Correspondence of $\mathbb{Z}_4 \times \mathbb{Z}_4$ Orbifold Model and $2^6$ Model

Tatsuo KOBAYASHI*

Department of Physics, College of Liberal Arts and Sciences
Kyoto University, Kyoto 606-01

We construct a 'shifted' $\mathbb{Z}_4 \times \mathbb{Z}_4$ orbifold model equivalent to a twisted $\mathbb{Z}_4 \times \mathbb{Z}_4$ orbifold model at an enhancement point. It is proved that an $\mathcal{N}=2$ superconformal algebra of a $2^6$ theory coincides with that of the 2-dim 'shifted' $\mathbb{Z}_4$ orbifold model. We relate chiral primary states of the $2^6$ model to 'shifted' ground states of the $\mathbb{Z}_4 \times \mathbb{Z}_4$ orbifold model. Yukawa couplings are also studied.

§1. Introduction

A huge number of 4-dim string vacua are obtained by several types of constructions, e.g., orbifold models,\textsuperscript{1} tensoring of coset constructions\textsuperscript{2} and so on. In other words, there are a lot of possibilities to construct the $c=9$ superconformal field theories (SCFT). For example, the $c=9$ superconformal field theories consist of the 6-dim string coordinates satisfying twisted boundary conditions and their superpartners (super-orbifold) in the orbifold models and the $c=9$ theories are also derived from tensoring of the minimal models. It is a great problem how to select uniquely a realistic vacuum among a huge number of the vacua. It is important to study phenomenological features (e.g., Yukawa couplings and structure of moduli) of models obtained by each construction, in order to derive a scenario leading to the low energy physics, e.g., via the supersymmetry breaking.

It is also very meaningful to relate models obtained by the different constructions one another and to understand the whole string vacua within a unified framework. Such analysis gives us geometrical aspects of the $\mathcal{N}=2$ minimal models as byproducts. It is well known that the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold models of the $(2, 2)$ compactifications have the same generation numbers in massless spectra as some Gepner's models.\textsuperscript{3} Recently, it has been proved that the $1^9$ model is equivalent to the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold model at the enhancement point with the $(2, 2)$ compactification from the point of view of the superconformal algebra (SCA).\textsuperscript{4} In Ref. 4), it has been shown that the $1^9$ model is directly equivalent to the shifted $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold model instead of the twisted $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold model. Similar correspondence between other models could be shown.

A purpose of this paper is to relate the $2^6$ model to the $\mathbb{Z}_4 \times \mathbb{Z}_4$ orbifold model with the $(2, 2)$ compactification. Both of these models have ninety 27 massless matter fields and no 27 massless matter field with respect to the $E_6$ gauge group. The $\mathbb{Z}_4 \times \mathbb{Z}_4$

\textsuperscript{3} Fellow of the Japan Society for the Promotion of Science.
orbifold model has $E_6 \times U(1)^2 \times E_8$ gauge group at general points of the moduli space, while the gauge group of the $2^6$ model is $E_6 \times U(1)^3 \times E_8$. At the only enhancement point, the gauge group of the $Z_4 \times Z_4$ orbifold model is enlarged and is the same as one of the $2^6$ model. But each state corresponding to the fixed point does not have definite enhanced $U(1)$ charge. So we construct "shifted" $Z_4 \times Z_4$ orbifold model which is equivalent to the twisted $Z_4 \times Z_4$ orbifold model at the enhancement point in a similar way to the case of the $Z_3 \times Z_3$ orbifold model. States of the shifted orbifold model have definite enhanced $U(1)$ charges, naturally. Then we compare the $2^6$ model with the $Z_4 \times Z_4$ orbifold model, paying our attention to comparison between the internal spaces represented by the $c=9$ superconformal field theories especially.

The paper is organized as follows. In § 2, we review on the twisted $Z_4 \times Z_4$ orbifold model and construct the shifted $Z_4 \times Z_4$ orbifold model which equivalent to the twisted orbifold model, transforming a basis of the Kac-Moody algebra in § 3. Then we compare the $N=2$ superconformal algebras of the $2^6$ theory and the 2-dim shifted $Z_4$ orbifold in § 4 and relate chiral primary states of the $2^6$ model to shifted ground states of the shifted $Z_4 \times Z_4$ orbifold model in § 5. In § 6, Yukawa couplings and phenomenological aspects connected with the couplings are discussed and the last section is devoted to conclusion and discussions.

