)$. From 3.4 it follows, that $u^*(y)$ is a solution of our problem. Different $u^*(y)$ correspond to different sets of $(h_1, \ldots, h_n)$. For each $\varepsilon > 0$ and for sufficiently small $h_i$ we have

\[ 0 < Y^*(q) - \phi(q) < \varepsilon. \]

4.2. Write $u^0(y)$ and $u^1(y)$ for any two different solutions; introduce

\[ Y^i(p) = \inf \{y, pu^i(y) \geq 1\}, \quad i = 0, 1. \]

Take an arbitrary value $\omega \in [0, 1]$, and introduce the following functions and numbers:

\[ Y^\omega(p) = (1 - \omega)Y^0(p) + \omega Y^1(p), \]
\[ y^\omega_k = Y^\omega(p_k + 0), \quad \bar{y}^\omega_k = Y^\omega(p_k - 0), \]
\[ s^\omega_k = (1 - \omega)Y^0(p_k - 0) + \omega Y^1(p_k - 0). \]

Now let us determine the function $u^\omega(y)$ in the following way:

for

\[ y^\omega_k < y < s^\omega_k \quad u^\omega(y) = u^0\left[\frac{y - y^\omega_k}{1 - \omega} + y^\omega_k\right], \]
\[ s^\omega_k < y < \bar{y}^\omega_k \quad u^\omega(y) = u^1\left[\frac{y - s^\omega_k}{\omega} + y^\omega_k\right]. \]

Then $u^\omega(y)$ is determined from $Y^\omega(p)$ outside the intervals $(y^\omega_k, \bar{y}^\omega_k)$. It follows from 3.4 that $u^\omega(y)$ is a solution for any $\omega \in [0, 1]$.

4.3. Suppose

\[ F^i_k(r) = \text{mes } \{y, y \leq Y^i(p_k + 0), u^i(y) \leq r\}, \quad i = 0, 1, \quad 1 \leq k \leq n + 1. \]

It follows from 3.3 that

\[ Y^0(p) = Y^1(p) \quad \text{for } p \in (p_k + 1, p_k) \]
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\[ F^0_{k+1}(r) = F^1_{k+1}(r) \quad \text{for} \quad r \leq p_k^{-1}. \]

The inverse statement is also correct:

\[ F^0_{k+1}(r) = F^1_{k+1}(r) \quad \text{for} \quad r \leq p_k^{-1} \]

if on some interval \([a, b] \subset (p_{k+1}, p_k)\) we have

\[ Y^0(p) = Y^1(p) \]

for \(p \in [a, b]\).

4.4. Let us write \(\{u_\nu(y)\}\) for the set of all solutions. We introduce the functions:

\[ Y_\nu(p) = \inf \{y, p u_\nu(y) \geq 1\}, \]
\[ V(p) = \sup Y_\nu(p), \quad N(p) = \inf Y_\nu(p), \]
\[ H_k = V(p_k - 0) - N(p_k - 0), \quad 1 \leq k \leq n. \]

It follows from 3.3 and 4.1 that

\[ N(p) = \phi(p), \quad p \in (0, 1). \]

It is evident that \(Y_\nu(p) \leq T(p)/p\); consequently

\[ V(p_k - 0) \leq \inf \{T(p)/p, p < p_k\} < \infty \]

for any \(k = 1, \ldots, n\).

It is easy to check that

\[ V(p) \leq N(p) + H_k \]

for \(p \in (p_{k+1}, p_k)\).

4.5. Let us consider the set \(G\) on the plane \(y, u\). The point \((y, u)\) belongs to \(G\), if \(1 \leq u \leq p_1^{-1}, y \in (Y(u_1^{-1} + 0), Y(u_1^{-1} - 0))\)

or if \(u > p_1^{-1}, N(u^{-1}) < y < V(u^{-1})\)

or if \(0 < u \leq p_1^{-1}, N(p_k + 0) < y < V(p_k - 0)\).

It follows from 4,

(a) The solution \(u_\nu(y)\), for which \(u_\nu(y_0) = u_0\) can be found for any point \((y_0, u_0) \in G\).

(b) For any \(u_\nu(y)\) and for any \(y \geq 0\) the point \((y, u_\nu(y)) \in G\).

5. Conclusion

The basic results of this paper will be summarized now in a less formal manner than above.

