Studies on the Stochastic Problem of Electron-photon Cascades

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A brief review of the well known fluctuation problem of cosmic radiation is given in a proper historical setting. We also present our latest numerical calculations for the second moments of the electron distribution as a sequel to those of Bhabha and Chakrabarty for the mean number.

Introduction

The cascade theory* of cosmic ray showers first put forward by Bhabha and Heitler (A. 1) and independently by Carlson and Oppenheimer (A. 2) and subsequently developed by many authors, adequately describes the average behaviour of electron-photon cascades initiated by a fast electron or photon in its passage through matter. It has been recognised from the beginning that, since radiation and pair creation processes are governed by probability laws, cascade multiplication is essentially a stochastic process, a study of which can be complete only when considered from a statistical point of view. Since the Bethe-Heitler cross-sections for radiation and pair creation are continuous in the energy variable it has been realised that the main difficulty in such a treatment lies in the development of techniques to deal with the very difficult mathematical problem relating to the stochastic variable representing the number of particles distributed in a continuous infinity of states characterised by the energy parameter. This problem has attracted the attention of many workers and many powerful mathematical techniques have been devised. More than forty papers have been published but unfortunately in many of them, inadequate references have been made to earlier or contemporary work. Out of the large number of papers on this subject, only two deal in detail with numerical results relating to the electron-photon cascade—that of Scott and Uhlenbeck (B. 6) and of Janossy and Messel (D. 4). Thus the object of the present paper is twofold:

1. To give a brief summary of the developments of the mathematical techniques in their proper historical setting;

2. To present our latest numerical results** based upon the paper of Bhabha and Ramakrishnan (C. 4).

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* At the end of this study, a comprehensive classified list of papers on this subject is given. In the course of the paper the references are indicated in brackets. These relate to the classified list.

** In a letter to this Journal (June 1953) we first announced some numerical results for the mean square number based on (C. 4). Since writing that letter, we discovered a mistake in (C. 4) which materially altered the results. The present calculations, made after correcting the mistake supersede the previous results.
Our results for the mean square number of electrons form a natural sequel to those for the mean number given by Bhabha and Chakrabarthy (A. 11). A brief comparison with the calculations of Scott and Uhlenbeck and those of Janossy and Messel will be made at the end of this paper.

PART I.
Mathematical techniques developed in the treatment of the fluctuation problem

Throughout this paper we ignore the lateral spread of the shower and treat the problem as one-dimensional as regards spatial distribution. The stochastic problem of cosmic ray showers can then be stated in the simplest mathematical terms as follows:

Statement of the problem

Given the initial energy spectrum of photons and electrons at thickness $t=0$ and that

(a) an electron of energy $E$ radiates a quantum and has residual energy between $E'$ and $E'+dE'$ with a probability per unit thickness $R^{(a)}(E',E)dE'$,

(b) a photon of energy $E$ forms an electron pair one of which has energy between $E'$ and $E'+dE'$ with probability per unit thickness $R^{(b)}(E',E)dE'=2R'(E',E)dE'$, (The factor 2 occurs since we do not distinguish between positive and negative electrons.)

(c) an electron with energy $E$ loses energy deterministically of magnitude $\beta(E)dt$ in its passage through a thickness $dt$.

We must calculate $\pi(n, E_1, m, E_2; t)$* the probability that there are $n$ electrons with energy greater than $E_1$ and $m$ photons with energy greater than $E_2$.

If we neglect process (c) i.e., ionisation, we shall call it Approximation A. If we include process (c) but assume $\beta$ to be independent of $E$ we shall it Approximation B.

It is realised that $\pi(n, E_1, m, E_2; t)$ does not specify the statistical distribution in a manner satisfactory to the mathematician, but from a physical point of view it adequately describes the essential features of the process. The numerical evaluation of $\pi(n, E_1, m, E_2; t)$ itself is a large task which has yet to be achieved. The present numerical calculations are confined to the second moments of the number of electrons or photons above a specified energy.

The method that immediately suggests itself is the classical method which consists in expressing the 'state' of the system at $t+dt$ in terms of the 'state' at $t$ or in other words in studying the behaviour of the system in an infinitesimal interval $dt$ in terms of the 'state' of the system at $t$. This method is applicable only to Markovian processes.

* Throughout this paper the symbol $\pi$ is used to denote a probability distribution function. When two or more functions are denoted by $\pi$, they will be distinguishable from the context.
Studies on the Stochastic Problem of Electron-photon Cascades

But the function \( \pi(n, E; t) \) does not define a Markov process since the information that there are \( n \) electrons above \( E_1 \) and \( m \) photons above \( E_2 \) at \( t \) is insufficient to predict their behaviour in \( dt \) and a more complete description of the 'state' at \( t \) is necessary before we can apply the classical method. If the energy space were discrete, characterised by \( E_1, E_2, \ldots, E_n, \ldots \), then it would be possible to define a function

\[
\pi(n_1, n_2, \ldots, n_i, \ldots, m_1, m_2, \ldots, m_i, \ldots; t)
\]

representing the probability that there are \( n_i \) electrons and \( m_i \) photons in \( E_i \), \( i = 1, 2, \ldots \). This would be an adequate description of the 'state' of the system to predict its behaviour in interval \( dt \). If the \( E \)-space is a continuum, no such function can be defined.

This difficulty was realised when the cascade theory was put forward, and so attempts were made to replace the cosmic ray process by simple statistical models. In their first paper, Bhabha and Heitler (A. 1) assumed a Poisson distribution for \( N(E) \), the number of electrons above a specified energy \( E \), i.e., if \( \pi(n, E; t) \) is the probability that there are \( n \) electrons above the energy \( E \) at \( t \)

\[
\pi(n, E; t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!},
\]

This was equivalent to the statement that the electrons in the various energy states are "uncorrelated"—an unreal assumption since the electrons are all generated from the initial parent, the incident particle. Furry (B. 1) was the first to take into account the multiplicative nature of the process and he obtained a distribution function on the basis of a simple multiplicative model which yielded fluctuations of much larger magnitude than the Poisson distribution. Assuming for simplicity, that there is only one type of particle having a probability \( B \) of splitting into two per unit distance of travel, he obtained an expression for the probability that there are \( n \) particles at \( t \),

\[
\pi(n; t) = \frac{1}{e\{N\}} \left(1 - \frac{1}{e\{N\}}\right)^{n-1},
\]

where

\[
e\{N\} = e^{nt},
\]

\[
e\{N^2\} - [e\{N\}]^2 = [e\{N\}]^2 - e\{N\}.\]

Other attempts were made by Euler (B. 3), Nordseik, Lamb and Uhlenbeck (B. 5) and a lengthy discussion of this problem is to be found in the book of Arley (B.4).

