Comment on the Geometric Interpretation of Ito Calculus on a Lattice

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The covariant nature of the Langevin equation in Itô calculus is clarified in the application of the stochastic quantization method to $U(N)$ lattice gauge theory. The stochastic process is expressed in a manifestly general coordinate covariant form as a collective field theory on the group manifold. A geometric interpretation is given in the sense of Itô for the Langevin equation and the corresponding Fokker-Planck equation.

Wilson’s lattice gauge theory ($\text{(LGT)}_d$) is, for this time, a very unique theory that provides a constructive definition of gauge theories in the path-integral method. As far as expectation values of observables are concerned, the gauge fixing procedure is not necessary, because of the finite volume of the gauge degrees of freedom. There exists an alternative approach which enables us to calculate the expectation values of observables without an explicit gauge fixing procedure, the so-called stochastic quantization method (SQM). The application of SQM to $\text{(LGT)}_d$ was first discussed in relation to the large $N$ reduced model, as well as in numerical studies of $\text{(LGT)}_d$. In particular, it has been shown that the integral over the group manifold is realized by the Haar measure at the equilibrium limit. In general, there are two different types of formulations for stochastic differential equations (i.e. Langevin equations) on a group manifold, the Itô formulation and the Stratonovich formulation. Their difference essentially comes from the definition of the equal stochastic time correlation between the dynamical variables and the random noise variables. In Itô calculus, the random noise is not correlated with the equal stochastic time dynamical variables. This is a significant advantage in the construction of a collective field theory of observables from the Langevin equation of a fundamental system in the context of SQM. In this short note, in applying SQM to $U(N)$ lattice gauge theory, we emphasize the importance of the Itô formulation and its covariant nature under general coordinate transformations on the group manifold. A geometrical meaning is given to the role of the contact term that appears in constructing a collective field theory. In particular, the Langevin equation and the corresponding Fokker-Planck equation are expressed in a manifestly general coordinate invariant form on the group manifold.

We apply SQM to $d$-dimensional $U(N)$ lattice gauge theory:

$$S[U] = - \sum_{x, \mu > \nu} \beta \frac{2}{kN} \text{Tr}(U_{\mu}(x)U_{\nu}(x + \mu)U_{\mu}^\dagger(x + \nu)U_{\nu}^\dagger(x) + \text{h.c.}) .$$

(1)

Here the link variables, $U_{\mu}$, are the $U(N)$ group elements, $\beta \equiv \frac{N}{g^2}$, the elements of $U(N)$.
the $U(N)$ algebra, $t_A$, satisfy $[t_A, t_B] = if_{AB}^C t_C$, and $\text{Tr}(t_A t_B) = k\delta_{AB}$.

To write a Langevin equation for $(\text{LGT})_d$, we introduce the left Lie derivative, $\hat{E}_A$, which satisfies

$$\hat{E}_A(x, \mu)U_\mu(x) = t_A U_\mu(x) ,$$
$$\hat{E}_A(x, \mu)U_\mu^\dagger(x) = -U_\mu^\dagger(x)t_A ,$$
$$\left[ \hat{E}_A(x, \mu), \hat{E}_B(y, \nu) \right] = -if_{AB}^C \hat{E}_C(x, \mu)\delta_{x,y}\delta_{\mu\nu} . \ (2)$$

It is defined by $\hat{E}_A(x, \mu) \equiv -iL_A^B(x, \mu)\frac{\partial}{\partial U_\mu^B}$, with $K_A^C L_C^B = L_A^C K_C^B = \delta_A^B$. The components of the Maurer-Cartan one-form, $L_A^B(x, \mu)$ and $K_A^B(x, \mu)$, satisfy the Maurer-Cartan equations:

$$L_A^C \partial_C L_B^D - L_B^C \partial_C L_A^D = +f_{AB}^C L_C^D ,$$
$$\partial_B K_C^A - \partial_C K_B^A = -f_{BC}^A K_B^C K_C^A . \ (3)$$