5.1. The problem of determination of the velocity–depth distribution \(v(r)\) from a travel-time curve in a spherically symmetric Earth is reduced to a problem of determination of velocity \(u(y)\) in a halfplane \(y > 0\). It is done by introducing the following variables:

\[ x = \frac{R \theta}{v(R)}, \quad y = \frac{R \ln R/r}{v(R)}, \quad u(y) = \frac{v(R e^{-yv(R)/R})}{v(R) e^{-yv(R)/R}}. \]

Then we have to consider a new travel-time curve \(\Gamma\) on \((x, t)\)-plane, instead of the real one on \((\theta, t)\)-plane.
5.2. The functions $X(p), T(p)$ are determined from a given $\Gamma$ in the following way: let $(2X, 2T)$ be the point of $\Gamma$ and let $I$ be the tangent to $\Gamma$ at this point; then we put: $p = \tan (l, x)$. The function $\tau(p) = T(p) - pX(p)$ is introduced. Geometrically $\tau(p)$ is the ordinate of the point where $I$ meets the $t$-axis. We denote the points of discontinuity of $\tau(p)$ by $p_1, ..., p_n$ and introduce:

$$\sigma_k = \tau(p_k - 0) - \tau(p_k + 0).$$

5.3. Our problem is to determine the velocity $u(y)$ from the given $X(p), T(p)$.

This problem does not have a unique solution: a whole set $\{u, (y)\}$ of solutions exists.

The necessary and sufficient conditions for $u, (y)$ to become the solution are described in section 3 (see 3.3 and 3.4).

The set $G$ on $(y, u)$-plane, which is formed by the plots of all solutions, is described in section 4 (see 4.5). A giraffe-like example of such a set for the case of two low-velocity zones is shown on Fig. 4.

![Fig. 4.](https://academic.oup.com/gji/article-abstract/11/1/165/720284)

We can precisely indicate the depth $y_1$, where the first waveguide starts, and one can strictly determine the velocity $u(y)$ for $y \in (0, y_1)$ by the Herglotz–Wiechert method. We further approximately indicate the depth at which for the first time (during the movement downward from the surface) any value of velocity (greater than on the Earth’s surface) is achieved. In order to present it more strictly let us introduce

$$N(p) = \frac{1}{\pi} \int_{\frac{1}{\sqrt{(q^2 - p^2)}}} X(q) dq,$$  

$$\bar{H}_k = \inf \left\{ \frac{T(p)}{p}, p < p_k \right\} - N(p_k - 0), \quad 1 \leq k \leq n,$$  

and let $u$ be the random number from the interval $(p_k^{-1}, p_{k+1}^{-1})$, then: (1) the velocity shall by all means acquire the value $u$ on the interval $N(u^{-1}) \leq y \leq N(u^{-1}) + \bar{H}_k$; (2) with $y < N(u^{-1})$ the velocity is strictly less than $u$. Finally, the position of the waveguides (numbers $y_2, ..., y_m, \tilde{y}_1, ..., \tilde{y}_n$) is also determined only approximately. However, we know the integral characteristic of the velocity $\sigma_k$, the critical value

*This definition is not quite strict: for example, it may be that $I$ does not exist. For strict analysis see 2.3.*
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$p_k^{-1}$, at which the velocity ceases to grow monotonously (for the waveguide $j_k$), and the number of the waveguides $n$ (equal to the number of discontinuities of $\tau(p)$). It is impossible to determine the minimum velocity in the waveguide.

5.4. In order to reduce the ambiguity in the determination of the velocity $u(y)$, we can use the travel-time curves from deep sources. The following theorem is true: if the travel-time curves from the surface and several deep sources are known, and if between any two neighbouring waveguides a source is located, then (see Fig. 5):

- the positions of the waveguides $j_k$ (numbers $y_1, \bar{y}_1, \ldots, y_n, \bar{y}_n$) are determined precisely;
- the velocity $u(y)$ between the waveguides (at $y \in (\bar{y}_k, y_{k+1})$, $0 \leq k \leq n$) is obtained unambiguously;
- $u(y)$ in the waveguides cannot be unambiguously determined, but the following functions are known:

$$F_k(z) = \text{mes} \{y, y \in j_k, \quad u(y) \leq z\}, \quad 1 \leq k \leq n,$$

in particular, for every $j_k$ we can find $\inf \{u(y), y \in j_k\}$.

The proof of this theorem (as well as the theorems formulated without proof in the article) can be found in our paper Determination of the velocity of seismic wave propagation by the travel-time curve, Trudy Inst. Fiz. Zemli, Vychislitel'naya Seismologiya, No. 3 (in press).

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