The first attempt at a statistical treatment of the development of the shower taking

\* There is no suggestion that \( E_1, E_2, \ldots, E_n, \ldots \) are ordered. They merely indicate the discrete values of the energy in the entire range.

\** Throughout this paper \( e \) denotes expectation value.
into account the energy distribution of the particles was made by Scott and Uhlenbeck. To avoid the well known complications outlined earlier, they made the following assumptions.

1. Electrons and photons exist in discrete states each with a definite energy.
2. A limiting process exists by which the number of such states can be made as large as we please.

This in fact corresponds to the well known procedure in quantum mechanics in which one considers all the phenomena to take place in a large box of dimension $L$. The states of the electrons and photons are then discrete and their number tends to infinity as $L$ tends to infinity. Scott and Uhlenbeck defined a 'Master function' $\pi(n_1, n_2, \ldots)$ representing the probability there are $n_i$ electrons and $m_i$ photons with energy $E_i$, $i=1, 2, \ldots$ and from these other statistical functions are defined by taking suitable average values. An identical procedure was adopted by Bhabha and Ramakrishnan* which ultimately led them independently to the formulation of the "theory of product densities" which deals directly with the continuous case and yields the same equations without the aid of a laborious limiting process. The equations of Bhabha and Ramakrishnan differ from those of Scott and Uhlenbeck only in that they take into account the Bethe-Heitler cross sections but their paper goes further in strictly carrying the calculations to the end.

Besides the theory of product densities, three other powerful techniques have been developed in the treatment of this problem which we shall consider in order of their historical development.

1. The method of product density functions

This method, independently put forward by Bhabha (C. 2) and Ramakrishnan (C. 1) can be briefly stated as follows. Here we use the notation of Ramakrishnan.

If particles are distributed in the $E$-space which is a continuum, we define a function $f_n(E_1, E_2, \ldots, E_n)$ called the product density of degree $n$, such that $f_n(E_1, E_2, \ldots, E_n) \cdot dE_1 dE_2 \cdots dE_n$ represents the probability that there is a particle in $dE_1$, a particle in $dE_2$, and a particle in $dE_n$. $f_n$ is not a probability density but $f_n(E_1, E_2, \ldots, E_n) \cdot dE_1 dE_2 \cdots dE_n$ is a probability magnitude.

If $N(E)$ is the stochastic variable representing the number of particles with energy above $E$ then

$$\epsilon \{[N(E) - N(E_0)]^r\} = \sum_{s=1}^{r} C^r_s \int_{E_0}^{E} \cdots \int_{E_0}^{E} f_n(E_1, E_2, \ldots, E_n) dE_1 dE_2 \cdots dE_n, \quad (4)$$

where $C^r_s$ are coefficients defined by the identity

$$M^r = \sum_{s=1}^{r} C^r_s M(M-1) \cdots (M-s+1), \quad (5)$$

$M$ being a positive integer. The coefficients are obtained by putting $M=1, 2, \ldots$.

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* For a full account of this, see—Alladi Ramakrishnan, Ph. D. Thesis (1951) University of Manchester.
It is clear from (4) that for the calculation of the $n$-th moment of $[N(E) - N(E_0)]$, we should know the product densities of degree less than or equal to $n$. In the cosmic ray problem since we are dealing with two types of particles, electrons and photons, we define $f_n(E_1, E_2, \ldots, E_n; t)$ and $g_n(E_1, E_2, \ldots, E_n; t)$ as the product densities of degree $n$ of electrons and photons respectively and $f_{n,m}(E_1, E_2, \ldots, E_n; E_{n+1}, \ldots, E_{n+m}; t)$ as the mixed product density of degree $(n,m)$ of electrons and photons, i.e., $f_{n,m}(E_1, E_2, \ldots, E_n; E_{n+1}, \ldots, E_{n+m}; t)dE_1dE_2\cdots dE_n dE_{n+1} \cdots dE_{n+m}$ is the probability that there is an electron in $dE_1$, an electron in $dE_2$, \ldots, and an electron in $dE_n$, a photon in $dE_{n+1}$, a photon in $dE_{n+2}$, \ldots, and a photon in $dE_{n+m}$.

Then it can be shown that since $f_n(E_1, E_2, \ldots, E_n; t) dE_1dE_2\cdots dE_n$ is a probability magnitude, its variation with $t$ can be expressed in terms of the $f_n$, $g_n$, and $f_{n,m}$ functions. Bhabha has formally written down the equations for $f_n$, $g_n$, and $f_{n,m}$ but equations involving product densities of order greater than two are not of immediate interest since the analytical solution of $f_n$, $g_n$ and $f_{1,1}$ is itself a long and tedious process which has been obtained by Bhabha and Ramakrishnan under approximation A. Section II of the paper deals with numerical results based upon their solution.

We must admit that even if we know $f_n$, $g_n$, and $f_{n,m}$ for every $n$, $m$, our knowledge of the statistical distribution of electrons or photons is still incomplete. A more complete description is achieved by the means of the characteristic functional which will be discussed a little later.

(2) Method of regeneration points

To get round the difficulties now outlined, Janossy (D. 2) employed an entirely different technique which, it was realised later, was employed previously by Bellman and Harris (D. 1) and C. Palm (D. 2) in relation to simpler stochastic processes. A complete account of the historical evolution of this "method of regeneration points" is given in a paper of Bartlett and Kendall (D. 5). But it is to Janossy we owe its ingenious application to the vexed problem of cosmic radiation.

Janossy* defined a function $\Phi^{(i)}(E, N; t; E_0)$ as the probability that at thickness $t$ there are $N$ electrons each with energy greater than $E$ where $E_0$ is the initial energy of the incident particle. $i=1$ refers to an electron-initiated shower while $i=2$ refers to a photon-initiated shower. The method of regeneration points uses the following argument: somewhere at a thickness less than $t$ (and this is called a regeneration point) the incident particle may radiate a pair and these secondaries and primary together become independent primaries of stochastic processes. By this argument we are led to the following integral equation

$$
\Phi^{(i)}(E, N; t; E_0) = \sum_{N'+N''=N} \int_0^t dt' \int_R e^{-a(E_0)t'} R^{(i)}(E', E_0) \Phi^{(i)}(E, N'; t-t'; E').
$$

* For an account of this problem and other applications of the method of regeneration points see Alladi Ramakrishnan (D. 6).
\[ \Phi^{(m)}(E, N'; t-E')dE + A(N-2+i)e^{-\alpha t(E_0)t}, \]

where \( m \) a non-negative integer; \( A(m)=1 \) if \( m=0 \), and otherwise zero.