We define the time evolution of the link variable, $U_\mu(\tau + \Delta \tau) \equiv U_\mu(\tau) + \Delta U_\mu(\tau)$, in terms of Itô calculus.\footnote{11} We use discretized notation to allow a clear understanding. We start from the Langevin equation\footnote{3}$^3$

$$U_\mu(\tau + \Delta \tau, x) = \exp \left( \Delta \tau (\hat{E}(\tau, x, \mu)S[U(\tau)]) + i \Delta W_\mu(\tau, x) \right) U_\mu(\tau, x) ,$$
$$\langle (\Delta W_\mu)_ij(\tau,x)(\Delta W_\nu)_{kl}(\tau,y) \rangle_{\Delta W_\tau} = 2k\Delta \tau \delta_{\mu\nu} \delta_{ij} \delta_{kl} \delta_{xy} . \ (4)$$

with $\hat{E}(\tau, x, \mu) \equiv t_A \hat{E}_A(\tau, x, \mu)$. Here, $\Delta W_\mu(\tau, x) \equiv t_A \Delta W_\mu^A(\tau, x)$ is a noise variable defined on the site. We note that, from (4), $U_\mu(\tau, x)$ and $\Delta W_\mu(\tau', x)$ are correlated only if $\tau' \leq \tau - \Delta \tau$. In the following, $\langle \ldots \rangle_{\Delta W_\tau}$ indicates that the expectation value is evaluated by means of the noise correlation at the stochastic time $\tau$ defined in (4).

Since $\hat{E}(x, \mu)S[U]$ is anti-hermitian, the form of the Langevin equation (4) ensures that the time development is such that the link variable remains within the element of $U(N)$. Up to order $\Delta \tau$, we obtain the following Langevin equation:

$$\left( \Delta U_\mu(x, \tau) \right) U_\mu(\tau, x) = \Delta \tau (\hat{E}(\tau, x, \mu)S[U(\tau)]) + i \Delta W_\mu(\tau, x) - \Delta \tau kN1 . \ (5)$$

The appearance of the term $\Delta \tau kN1$ is more understandable if we note that the constraint $U_\mu(\tau + \Delta \tau)U_\mu(\tau + \Delta \tau)^\dagger = 1$ implies $\Delta U_\mu U_\mu^\dagger + U_\mu \Delta U_\mu^\dagger = -2\Delta \tau kN1 + O(\Delta \tau^3/2)$, where $1$ is the $N \times N$ unit matrix.

The time evolution of an arbitrary function of the elements $U_\mu$, $F[U]$, is deduced from (5),

$$\Delta F[U] = \sum_{x, \mu} \left( \Delta \tau (\hat{E}(x, \mu)S[U]) + i \Delta W_\mu(x, \mu) \right)^A \hat{E}_A F[U] - \sum_{x, \mu} \Delta \tau \hat{E}_A^A \hat{E}_A F[U] , \ (6)$$

where $(\ldots)^A \equiv (1/k)\text{Tr}(\ldots t_A^A)$. Here, we have used the identity

$$\hat{E}_A(x, \mu)\hat{E}_B(y, \nu)F[U] = (t_A U_\mu(x))_{ij}(t_B U_\nu(y))_{kl} \frac{\delta^2 F[U]}{\delta (U_\mu(x))_{ij}\delta (U_\nu(y))_{kl}} \delta_{ij}(t_B t_A U_\mu(x))_{ij}$$
$$+ \delta_{\mu\nu} \delta_{xy}(t_B t_A U_\mu(x))_{ij} \frac{\delta F[U]}{\delta (U_\mu(x))_{ij}} . \ (7)$$
In order to check the consistency of this time evolution equation, we consider the expectation value of \( (U_\mu)_{ij}(\tau,x)(U_\nu)_{kl}^\dagger(\tau,y) \), which defines a \( U(N) \) group integral under the condition \( S[U] = 0 \). From (6), the differential equation to determine the expectation value is

\[
d\langle (U_\mu)_{ij}(\tau,x)(U_\nu)_{kl}^\dagger(\tau,y) \rangle \Delta W \\
= -2kN\langle (U_\mu)_{ij}(\tau,x)(U_\nu)_{kl}^\dagger(\tau,y) \rangle \Delta W + 2k\delta_{il}\delta_{kj}\delta_{\mu\nu}\delta_{xy}.
\]

