A Remark on the Infinities due to the New Complex Poles of Modified Propagators

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In this short note we shall show that the new pole of modified propagators pointed out by G. Feldman is not independent on the divergence of perturbation expansion.

Among many attempts\(^1\) to improve on the results of the perturbation method in meson theory, the introduction of Green functions \(G\) and \(S\) (or modified propagators) in place of \(A\) and \(S\) seems to be most reasonable and promising from the physical point of view. It was pointed out by G. Feldman\(^1\), however, that the introduction of modified propagators gives rise to a new type of infinities which can not be removed by renormalization. These new infinities are due to new complex poles of the modified propagators. This situation was shown by Feldman by considering a simple model of a modified propagator. The outline of his line of reasoning is as follows.

He adopted a simple propagator having the expression

\[
Q(k^2) = \sum_{n=0}^{\infty} \left( \frac{-\alpha^2 \xi \theta}{k^2 + \alpha^2 - i\xi} \right)^n
\]  

(1)

in momentum space, and discussed convergence of an integral
which is considered to correspond to some type of Dyson graphs. In (1) \( \alpha, \theta, \) and \( \epsilon \) are some real positive constants. The symbol \( \rightarrow \) in (2) means that the integration with respect to \( k^0 \) should be carried out along the real axis in a complex \( k^0 \)-plane. According to Feldman, (1) can be summed up and put equal to

\[
Q(k^0) = \frac{k^0 + \alpha^2 - i\epsilon}{k^0 + \alpha^2 - i\epsilon + \alpha e^i \theta}.
\]

Substituting with (1)' into (2), one encounters new complex poles

\[
k^0 = \pm \sqrt{\frac{2}{\epsilon} + \alpha^2 + \alpha^2 \cos \theta + (\alpha^2 \sin \theta - \epsilon)^2} \pm \frac{(\alpha^2 \sin \theta - \epsilon)}{\sqrt{\epsilon} + \sqrt{\alpha^2} (1 + \cos \theta)}.
\]

Thus \( I \) can be transformed into

\[
I = \int d^4k \int d^4k^0 Q(k^0) \left( \frac{1}{(k^2 + \alpha^2 - i\epsilon')^3} + P \right)
\]

where the symbol \( \uparrow \) means that the \( k^0 \)-integration should be carried out along the imaginary axis in the \( k^0 \)-plane. \( P \) in (4) stands for a divergent quantity corresponding to the new complex poles (3).

On the contrary, using the ordinary perturbation method, one gets an integral

\[
I' = \int d^4k \int d^4k^0 Q(k^0) \left( \frac{1}{(k^2 + \alpha^2 - i\epsilon')^3} \right) \]

instead of \( I \). Since every integrand of (5) has only ordinary poles, the path of integration with respect to \( k^0 \) can be transformed to that along the imaginary axis. Then \( I' \) can be written in a compact form as

\[
I' = \frac{1}{\sqrt{\epsilon}} \int d^4k Q(k^0) \left( \frac{1}{(k^2 + \alpha^2 - i\epsilon')^3} \right) < \infty,
\]

on account of the fact that after the transformation of the path of integration denominators of the integrand in (5) are positive definite in the limit \( \epsilon, \epsilon' \rightarrow 0 \). Thus Feldman gets

\[
I = I' + P = \infty,
\]

that is, he arrives at the conclusion that although the expression \( I' \) derived by perturbation method is convergent, the other expression \( I \) is divergent, thus the new type of divergence is independent on the divergence of perturbation expansion.

From our point of view, however, \( Q(k^0) \) defined by (1) is in general not equal to (1)'. Only in the case of

\[
\epsilon > \alpha^2,
\]

(1)' holds for every set of real values of \( k^0 \) and in this case the series (1) turns out to be uniformly convergent with respect to \( k^0 \) and the argument \( \phi \) of the pole of \( Q \) becomes as

\[
-\pi/4 < \phi < 0,
\]

that is, the pole of \( Q \) becomes an ordinary one. In such a case, of course, one can easily show the relation

\[
I = I' = \int d^4k \int d^4k^0 Q(k^0) \left( \frac{1}{(k^2 + \alpha^2 - i\epsilon')^3} \right).
\]

The appearance of the curious infinity \( P \) is essentially due to the fact that the series (1) is not uniformly convergent. In the proof of the equality (7) we have changed the order of integration and summation. As to this point some care should be taken because the domain of integration is infinite, but in our model it is easily seen that this change of order can be permitted.

Although our conclusion exclusively depends on the special form of our model, it is easily conjectured that similar situation will occur in the ordinary meson theory by considering the following fact. If the new type of poles of modified propagators appear in course of calculation and at the same time if the perturbation expansion of modified propagators uniformly converge with respect to momentum variables, we could scarcely understand from what origin these new poles results.

The authors wish to express their cordial thanks to Dr. G. Feldman for communicating his work before publication.

1) G. Feldman, "Modified Propagators in Field Theory". This work will be published in near future.