At \( t=0^+ \)

\[ \Phi^{(1)}(E, 1; t; E_0) = 1, \]
\[ \Phi^{(1)}(E, N; t; E_0) = 0 \text{ if } N+1, \]
\[ \Phi^{(2)}(E, 0; t; E_0) = 1, \]
\[ \Phi^{(2)}(E, N; t; E_0) = 0 \text{ if } N+0. \]

Differentiating the above equation with respect to \( t \), we have

\[ \frac{\partial \Phi^{(0)}(E, N; t; E_0)}{\partial t} = -\Phi^{(0)}(E, N; t; E_0)u(E-t) + B \int_{E'}^{E_0} \Phi^{(i)}(E, N'; t; E') \Phi^{(0-i)}(E, N''; t; E_0-E')R^{(0)}(E', E_0)dE'. \]

It was noticed by Ramakrishnan** that for processes homogeneous and Markovian with respect to \( t \), the above method consists in considering the change in the first infinitesimal interval \( 0 \) to \( \Delta \), and expressing the state of the system at \( t \) in terms of the state at \( \Delta \) as contrasted with the classical method of expressing the state at \( t+\Delta \) in terms of the state at \( t \). The main advantage of this method is that it is simpler to study the change in the system during the first infinitesimal interval \( \Delta \). The striking contrast between these two approaches was observed much earlier by Kolmogoroff*** when he derived the backward and forward differential equations of the function \( \pi(s_j; t; s_i) \) representing the probability that a system is in the state \( s_j \) at \( t \) given that it was in \( s_i \) at \( t=0 \). In any stochastic process if we interpret the 'state' of the system suitably, the backward and forward differential equations may be written down directly. \( \pi(s_j; t; s_i) \) is defined by the Kolmogoroff equations only if we know \( R(s_j, s_i) \) where \( \pi(s_j; t; s_i) \rightarrow R(s_j, s_i)t \) as \( t \rightarrow 0(j \neq i) \). But there may occur stochastic processes where it is not possible to know the limit \( R(s_j, s_i) \) for all but a few simple \( s_i \)'s.

At \( t=0 \), the state of the system \( s_i \) will be defined in a simpler manner than at some \( t \), and so it may be possible to study the transitions from this initial state but not those from the state at \( t \). In such a case we can write only the backward differential equations of the process.

It is to be noted that even for product densities backward differential equations can be obtained. But in this case, it is easier to solve the forward equations of Bhabha and Ramakrishnan since the initial conditions of the product densities is the same inside and outside the integral sign in their integro-differential equations.

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* In reference (D. 6) the initial conditions stated are wrong, but fortunately the mistake was not carried over into the subsequent calculations.

** See M. S. Bartlett, reference (D. 4).

*** For an account of this method see reference (H. 4).
Studies on the Stochastic Problem of Electron-Photon Cascades

(3) The method of Janossy functions

In an attempt to treat the fluctuation problem of nucleon cascades* by a method different from that outlined just now, Janossy (E. 1) defined a function \( \phi_h(E_1, E_2, \ldots, E_n) \cdot dE_1 \cdot dE_2 \cdots dE_h \) representing the probability that there is one particle in \( dE_1 \), one in \( dE_2 \), \ldots, and one in \( dE_h \) and none in the other energy states. Note that \( \phi_h \) is not a probability density. Janossy in fact dealt with \( \phi_h/h! \) but it is \( \phi_h \) that has a physical significance since the particles are indistinguishable.

The Janossy functions \( \phi_h \) bear a close relationship to the product density function** \( f_h \) and we give here without proof the following result

\[
f_h(E_1, E_2, \ldots, E_n) = \sum_h \frac{1}{(n-h)!} \int_{E_{n+1}} \cdots \int_{E_{n+m}} \phi_n(E_1, E_2, \ldots, E_n) dE_{n+1} \cdots dE_{n+m}.
\] (9)

Forward differential equations for \( \phi_h \) can be written down directly but equations for \( \phi_h \) will involve \( \phi_m(m > h) \). In the case of product densities equations for \( f_h \) will involve densities of order less than \( h \) and hence successive solution of \( f_h \) is possible.

The equations of the nucleon cascade problem have been derived using the two methods and their equivalence established by Ramakrishnan (G. 1) and by Messel and Potts (G. 2).

In the case of the electron-photon cascade, it is clear that we have to define \( \phi_{n,m}(E_1, E_2, \ldots, E_n; E_{n+1}, E_{n+2}, \ldots, E_{n+m}) dE_1 \cdots dE_{n+m} \) as the probability that there is an electron in \( dE_1 \), one in \( dE_2 \), \ldots, one in \( dE_n \), a photon in \( dE_{n+1} \), a photon in \( dE_{n+2} \), \ldots, and a photon in \( dE_{n+m} \). This function was used extensively by Messel and his collaborators.

(4) The characteristic functional method

The characteristic functional is a powerful mathematical device to study the joint distribution of a continuous infinity of stochastic variables. If the stochastic variable \( N(E) \) represents the number of particles with energy above \( E \) then symbolically \( -dN(E) \) is a stochastic variable which represents the number in \( dE \). Then, if

\[
C[\theta(E), \xi] = \exp \{ -i \int_{E} \theta(E) dN(E, \xi) \},
\] (10)

\( C[\theta(E), \xi] \) is called the characteristic functional and \( \theta(E) \) its argument. "The characteristic functional, when it can be determined contains in portmanteau form all the probability relationships associated with the state of the 'population' of particles at \( t \)."

The characteristic functional in the cosmic ray problem must necessarily involve two argument functions and so we shall write

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* The problem of nucleon cascades differs from that of electron-photon cascades in that we deal with only one type of particle.
** For proof of (9), see reference (G. 1).
\[ C^{(\alpha)}[\theta(E), \chi(E); t] = e^{-i \int_E^\infty \{ \theta(E) \, dN^{(\alpha)}(E, \tau) + \chi(E) \, dM^{(\alpha)}(E, \tau) \}}; \]

(11)

\(-dN^{(\alpha)}(E, \tau)\) and \(-dM^{(\alpha)}(E, \tau)\) are stochastic variables representing respectively the number of electrons and photons in \(dE\). \(i = 1\) refers to an electron-initiated shower; \(i = 2\) to a photon-initiated shower.

By using the now familiar arguments of the regeneration point method, Bartlett and Kendall (D. 5) obtained the equation

\[
\begin{align*}
C^{(\alpha)}[\theta(E), \chi(E); t; E_0] &= \int_0^{E_0} \int_0^\infty e^{-a_\tau(E_0) \tau} C^{(\alpha)}[\theta(E), \chi(E); t - \tau; E'], \\
C^{(\alpha-\delta)}[\theta(E), \chi(E); t - \tau; E_0 - E'] \, R^{(\alpha)}(E', E_0) \, dE' \, d\tau + e^{-a_\tau(E_0) \tau} \, e^{i \theta(E_0)}.
\end{align*}
\]

(12)

Putting

\[
\chi(E) \equiv 0
\]

and

\[
\theta(E') \equiv 0, \ (E' < E)
\]

\[
\equiv -i \log u, \ (E' > E)
\]

(13)

we get Janossy's equation (6).

