Conservation Laws in Classical and Quantum Physics*

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The relation between the conservation laws and the symmetry properties of space and time is examined. If the simplest possible equation of motion takes the place of Newton's equation, the usual conservation laws are not obtained. In the quantum physics, on the contrary, there is a superabundance of conservation laws and their physical significance is not obvious.

As will be outlined in the ensuing article by Murai, there are several reasons for investigating the consequences of an extension of the inhomogeneous Lorentz group to include the transition to coordinate systems in uniformly accelerated motion. The only reason for investigating the reduction of the inhomogeneous Lorentz group to be discussed below is one of curiosity and a desire to be reminded of the fundamental role which Newton's equation of motion continue to play in physics.

We shall consider the consequences of the simplest possible equation of motion: "Every body remains at rest if no force acts on it." The equations of motion which correspond to this modified law are

\[ m_a \dot{x}_a = -\partial f/\partial x_a ; \quad m_a \dot{y}_a = -\partial f/\partial y_a ; \quad m_a \dot{z}_a = -\partial f/\partial z_a. \] (1)

This applies then to every particle \( \alpha \), the \( f \) is a kind of potential, the gradient of which determines, however, not the acceleration but the velocity. Naturally, the results which we may obtain from (1) have no physical significance. However, by exploring the conservation laws which (1) leads to, we shall see that, in mechanics, space symmetry without the specific form of Newton's law does not lead to all conservation theorems which we could expect. In quantum theory, on the other hand, the equivalent of (1) will entail these conservation laws even though their interpretation will not be too obvious.

If (1) holds, two coordinate systems which are in uniform motion with respect to each other are not equivalent any more, and there is no way to introduce a theory of relativity if one starts with (1). However, the symmetry properties of space alone remain preserved, its homogeneity and isotropy, and so does the homogeneity of time. As a

* The ensuing article of Y. Murai, the subject of which he has discussed with me repeatedly, encourages me to publish the following simple considerations which I had presented, some time ago, to the New Jersey Science Teachers Association (November 1951 meeting at Atlantic City).
result, we expect that there will be a conservation theorem for energy (due to the homogeneity of time) and a momentum and angular momentum law. These expectations are fulfilled in our mechanics (1) only partially: if \( f \) is invariant with respect to displacements a momentum law follows from (1) which states that the centre of mass of an isolated system is at rest;

\[
m_1 x_1 + m_2 x_2 + \cdots + m_n x_n = \text{Const.}
\]  

(2)

Similar equations hold for the \( y \) and \( z \) components. However, there does not seem to be an energy principle, nor a conservation law for angular momentum. This result does not conflict with the results of Hamel and Engel\(^1\). The connection between conservation laws and symmetry is based, in mechanics, on the Hamiltonian formulation, and the equations of motion (1) do not allow this formalism.

The quantum equation which corresponds to (1) can be derived by an adaptation of Ehrenfest's principle\(^2\). This demands that it shall follow from

\[
\frac{\partial}{\partial t} \Psi(x_1, \ldots, x_n) = Q \Psi
\]

that the motion of the centre of mass

\[
\bar{x}_a = \int |\Psi|^2 x_a d\tau
\]

(\( \int \cdots d\tau \) indicates integration over the whole configuration space) obeys the law (1) where \( \frac{\partial f}{\partial x_a} \) is again replaced by its average. Hence

\[
m_a \frac{d\bar{x}_a}{dt} = m_a \int x_a (Q \Psi + \Psi(Q^*)^*) d\tau = -\int \frac{\partial f}{\partial x_a} \Psi \Psi^* d\tau.
\]

(3a)

It follows from the requirement that the total probability \( \int |\Psi|^2 d\tau \) be independent of time, just as in ordinary quantum theory, that \( Q \) is skew hermitean. Hence, one can transform the second member of (3a) into

\[
m_a \int \Psi^* [x_a Q \Psi - Q(x_a \Psi)] d\tau.
\]

If this is to be equal to the right side of (3a) for all \( \Psi^* \), one must have

\[
x_a Q \Psi = Q x_a \Psi = -\frac{1}{m_a} \frac{\partial f}{\partial x_a} \Psi
\]

(3b)

This will be valid for all \( \Psi \) if the commutator

\[
[C, x_a] = \frac{1}{m_a} \frac{\partial f}{\partial x_a}.
\]

(3c)

The most general skew hermitean \( C \), which satisfies (3a) and the similar equations for the \( y \) and \( z \) components is
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\[ Q = \sum \frac{1}{m_a} \left( \frac{\partial f}{\partial x_a} + \frac{\partial f}{\partial y_a} + \frac{\partial f}{\partial z_a} + \frac{1}{2} \Delta_a f \right) + ig \]  

(4)

The real function \( g \) can be quite arbitrary except that it must be invariant with respect to displacements and rotations, i.e., must depend only on the distances of the particles.