(8)

Here we have taken the limit \( \Delta \tau \to 0 \). The expectation value \( \langle ... \rangle_{\Delta W} \) is defined with all the noise correlations for \( \tau' \leq \tau - \Delta \tau \). The solution is given by

\[
\langle (U_\mu)_{ij}(\tau,x)(U_\nu)_{kl}^\dagger(\tau,y) \rangle \Delta W \\
= e^{-2kN\tau}(U_\mu)_{ij}(0,x)(U_\nu)_{kl}^\dagger(0,y) + \left( 1 - e^{-2kN\tau} \right) \frac{1}{N}\delta_{il}\delta_{kj}\delta_{\mu\nu}\delta_{xy}.
\]

(9)

Hence, we conclude that the group integral with respect to the Haar measure \( d\mu(U) \), which is normalized according to \( \int d\mu(U)U_{ij}U_{kl}^\dagger = \frac{1}{N}\delta_{il}\delta_{kj} \), is reproduced in the equilibrium limit.\(^4,5\) We also confirm the importance of the contribution \(-\Delta \tau kN1\) in (5) to realize the Haar measure in the equilibrium limit in the sense of Itô calculus. Also, this clearly shows that, apart from the initial value dependence, the integral measure possesses a stochastic time dependent damping factor at finite stochastic time. This comes from the fact that the stochastic process described by the Langevin equation (5) under the condition \( S[U] \to 0 \) is a diffusion process on the group manifold. The damping factor simply reflects that it takes an infinite stochastic time for the diffusion process to reproduce the group integration with the weight given by the Haar measure.

The time evolution of the expectation value of the observable \( F[U] \) is assumed to be described by the probability distribution \( P(\tau,U) \) defined by

\[
\langle F[U(\tau)] \rangle_{\Delta W} \equiv \int d\mu(U)F[U]P(\tau,U),
\]

(10)

where \( U(\tau) \) on the l.h.s. represents the solution of the Langevin equation (5) with the initial condition \( U_\mu(0,x) \). The Fokker-Planck equation follows from (6) and (10):

\[
\frac{\partial}{\partial \tau}P(\tau,U) = -\frac{1}{k}\sum_{x,\mu}\text{Tr}\left\{ \hat{E}(x,\mu)\left( \hat{E}(x,\mu) + (\hat{E}(x,\mu)S[U]) \right) \right\} P(\tau,U).
\]

(11)

This shows that \( U(N) \) (LGT)_d is reproduced in the equilibrium limit. The probability distribution \( P(\tau,U) \) behaves as \( \lim_{\tau \to \infty} P(\tau,U) = e^{-S[U]} \) in the time evolution governed by the Langevin equation (5).

If we choose \( SU(N) \) instead of \( U(N) \), since the noise correlation in (4) is proportional to \( \left( \delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right) \), the order \( \Delta \tau \) Langevin equation is obtained by replacing \( N \) with \( (N^2 - 1)/N \) on the r.h.s of (5). Then, we obtain similar results for the Langevin equation and the Fokker-Planck equation for \( SU(N) \) (LGT)_d.
Here we comment on the local gauge covariance of the Langevin equation (5). The local gauge transformation of the link variable $U_\mu$ is defined by

$$U_\mu(\tau, x) = e^{i\Lambda(x)} U_\mu(\tau, x) e^{-i\Lambda(x+\mu)}, \quad (12)$$

where $\Lambda^\dagger(x) = \Lambda(x)$.

For the covariance of the Langevin equation, the noise variable defined on a given site must transform as

$$\Delta W_\mu(\tau, x) \rightarrow e^{i\Lambda(x)} \Delta W_\mu(\tau, x) e^{-i\Lambda(x)}.$$

(13)

Since the noise correlation in (4) is invariant under the transformation (13), the Langevin equations and the noise correlations manifestly preserve the local gauge symmetry. We note that the Fokker-Planck equation (11) is also invariant under local gauge transformations. These results are essentially not new. In previous works,\(^3\)–\(^6\) the contribution of the third term in the r.h.s. of (5) was not discussed seriously. However, it is a manifestation of Itô calculus that comes from the constraint $U_\mu(\tau + \Delta \tau)^{-1} = U_\mu(\tau + \Delta \tau)^\dagger$. In the following, the covariant nature of Itô calculus shows up the internal geometry in (LGT)\(_d\).