Differentiating with respect to \(t\) we get

\[
\frac{\partial C^{(\alpha)}[\theta(E), \chi(E); t; E_0]}{\partial t} = -C^{(\alpha)}[\theta(E), \chi(E); t; E_0] \, \alpha^{(\alpha)}(E_0)
\]

\[
+ \int_0^{E_0} C^{(\alpha)}[\theta(E), \chi(E); t; E'] \, C^{(\alpha-\delta)}[\theta(E), \chi(E); t; E_0 - E'] \, R^{(\alpha)}(E', E_0) \, dE'.
\]

(14)

(5) **Analytical solutions of the cascade equations**

The analytical solutions of the equations by various methods have been attempted only in the case when \(R^{(\alpha)}(E', E) \, dE'\) can be expressed in the form \(R^{(\alpha)}(q) \, dq, \ q = E'/E\). The method of attack essentially consists in using the Mellin's transform technique and reducing the integro-differential equations to differential equations. We here summarise the various attempts that have been made.

Bhabha and Ramakrishnan have obtained explicit analytical solutions for product-densities of degree two. The solutions of equations involving product densities of higher order have yet to be obtained.

Janossy attempted to solve his equation by using the method of generating functions but has succeeded only in obtaining formal expressions for the moments of the distribution. Messel and Gardner (F. 12) have claimed that the formal solution of Janossy's equation can be obtained without the aid of generating functions.

Messel and his co-workers have made a series of contributions to the analytical solution of the cascade equations (including ionisation loss) based on Janossy's functions by an ex-
The solution of the characteristic functional equation is yet to be attempted.

PART II.

The analytical solution of Bhabha and Ramakrishnan

As has been pointed out in the introduction, while the formal mathematical treatment of the fluctuation problem has been completed, numerical calculations have been confined only to the second moments of the electron distribution in the case of Approximation A*. We present here our latest numerical results based on the product density equations of Bhabha and Ramakrishnan (C. 4). We shall summarise their results to the extent necessary for a proper appreciation of our numerical calculations. For the sake of completeness we will reproduce the product density equations.

If \( f_1 \) and \( f_2 \) are the product densities of electrons of order one and two, \( g_1 \) and \( g_2 \) of photons of order one and two, and \( f g_{1,1} \) the mixed product density of electrons and photons, they satisfy the following equations**.

**Product-density-Degree 1:**

\[
\begin{align*}
\frac{\partial f_1(E; t)}{\partial t} &= -f_1(E; t) \int_0^E R^{(1)}(E', E) dE' + \int_0^E f_1(E'; t) R^{(1)}(E, E') dE' \\
&\quad + 2 \int_0^E g_1(E'; t) R^{(2)}(E, E') dE', \\
\frac{\partial g_1(E; t)}{\partial t} &= -g_1(E; t) \int_0^E R^{(2)}(E', E) dE' + \int_0^E f_1(E'; t) R^{(1)}(E' - E, E') dE',
\end{align*}
\]

(15)

(16)

As stated earlier, \( R^{(2)}(E', E) = 2R'(E', E) \).

**Product density-Degree 2:**

\[
\begin{align*}
\frac{\partial f_2(E_1, E_2; t)}{\partial t} &= -f_2(E_1, E_2; t) \left\{ \int_0^{E_1} R^{(2)}(E, E_1) dE + \int_0^{E_2} R^{(2)}(E, E_2) dE \right\} \\
&\quad + \int_0^{E_1} f_2(E, E_2; t) R^{(1)}(E_1, E) dE + \int_0^{E_2} f_2(E, E_1; t) R^{(1)}(E_2, E) dE \\
&\quad + 2 \int_0^{E_1} f g_{1,1}(E_1, E; t) R^{(2)}(E_2, E) dE + 2 \int_0^{E_2} f g_{1,1}(E_2, E; t) R^{(2)}(E_1, E) dE \\
&\quad + 2g_1(E_1 + E_2; t) R^{(2)}(E_1, E_1 + E_2),
\end{align*}
\]

(17)

* Mesel and Gardner (F. 12) have recently announced that they are planning a programme of numerical work on the Ferranti electronic computer at the University of Toronto computation centre.

** For the derivation of these equations see (C. 1). It is to be noted that we have used a notation for the cross-sections different from that of (C. 1) and also of (A. 11).
The initial conditions of the problem are defined thus:

At $t=0$, there is only one primary of energy $E_0$. Therefore at $t=0$

$$f_1(E; 0) = \delta(E - E_0), \quad g_1(E; 0) = 0,$$

$$f_2(E_1, E_2; 0) = 0, \quad g_2(E_1, E_2; 0) = 0, \quad f_{g1,1}(E_1, E_2; 0) = 0 ,$$

$R(E'E)$ and $R'(E', E)$ are the Bethe-Heitler cross-sections given by

$$R(E', E) = \left( \frac{E - E'}{E} \right) \frac{1}{E} \left( 1 - \left( \frac{4}{3} + \alpha \right) \frac{E}{E - E'} \right),$$

$$R'(E', E) = \left( 1 - \left( \frac{4}{3} + \alpha \right) \frac{E'}{E} \right) \frac{1}{E} \left( \frac{E - E'}{E} \right).$$

(15) and (16) are the well known equations of the cascade theory for the mean number. (17) to (20) have been completely solved by Bhabha and Ramakrishnan using the Mellin's transforms,

$$\nu(r; t) = \int_0^\infty E^{r-1} f_1(E; t) dE, \quad \gamma(r; t) = \int_0^\infty E^{r-1} g_1(E; t) dE,$$

$$\nu_2(r, s; t) = \int_0^\infty dE \int_0^\infty dE' E^{r-1} E'^{-1} f_2(E, E'; t),$$

$$\gamma_2(r, s; t) = \int_0^\infty dE \int_0^\infty dE' E^{r-1} E'^{-1} g_{1,1}(E, E'; t).$$

(23)
They obtained the following equations* (written in matrix form)

\[
\frac{d}{dt} \phi(r; t) = \tau(r) \phi(r; t),
\]

(24)

where

\[
\phi(r; t) = \begin{pmatrix} \nu(r; t) \\ \gamma(r; t) \end{pmatrix},
\]

and

\[
\tau(r) = \begin{pmatrix} -A_r & B_r \\ C_r & -D \end{pmatrix},
\]

(25)

and

\[
A_r = \left( \frac{4}{3} + \alpha \right) \left( \frac{d}{dr} \log \Gamma(r) + \gamma - 1 + \frac{1}{r} \right) + \frac{1}{2} - \frac{1}{r(r+1)},
\]

\[
B_r = 2 \left( \frac{1}{r} - \left( \frac{4}{3} + \alpha \right) \frac{1}{(r+1)(r+2)} \right),
\]

\[
C_r = \frac{1}{r+1} + \left( \frac{4}{3} + \alpha \right) \frac{1}{r(r-1)},
\]

\[
D = \frac{7}{9} - \frac{1}{6} \alpha.
\]