One can easily formulate the quantum mechanical conservation laws on the basis of (3) and (4). The energy principle becomes

\[ E = i\hbar \int \Psi^* \left[ \sum \frac{1}{m_a} \left( \text{grad}_a f \cdot \text{grad}_a \Psi + \frac{1}{2} \Delta_a f \right) + ig \Psi \right] d\tau : \]  

(5)

the angular momentum

\[ M_a = i\hbar \int \Psi^* \left( x_a \frac{\partial}{\partial y_a} - y_a \frac{\partial}{\partial x_a} \right) \Psi d\tau. \]  

(6)

This last expression is identical with the one of ordinary quantum mechanics.

However, the connection between conservation laws and symmetry is not unique in quantum theory either. In contrast to the situation in the non-quantum formulation, there is a superabundance of conservation laws. In addition to the quantum momentum

\[ P_a = -i\hbar \int \Psi^* \frac{\partial \Psi}{\partial x_a} d\tau, \]  

(7)

there is the analogue of (2)

\[ \int (\Delta m_a x_a) \Psi^* \Psi d\tau. \]  

(7a)

The modified Newton’s law (1), which was used above to demonstrate the directness of the connection, in quantum theory, between conservation laws and symmetry, is the simplest law of motion that one can imagine. As was mentioned already, this law of motion precludes introducing any principle similar to the principle of relativity. It may be worth while to remark that, in addition, the mechanical law of motion is in this case, strictly valid also in quantum theory. It follows from (3) and (4) also that

\[ \frac{\partial |\Psi|^2}{\partial t} = \sum \frac{1}{m_a} \left[ \frac{\partial f}{\partial x_a} \left( \frac{\partial f}{\partial x_a} |\Psi|^2 \right) + \frac{\partial f}{\partial y_a} \left( \frac{\partial f}{\partial y_a} |\Psi|^2 \right) + \frac{\partial f}{\partial z_a} \left( \frac{\partial f}{\partial z_a} |\Psi|^2 \right) \right] \]  

(8)

which is the continuity equation for particles which have the velocity components \( (\partial f/\partial x_a)/m_a \) when they are at the point described by the variables of \( f \); (8) expresses the fact that the time rate of change of \( |\Psi|^2 \)—which is the density of the particles—is equal to the negative divergence of the current. This last quantity is equal to the product of density, \( |\Psi|^2 \), and the velocity. It is well known that not only is the equation (8) a consequence of the equations of motion but that, conversely, the equations of motion (1) can be deduced from the validity of (8) for a set of particles with density distribution \( |\Psi|^2 \). Thus the substitution of the quantum equation (3) and (4) really did not invalidate the mechanical equation (1), a circumstance which we could have inferred from the absence of \( \hbar \) in (4). The equation of motion (1) would have made the
physics of the past fifty years very much easier: they would have made it impossible to introduce the theory of relativity, and quantization of the equations would not have changed their physical content. The only new feature which the quantum theory introduces is the complex phase of our wave function $\psi$ and it is questionable whether this quantity could be attributed any physical significance. Since the quantum conservation laws (5), (6) and (7) are all based on this complex phase, and vanish for a real wave function $\psi$, their physical interpretation is open to question.

The above example was meant as a warning against a facile identification of symmetry and conservation laws. It reminds us that the Hamiltonian formulation is necessary for that connection to hold in ordinary mechanics and that while it is always possible in quantum theory to deduce conservation laws from a symmetry condition, the interpretation of these conservation laws, and their significance, might be quite problematical.

The fact that the quantization of the equations of motion would not lead to a real quantum theory might have been foreseen from the fact that the uncertainty principle is hardly compatible with (1). If all the coordinates have sharp values, this holds also for the forces $-\nabla f$. This is true also in current quantum theory. However, in the present theory, the forces determine the velocities, rather than the accelerations, and these become determined also.

References

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