The geometrically covariant nature of the Langevin equation in the sense of Itô calculus was first pointed by Graham\(^13\) in analysis of a stochastic process (the Brownian motion of a particle) in curved spaces. Here, to obtain a geometric interpretation of the Langevin equation of (LGT)\(_d\), we define a metric on the group manifold:

$$G_{AB}(x, \mu) \equiv L_A C L_B C$$

and

$$G_{AB}(x, \mu) \equiv K_A C K_B C.$$  

$G_{\mu}$ denotes $\text{det}G_{AB}$. By using the components of the Maurer-Cartan one-form, the local gauge transformation (12) can be expressed for the $U(N)$ gauge field defined by $U_\mu(\tau, x) \equiv e^{iV_\mu A (\tau, x)t_A}$ as

$$\delta V_\mu^A (\tau, x) = A^B(x)L_B^A (\tau, x, \mu) - L_B^A (\tau, x, \mu) A^B (x + \mu), \quad (14)$$

where $L_B^A = (L_B^A)^\dagger$. The continuum limit is taken by introducing the lattice spacing $\epsilon$ according to $V_\mu^A \rightarrow \epsilon V_\mu^A$. The transformation (14) corresponds to the covariant derivative of $A$ in the continuum limit. For the metric $G^{AB}$ and $G_{AB}$, the variation $\delta V_\mu^A$ in (14) satisfies the Killing vector equation in the “superspace” $\{V_\mu^A, G_{AB}\}$:

$$\delta G^{AB} = \partial G^{AB}/\partial V_\mu^C \delta V_\mu^C = G^{AC} \partial \delta V_\mu^B /\partial V_\mu^C + G^{CB} \partial \delta V_\mu^A /\partial V_\mu^C. \quad (15)$$

This relation is easily derived using the Maurer-Cartan equation (3). From the Killing vector relation, the symmetry properties of the $d$ dimensional system can be extended to $d + 1$ dimensions in SQM. It is also possible to introduce the BRS symmetry to the $d + 1$ dimensional system.\(^14\)

In order to describe the internal geometry on the lattice, we derive the Langevin equation for the gauge field $V_\mu(x)$. This follows from (5), and we obtain

$$\Delta V_\mu^A (\tau, x) = -i(\Delta U_\mu) U_\mu^B L_B^A + \frac{i}{2} \Delta V_\mu^B \Delta V_\mu^C ((\partial_B \partial_C U_\mu) U_\mu^A) A^L A^B + O(\Delta \tau^{3/2})$$

(13)
\[
= \left( -i\Delta \tau \tilde{E}(\tau, x, \mu)S[U] + \Delta W_\mu(\tau, x) + i\Delta \tau kN1 \right)^B L_B^A \\
+ i\Delta \tau L_B^B L_B^C \left( (\partial_B \partial_C U_\mu)U^\dagger_\mu \right) L_C^A + O(\Delta \tau^{3/2}) ,
\]
\[
= \left( -i\Delta \tau \tilde{E}(\tau, x, \mu)S[U] + \Delta W_\mu(\tau, x) \right)^B L_B^A(\tau, x, \mu) \\
+ \Delta \tau (\partial_B L_C^A)L_C^B(\tau, x, \mu) + O(\Delta \tau^{3/2}) .
\]

The contribution \(\Delta \tau (\partial_B L_C^A)L_C^B\) in (16) plays an essential role for the covariance of the Langevin equation.