(26)

\( \alpha \) is a constant characteristic of the material** on which the initial electron is incident. \( \gamma \) inside (26) represents the Euler-Mascheroni constant. The matrix \( \tau(r) \) has two eigenvalues \(-\lambda_r\) and \(-\mu_r\) given by

\[
\begin{pmatrix} \lambda_r \\ \mu_r \end{pmatrix} = \frac{1}{2} \left( A_r + D \right) \pm \frac{1}{2} \left\{ (A_r - D)^2 + 4B_rC_r \right\}^{1/2}.
\]

(27)

If

\[
\phi^{(2)}(r, s; t) = \begin{pmatrix} \nu_3(r, s; t) \\ \gamma_3(r, s; t) \\ \nu_3(r, s; t) \\ \gamma_3(r, s; t) \end{pmatrix},
\]

(28)

then

\[
\frac{d}{dt} \phi^{(2)}(r, s; t) = \left\{ \tau(r) \times 1 + 1 \times \tau(s) \right\} \phi^{(2)}(r, s; t) + \phi(r, s; t),
\]

(29)

where \( 1 \) is the \( 2 \times 2 \) unit matrix and \( \psi \) is a column vector defined by

---

* As regards the transform equations, the notation is identical with that used in reference (C. 4). The paper of Bhabha and Ramakrishnan contains a number of mistakes which have all been pointed out in the Appendix.

** In this paper we take the value of \( \alpha \) to be .0246.
\[ \phi(r, s; t) = \psi_1(r, s) E_0^{r+s-2} e^{-\lambda_r t} + \psi_2(r, s) E_0^{r+s-2} e^{-\mu_t t}, \]  

where

\[ \psi_1(r, s) = \frac{1}{\mu_{r+s+1} - \lambda_{r+s+1}} \begin{pmatrix} a_1(r, s) C_{r+s+1} \\ a_2(r, s) (D - \lambda_{r+s+1}) \\ 0 \end{pmatrix}, \]
\[ \psi_2(r, s) = \frac{1}{\mu_{r+s+1} - \lambda_{r+s+1}} \begin{pmatrix} -a_1(r, s) C_{r+s+1} \\ a_2(s, r) (\mu_{r+s+1} - D) \\ 0 \end{pmatrix}, \]  

and

\[ a_1(r, s) = 2 \frac{\Gamma(r) \Gamma(s)}{\Gamma(r+s)} \left\{ 1 - \left( \frac{4}{3} + a \right) \frac{rs}{(r+s+1)(r+s)} \right\}, \]
\[ a_2(r, s) = \frac{\Gamma(r) \Gamma(s)}{\Gamma(r+s)} \left\{ \frac{s}{r+s} + \left( \frac{4}{3} + a \right) \frac{r}{s-1} \right\}. \]  

The initial conditions of the problem require that
\[ \nu(s; 0) = E_0^{s-1}, \quad \gamma(s; 0) = 0. \]  

Thus solving (24) we obtain
\[ f_1(E; t) = \frac{1}{2\pi i E_0} \int_{-\infty}^{\infty} \left( \frac{E_0}{E} \right)^s \left\{ \frac{D-\lambda_s}{\mu_s - \lambda_s} e^{-\lambda_s t} + \frac{\mu_s - D}{\mu_s - \lambda_s} e^{-\mu_s t} \right\} ds. \]  

The mean number of electrons with energy greater than \( E \) is given by
\[ \epsilon \{ N(E) \} = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \left( \frac{E_0}{E} \right)^{s-1} \left\{ \frac{D-\lambda_s}{\mu_s - \lambda_s} e^{-\lambda_s t} + \frac{\mu_s - D}{\mu_s - \lambda_s} e^{-\mu_s t} \right\} ds. \]  

For all but small thickness we neglect the term involving \( e^{-\mu_s t} \).

The solution of the inhomogeneous equation (29) is given by
\[ \phi^{(2)}(r, s; t) = \zeta(\zeta^{-1}(r, s; t)) \int_0^t e^{-\tau(s) t} \phi(r, s; \tau) d\tau. \]  

when \( \zeta \) is the solution of the homogeneous equation
\[ \frac{d}{dt} \zeta(r, s; t) = [\tau(r) \times 1 + 1 \times \tau(s)] \zeta(r, s; t). \]  

Corresponding to the two parts \( \psi_1 \) and \( \psi_2 \) of \( \phi \), \( \phi^{(2)} \) itself splits into two parts \( \phi_1^{(2)} \) and \( \phi_2^{(2)} \). Since the only \( t \) dependent factors of the integral in (36) are the exponentials, we find easily
\[ \phi_1^{(2)}(r, s; t) = E_0^{r+s-2} \zeta_0(r, s) \eta(\lambda_{r+s+2}) \zeta_0^{-1}(r, s) \phi_1(r, s), \]
Studies on the Stochastic Problem of Electron-photon Cascades

\[ \phi_2(r, s; t) = E_0^{r+s-2} \xi_0(r, s) \gamma(\mu_{r+s-1}^-) \zeta_0^-(r, s) \psi_2(r, s), \]  

(39)

where \( \gamma(x) \) is the diagonal matrix with the four elements

\[ \begin{align*}
-\lambda_r - \lambda_s - x & \quad \text{if } x < \lambda_r + \lambda_s - x \\
- \mu_r - \mu_s - x & \quad \text{if } x > \mu_r + \mu_s - x \\
-\lambda_r - \lambda_s - x & \quad \text{if } \mu_r + \mu_s - x
\end{align*} \]

(40)

We note a very important feature of this matrix. It remains finite even if any of the denominators of its elements tends to zero. This has to be taken into account in tabulating \( \phi_2 \) for numerical calculations using the saddle point method.

From the theory of product densities the mean square number of electrons with energy greater than \( E \) is given by

\[ \varepsilon \{ N^2(E; t) \} = \varepsilon \{ N(E; t) \} + \int_B \int_B f_2(E_1, E_2; t) dE_1 dE_2, \]

(41)

\[ f_2(E_1, E_2; t) = \frac{1}{(2\pi i E_0)^2} \int_{\sigma - i \infty}^{\sigma + i \infty} \int_{\sigma - i \infty}^{\sigma + i \infty} \nu_2(r, s; t) \rho(E_1, E_2; t) dE_1 dE_2. \]

(42)

Bhabha and Ramakrishnan have calculated this (see equation (41a) of their paper) and we here give the suitable approximation for all but small thicknesses. This approximation results from the fact that terms involving \( e^{-x} \) and \( e^{-y} \) exceed the others (i.e., those involving \( \mu \) in the exponent i.e. \( e^{-x} \) and \( e^{-y} \)) by an order of magnitude for all but small \( t \). However, it may seem that even for small thicknesses the terms involving \( \mu \) in the exponent have to be taken into account when the denominators of the matrix occurring in (40) vanish. Fortunately in the evaluation of (43) by the double saddle-point method, we find that only \( \lambda_r + \lambda_s - \lambda_{r+s-1} \) vanishes at some point in the domain of the saddle-point corresponding to the physically important values of \( E \) and so terms involving \( \mu \) in the exponent can be neglected.

\[ \varepsilon \{ N^2(E; t) \} = \varepsilon \{ N(E; t) \} + \frac{1}{(2\pi i E_0)^2} \int_{\sigma - i \infty}^{\sigma + i \infty} \int_{\sigma - i \infty}^{\sigma + i \infty} \left( \frac{E_0}{E} \right)^{2-2} G(r, s; t) e^{-x} dE_1 dE_2, \]

(43)

where

\[ G(r, s; t) = \frac{G_1(r, s) + G_2(r, s) e^{-x}}{\lambda_r + \mu - \lambda_{r+s-1}}, \]

(44)

and*

\[ G_1(r, s) = D - \lambda_r, \quad G_2(r, s) = D - \lambda_s, \quad \frac{1}{\lambda_r + \mu - \lambda_{r+s-1}} \left\{ a_1(r, s) C_{r+s-1} \right\}, \]

(45)

* In their paper Bhabha and Ramakrishnan omitted by mistake the factor \( \left( \mu_2 - \lambda_2 \right) \left( \mu_r - \lambda_r \right) \) occurring in (45). Their expression (41) must be multiplied by that factor.
We have expressed $G(r, s; t)$ in the form (44) to facilitate tabulation of $G'$. The factor \( \frac{1}{r + \lambda_s - \lambda_{r+s-1}} \) has been separated out since it becomes infinite at points where \( \lambda_r + \lambda_s = \lambda_{r+s-1} \). At these points, as has already been mentioned, \( G_1 + G_2 e^{(\lambda_r + \lambda_s - \lambda_{r+s-1})t} \) tends to zero in such a manner that $G$ is finite.

**Numerical evaluation of $\varepsilon \{ N^2(E; \ell) \}$**

$\varepsilon \{ N(E; \ell) \}$ and $\varepsilon \{ N^2(E; \ell) \}$ are functions of \( E/E_0 \) if $E_0$ is the energy of the incident electron. We then write them as $\varepsilon \{ N(y; \ell) \}$ and $\varepsilon \{ N^2(y; \ell) \}$ respectively where $y = \log(E/E_0)$. Thus we have

\[
\varepsilon \{ N^2(y; \ell) \} = \varepsilon \{ N(y; \ell) \} + I(y; \ell),
\]

where

\[
I(y; \ell) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{y(r+s-2)} G(r, s; \ell) e^{-(\lambda_r + \lambda_s)\ell} dr ds.
\]

To apply the saddle-point method for the numerical evaluation of the complex integral, we write $I(y; \ell)$ in the form

\[
I(y; \ell) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\omega(r,s;\ell)} dr ds,
\]

where

\[
\omega(r, s; \ell) = y(r+s-2) - t(\lambda_r + \lambda_s) + \log G(r, s; \ell) - \log(r-1) - \log(s-1).
\]

The saddle point \((\lambda_0, s_0)\) is then determined by the equations

\[
\frac{\partial \omega(r, s; \ell)}{\partial r} = \frac{\partial}{\partial r} \log G(r, s; \ell) - \frac{1}{r-1} - t \frac{d\lambda_r}{dr} + y = 0,
\]

\[
\frac{\partial \omega(r, s; \ell)}{\partial s} = \frac{\partial}{\partial s} \log G(r, s; \ell) - \frac{1}{s-1} - t \frac{d\lambda_s}{ds} + y = 0.
\]

$G$ is symmetric in $r$ and $s$. So also is $\omega$. Hence $r_0 = s_0$ and the saddle points for variations of $y$ and $t$ lie on $r = s$. 

---

\[G_2(r, s) = \left\{ \begin{array}{l}
(D(D-x)(x_r+x_s) + \lambda_r \mu_r x_r + \lambda_s \mu_s x_s - xx_r x_s) a_1(r, s) C_{r+s-1} \\
+ (D-\lambda_r) (\mu_r - D) \left\{ D(x_r + x_s) - \lambda_r \mu_r + (\lambda_s - x) (\mu_s - x) \right\} a_2(r, s) (D-\lambda_{r+s-1}) \\
+ (D-\lambda_r) (\mu_r - D) \left\{ D(x_r + x_s) - \lambda_r \mu_s + (\lambda_s - x) (\mu_r - x) \right\} a_2(s, r) (D-\lambda_{r+s-1}) \\
\end{array} \right. \]

\[
(\lambda_r + \mu_r - x) (\lambda_s + \mu_s - x) (\mu_r + \mu_s - x) \right\}^{-1},
\]

with $x = \lambda_{r+s-1}$ and $x_r = \lambda_r + \mu_r - x$. 

We have expressed $G(r, s; \ell)$ in the form (44) to facilitate tabulation of $G'$. The factor \( \frac{1}{\lambda_r + \lambda_s - \lambda_{r+s-1}} \) has been separated out since it becomes infinite at points where \( \lambda_r + \lambda_s = \lambda_{r+s-1} \). At these points, as has already been mentioned, \( G_1 + G_2 e^{(\lambda_r + \lambda_s - \lambda_{r+s-1})t} \) tends to zero in such a manner that $G$ is finite.

\[
\varepsilon \{ N^2(y; \ell) \} = \varepsilon \{ N(y; \ell) \} + I(y; \ell),
\]

\[
I(y; \ell) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{y(r+s-2)} G(r, s; \ell) e^{-(\lambda_r + \lambda_s)\ell} dr ds.
\]

\[
\omega(r, s; \ell) = y(r+s-2) - t(\lambda_r + \lambda_s) + \log G(r, s; \ell) - \log(r-1) - \log(s-1).
\]

\[
\frac{\partial \omega(r, s; \ell)}{\partial r} = \frac{\partial}{\partial r} \log G(r, s; \ell) - \frac{1}{r-1} - t \frac{d\lambda_r}{dr} + y = 0,
\]

\[
\frac{\partial \omega(r, s; \ell)}{\partial s} = \frac{\partial}{\partial s} \log G(r, s; \ell) - \frac{1}{s-1} - t \frac{d\lambda_s}{ds} + y = 0.
\]
The first step was to find the saddle-points of $e^{\omega(r,s;t)}$ for different values of $y$ for each $t$. For finding this point, the differential co-efficients $\frac{\partial}{\partial r} (\log G)$ were evaluated at $r=s$. A table of $t \frac{d^2 \omega}{dr^2} + \frac{1}{r-1} \frac{\partial}{\partial r} (\log G)$ against $r$, $(r=s)$ was formed and inverse interpolation carried out to determine the saddle points for various values of $y$ and $t$. The choice of $y$ is restricted by the fact that the saddle point should fall in the region of $r$, $s$ in which tabulation is done. The range depends on $t$.