The key observation to understand a precise covariant nature of the Langevin equation for \(U(N)\) (LGT)\(_d\) is that \(\Delta V_\mu^A\) is not a covariant quantity in Itô calculus. Under the general coordinate transformation on the group manifold \(V_\mu^A \rightarrow V_\mu^A\) (16) shows that \(\Delta V_\mu^A(x)\) is transformed as \(\Delta V_\mu^A \rightarrow (\partial_B V_\nu^A)\Delta V_\mu^B + \Delta \tau G^{BC} \partial_B \partial_C V_\nu^A\). This is not a disadvantage of Itô calculus. From the metric tensor, we define a covariant form of \(\Delta V_\mu^A\) as \(\Delta_{\text{cov}}V_\mu^A \equiv \Delta V_\mu^A + \Delta \tau \Gamma^A_{\nu BC}G^{BC}\), which is a contravariant vector under general coordinate transformations. The second term yields \(\Delta \tau \Gamma^A_{\nu BC}G^{BC} = -\Delta \tau \frac{1}{\sqrt{G}} \partial_B(\sqrt{G}G^{AB}) = -\Delta \tau (\partial_B L_C^A)L_C^B\). Now, the covariance is obvious if we write the Langevin equation (16) and the noise correlation as

\[
\Delta V_\mu^A(\tau, x) = -\Delta \tau G^{AB}(\tau, x, \mu) \frac{\partial S}{\partial V_\mu^B(x)}(\tau) - \Delta \tau \Gamma^A_{\nu BC}G^{BC}(\tau, x, \mu) + \Delta W \Xi_\mu^A(\tau, x) ,
\]

\[
\langle \Delta W \Xi_\mu^A(\tau, x)\Delta W \Xi_\nu^B(\tau, y) \rangle_{\Delta W} = 2\Delta \tau G^{AB}(\tau, x, \mu)\delta_{\mu\nu}\delta_{xy} ,
\]

where we have introduced the collective noise field \(\Delta W \Xi_\mu^A(\tau, x) = \Delta W_\mu^B(\tau, x) \times L_B^A(\tau, x)\). In (17), the correlation is not the definition of the collective noise, but it follows from (5). In particular, on the r.h.s. of the correlation, \(G^{AB}(\tau, x, \mu)\) as a function of the link variables is not the expectation value. The corresponding Fokker-Planck equation also appears in a manifestly general coordinate invariant form:

\[
\frac{\partial}{\partial \tau} P(\tau, V) = \sum_{x, \mu} \frac{1}{\sqrt{G}} \frac{\partial}{\partial V_\mu^A(x)} \left( \sqrt{G}G^{AB}(x, \mu) \left( \frac{\partial}{\partial V_\mu^B(x)} + \frac{\partial S}{\partial V_\mu^B(x)} \right) P(\tau, V) \right) .
\]

Here, \(P(\tau, V)\) is a scalar probability density defined by

\[
\langle F[V(\tau)] \rangle_{\Delta W} = \int F[V] P(\tau, V) \sqrt{G} dV .
\]
weight is not reproduced at finite stochastic time. This causes the relaxation behavior in the expectation value (9).

In conclusion, we have derived the Langevin equation (17) for $U(N)$ (LGT)$_d$. For the $SU(N)$ case, we have obtained similar results. We again emphasize the importance of the contribution $\Delta r_k N \mathbf{1}$ in the Langevin equation (5), which is a consequence of Itô calculus. Without it, we would have arrived at neither the group integral defined by the Haar measure (9) nor the manifestly general coordinate covariant form of the Langevin equation (17). The corresponding Fokker-Planck equation is also expressed in the general coordinate invariant form (18). This equation describes a diffusion process on the group manifold. Therefore, the definition of the probability with the Haar measure is necessary to describe the diffusion process in terms of an invariant Fokker-Planck equation, though the Haar measure weight is not reproduced within a finite stochastic time. We note that the covariant nature of the Langevin equation is a direct consequence of Itô calculus.

The Langevin equation (5) can be regarded as a collective field theory if we consider the Langevin equation (17) to be fundamental. Conversely, (17) can be regarded as a collective field theory constructed from (5). It is interesting that the manifest geometric structure is realized as a collective field theory of the underlying system. The structure that is observed for the Langevin equation of (LGT)$_d$ in (17) also appears in the stochastic quantization of $N = 1$ super Yang-Mills theory in the superfield formalism.\(^{15}\)

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