The saddle points having been determined thus, it was necessary to find $\frac{\partial^2 \omega}{\partial r^2}$ and $\frac{\partial^2 \omega}{\partial r^2}$ at these points.

We notice

$$\frac{\partial^2 \omega(r,s;t)}{\partial r^2} = \frac{\partial^2 \omega(r,s;t)}{\partial s^2} \text{ at } r=s, \quad (53)$$

$$\left[ \frac{\partial^2 \omega(r,s;t)}{\partial r^2} \right]_{r=s} = \left[ \frac{\partial^3 \log G(r,s;t)}{\partial r^2} \right]_{r=s} - t \frac{d^2 \omega}{dr^2} + \frac{1}{(r-1)^2}. \quad (54)$$

To calculate $\frac{\partial^2 \omega}{\partial r \partial s}$, it is not necessary to tabulate $\frac{\partial \omega}{\partial r}$ at all $r$, $s$ if we use the symmetry property of $\omega(r,s;t)$.

If we define

$$\Omega(r,t) = \omega(r,r;t) = \log G(r,r;t) - 2t + 2y(t-1) - 2 \log(r-1), \quad (55)$$

$$\frac{\partial \Omega(r,t)}{\partial r} = 2 \left[ \frac{\partial \omega(r,s;t)}{\partial r} \right]_{r=s}, \quad (56)$$

$$\frac{\partial^2 \omega(r,s;t)}{\partial r \partial s} \bigg|_{r=s} = \frac{1}{2} \frac{\partial^2 \Omega(r,t)}{\partial r^2} \bigg|_{r=s} - \left[ \frac{\partial^2 \omega(r,s;t)}{\partial r^2} \right]_{r=s} \bigg|_{r=s}, \quad (57)$$

$\frac{\partial^2}{\partial r^2} (\log G(r,r;t))$ can be evaluated directly by using the standard formula for obtaining the differential co-efficients of a tabulated function for any value of the argument.*

The values of $e^{\omega(r,s;t)}$ at the saddle points were obtained from the tables of $\log G$, using the Bessel’s interpolation formula.

In this paper, tables of $\varepsilon \{N^2\}$, $\varepsilon \{N\}$,

$$\sigma^2 = \varepsilon \{N^2\} - [\varepsilon \{N\}]^2, \sigma^2/\varepsilon \{N\} \text{ and } \sigma^2 / ([\varepsilon \{N\}]^2 - \varepsilon \{N\})$$

are given for various values of $y$ and $t$. We have also tabulated $G_1(r,s)$ and $G_2(r,s)$ to enable anyone interested to compute $\varepsilon \{N^2 \{y,t\}}$ for any value of $y$ and $t$.

* See for example, Interpolation and Allied Tables (H. M. S. Office).
Table I

\( G_i(r, \kappa) \)

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Studies on the Stochastic Problem of Electron-photon Cascades

Table II.

\( G_2(r, s) \)

The starred values of \( G_2(r, s) \) in the table are those in the neighbourhood of the points where \( \lambda_r + \mu_s - \lambda_{r+1} + \mu_{s-1} \) or \( \lambda_s + \mu_r - \lambda_{s+1} + \mu_{r-1} \) vanishes. These points do not play any role in the saddle point integration and so there is no need to include terms involving \( e^{-(\lambda_r + \mu_s) t} \) and \( e^{-(\lambda_s + \mu_r) t} \), which, as stated earlier, render \( G_2 \) finite even at points where \( \lambda_r + \mu_s = \lambda_{r+1} + \mu_{s-1} \) or \( \lambda_s + \mu_r = \lambda_{s+1} + \mu_{r-1} \).

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Studies on the Stochastic Problem of Electron-photon Cascades

Table V:
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\[ \sigma^2/\langle N \rangle \text{ is given in brackets} \]

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<td>(35.8)</td>
<td>(35.7)</td>
<td>(35.8)</td>
<td>(35.9)</td>
</tr>
</tbody>
</table>

We have calculated \( \varepsilon \langle N(y; t) \rangle \) using the tables of Bhabha and Chakrabarthy**. It is to be noted that our values for \( \varepsilon \langle N(y; t) \rangle \) differ from those given in Heitler's "Quantum theory of radiation" (H.2).

Discussion of results

In discussing our results from the standpoint of the general theory of multiplicative stochastic processes we meet with one serious difficulty: the total cross-section for radiation by an electron is infinite if we use the exact Bethe-Heitler expression for \( R^{(n)} \) given by (22). Consequently, all the moments of the total number of particles (i.e. \( N(y; t) \)) as \( y \) tends to infinity or in other words, the number of particles above zero energy) become infinite. This is contrary to physical facts. So in (D. 6) Ramakrishnan assumed the total cross-section to be finite and obtained an approximately Furry distribution for the total number of electrons.

The above difficulty does not arise if we consider the number of particles above a certain energy i.e. for finite \( y \) (which can be chosen as large as we please). Fortunately, it is the case of finite \( y \) that is of physical significance and the calculation of the moments of the total number is only of academic interest. However, the question arises:

** We have in fact used the tables of Leonie Janossy and Messel (Proc. Roy. Irish Acad. Sci. 54A (1951), 217) who have extended those of Bhabha and Chakrabarthy to more decimal places.
is there any essential difference from a stochastic point of view, between the case where the total cross-section is finite and where it is infinite? In a later paper, one of us (R) proposes to discuss two simplified models of multiplicative processes, one in which the total cross-section is finite and the other in which it is infinite. It is hoped that a comparison between these two models may reveal any marked differences in their stochastic features which are obscured in more complicated processes by the very complexity of the problem.

In the present paper we have tabulated \( \sigma^2 / \epsilon \{ N \} \) and \( \sigma^2 / (\{ e \{ N \} \}^2 - \epsilon \{ N \}) \) for convenience of comparison with the Poisson and Furry distributions. It should be noted that \( \sigma^2 / \epsilon \{ N \} \) and \( \sigma^2 / (\{ e \{ N \} \}^2 - \epsilon \{ N \}) \) are complicated functions of \( y \) and \( t \) as is obvious from the fact that we are not able to invert the double complex integral (43). An examination of the tables reveals the following general features of the process.

1. For a given \( y \) the maxima for \( \epsilon \{ N \} \), \( \epsilon \{ N^2 \} \), \( \sigma^2 \), \( \sigma^2 / \epsilon \{ N \} \) and \( \sigma^2 / (\{ e \{ N \} \}^2 - \epsilon \{ N \}) \) are all different as expected.

2. Comparison with the Furry distribution is meaningful only if the number of particles is more than, say, 10. The negative values for \( \sigma^2 / (\{ e \{ N \} \}^2 - \epsilon \{ N \}) \) correspond to the negative values for \( (\{ e \{ N \} \}^2 - \epsilon \{ N \}) \) i.e. when \( \epsilon \{ N \} < 1 \). For very large values of \( y \) but before the shower maximum the deviation is comparable to (about .1 to .2 times) the Furry deviation.

3. Comparing with the Poisson distribution, we find that \( \sigma^2 / \epsilon \{ N \} \) increases with \( y \) for a given value of \( t \). At the shower maximum the deviation is many times the Poisson value—a result at variance with that of Scott and Uhlenbeck and Janossy and Messel. For example, for \( y = 8 \) the maximum occurs between \( t = 6 \) and \( t = 8 \) and the deviation is about 12 times the Poisson value or .1 times the Furry deviation.

While we do not hazard a detailed criticism of the results of Scott and Uhlenbeck or Janossy and Messel we here attempt to indicate the reasons for such a discrepancy.

(i) It was pointed out by Bhabha and Ramakrishnan that Scott and Uhlenbeck, in the course of their calculations erroneously omitted certain terms as negligible.

(ii) The calculations of Janossy and Messel were based on the equation for the second moment obtained from the fundamental integral equations of Janossy. This equation did not yield an explicit Mellin's transform solution for \( \epsilon \{ N^2 \} \) (see equations (22), (23) and (24) of their paper) and this is as should be expected since they were dealing with the direct Mellin's transform of \( \epsilon \{ N^2 \} \) with respect to \( y \) while Bhabha and Ramakrishnan obtained \( \epsilon \{ N^2 \} \) as the inverse of a two-fold Mellin's transform.

(iii) There seems to be no reason, from a logical and stochastic point of view, to expect a Poisson distribution at the shower maximum. This expectation by previous authors was probably based on the assumption that in a 'stationary state,' particles behave as if they are independent and hence the number of particles \( N(y; t) \) has a Poisson distribution at thicknesses where \( \epsilon \{ N(y; t) \} \) is a maximum. But \( \epsilon \{ N(y; t) \} \) \_max does in no way define a 'stationary state' in a stochastic sense. In the theory of stochastic processes we call a 'state' stationary only if the differential coefficients (with respect to \( t \)) of all orders of all the moments vanish.
In conclusion we wish to assert that our calculations are as accurate as can be obtained by the use of the double saddle-point formula. Our equations for the product density of degree two are a natural sequel to the equations of Bhabha and Chakrabarthy for the mean i.e. product density of degree one, since both are 'last collision' diffusion equations – as contrasted with the 'first collision' equations of Janossy and Messel.

Appendix

Here we point out the mistakes occurring in the paper of Bhabha and Ramakrishnan*.

1. On page 144 of that paper it was stated (see equation (5)) that in the absence of the last terms in equations (4a) and (4b), equation (4) would clearly be satisfied by

\[ f_x(E_1, E_2; t) = f_x(E_1; t) f_x(E_2; t), \]
\[ f y_1,1(E_1, E_2; t) = f_1(E_1; t) g_y(E_2; t), \]
\[ f y_1,1(E_2, E_1; t) = f_1(E_2; t) g_y(E_1; t), \]
\[ g_2(E_1, E_2; t) = g_y(E_1; t) g_y(E_2; t), \]
\[ g_2(E_1, E_2; t) = g_y(E_2; t) g_y(E_1; t). \]

The correct statement should be

\[ f_x(E_1, E_2; t) = k_1 f_x(E_1; t) f_x(E_2; t), \]
\[ f y_1,1(E_1, E_2; t) = k_2 f_1(E_1; t) g_y(E_2; t), \]
\[ f y_1,1(E_2, E_1; t) = k_2 f_1(E_2; t) g_y(E_1; t), \]
\[ g_y(E_1, E_2; t) = k_4 g_y(E_1; t) g_y(E_2; t), \]
where \( k_1, k_2, k_3, k_4 \) are constants. The initial condition in the case of a shower initiated by a single electron of energy \( E_0 \) is

\[ f_x(E_1, E_2; t) = 0 \text{ at } t = 0, \]
\[ f y_1,1(E_1, E_2; 0) = 0 \]
\[ g_y(E_1, E_2; 0) = 0 \text{ for all } E_1, E_2. \]

This follows directly from the definition of product densities. Thus \( k_1 = k_2 = k_3 = k_4 = 0. \)

Fortunately this error was not carried over into the calculations since \( \phi^{(2)}(r, s; t) \) the vector representing the transforms of the product densities was expressed as an integral \( 0 \) to \( t \) according to (25) of that paper. This means that at \( t = 0 \), the vector is null.

2. The expression for \( R(E', E) \) as given by (6a) is not correct. It should read

\[ R(E', E) = \left( \frac{E - E'}{E} \right)^{-\left( \frac{4}{3} + \nu \right)} \left( 1 - \frac{E'}{E - E'} \right) \frac{1}{E} = R^{(0)}(E', E), \]

since in our definition of \( R^{(0)}(E', E) \), \( E' \) represent the energy of the electron after radiation and not that of the photon. But this error was not carried over into the calcu-

* The equations referred to in the Appendix are those of the paper of Bhabha and Ramakrishnan, though the product-density notation is that used in the present study. Hence we write \( f_\alpha, g_{\alpha, \beta} f_\beta, g_\beta \) for \( n, m, nm, n_2 \) and \( m_2 \) functions used in that paper.
lations since the correct expression for $A_\nu$, $A_\nu$ was written down.

3. The vector $\nu'$ of equation (35) of that paper has to be multiplied by a factor
$$
\frac{1}{(\mu_\nu - \lambda_\nu) (\mu_\nu - \lambda_\sigma)}
$$
This factor was omitted by mistake.

4. On page 150, the same expression is numbered as (36) and (37).

5. In eq. (40) there is no need to separate out $f_0$ and $f''$. $f_0 + f''$ can be written as $f'$ and so in (41), we write $f'$ and $\nu'$ instead of $f''$ and $\nu''$. Using the usual approximation for all but small $t$ we get (47) of the present paper.

References
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5. R. Serber, Phys. Rev. 54 (1938), 317.
9. L. W. Nordheim and M. H. Hebb, Phys. Rev. 56 (1939), 494

B. Cascade theory (Fluctuation problem with simplified models)


C. Method of product densities


D. Method of product densities

2. C. Palm, Ericsson Technics No. 44 (1943), 1-189.
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E. Method of Janossy functions


F. Further contributions using the above methods


G. Articles comparing the methods or in the nature of reviews


H. Books