§ 2. $Z_4 \times Z_4$ orbifold model

At first, we review the $Z_4 \times Z_4$ orbifold model. The $E_8 \times E_8$ heterotic string consists of the $(10+16)$-dim left-moving bosonic string and the 10-dim right-moving superstring. It represents the 10-dim space-time. Momenta $P^t(I=1\sim 26)$ of the 16-dim left-moving bosonic string span an $E_8 \times E_8$ root lattice $A_{E_8 \times E_8}$. For the right-moving fermionic string, we treat bosonized fields, whose momenta $p^t(t=1\sim 4)$ span an $SO(8)$ weight lattice $A_{SO(8)}$ in the light-cone gauge.

We should compactify the 6-dim space in order to obtain 4-dim string vacua from the 10-dim string theory. Hereafter, we represent string coordinates of the 6-dim compact space as $X^i (i=1\sim 6)$. Here, we consider the 6-dim $Z_4 \times Z_4$ orbifold as a 6-dim compact space. The $Z_4 \times Z_4$ orbifold is a quotient of the $R^6$ by an $SO(4)^3$ root lattice and two independent twists $\theta$ and $\omega$, whose eigenvalues on each 2-dim space $(X^{2j-1}, X^{2j})$ ($j=1\sim 3$) are $\exp(-2\pi i/4)$, $1$, $\exp(2\pi i/4)$ and $1$, $\exp(-2\pi i/4)$, $\exp(2\pi i/4)$, respectively. These twists $\theta$ and $\omega$ are, of course, automorphism of the $SO(4)^3$ lattice. When we twist the 6-dim string coordinates $X^i$ ($i=1\sim 6$), some transformation should take place on the right-moving superpartner in order to preserve the world-sheet supersymmetry. Namely, in connection with the twists $\theta$ and $\omega$, the $SO(8)$ weight lattice is shifted by two independent shifts $v_1^t$ and $v_2^t$, respectively. Elements of these shifts are

$$v_1^t= \left( \frac{1}{4}, 0, -\frac{1}{4}, 0 \right), \quad v_2^t= \left( 0, \frac{1}{4}, -\frac{1}{4}, 0 \right).$$

Further, in the case of the $(2, 2)$ compactification, the $E_8 \times E_8$ root lattice is shifted by two independent shifts $V_1^t=(1/4, 0, -1/4, 0, \cdots, 0)$ and $V_2^t=(0, 1/4, -1/4, 0, \cdots, 0)$ corresponding to $\theta$ and $\omega$, respectively. The shifts $v_1^t$ and $v_2^t$ on the $A_{SO(8)}$ break
$N=4$ supersymmetry in the 4-dim space-time into $N=1$ supersymmetry corresponding to the $SO(8)$ quantum numbers $\pm (1/2, 1/2, 1/2, 1/2)$ and the shifts $V_i^t$ and $V_2^t$ on the $\Lambda_{E_6 \times E_8}$ break the $E_6 \times E_8$ gauge group into the $E_6 \times U(1)^2 \times E_8$ group.

It is well known that closed strings on the orbifold are classified into two types. One is the untwisted sector and the other is the twisted sector. Massless conditions of the right-moving and left-moving untwisted sectors are obtained as

$$\frac{1}{2} \sum_{i=1}^{4} (p^t)^2 + N_R = \frac{1}{2} ,$$  \hspace{1cm} (2.1)

$$\frac{1}{2} \sum_{j=1}^{3} ((P_L^{2j-1})^2 + (P_L^{2j})^2) + \frac{1}{2} \sum_{i=1}^{26} (P^t_i)^2 + N_L = 1 ,$$  \hspace{1cm} (2.2)

where $N_{R(L)}$ is an oscillation number and $P_i^t (i=1 \sim 6)$ are left-moving momenta of the 6-dim compactified string coordinates $X^i$. States with nonzero momenta $P_i^t$ do not satisfy the massless condition (2.2) at the general points of the moduli space, i.e., in the case of the general radii of the $SO(4)$ torus, but they satisfy Eq. (2.2) at a specific radius, i.e., an enhancement point.

For the states without quantum numbers in the 6-dim space, conditions to combine the right and the left massless states with momenta $p^t$ and $P^t$ are obtained as

$$\exp 2\pi i (\sum_{i} p_i^t p^t - \sum_{i} V_i^t P^t^t) = 1 ,$$  \hspace{1cm} (2.3a)

$$\exp 2\pi i (\sum_{i} v_i^t p^t - \sum_{i} V_2^t P^t^t) = 1 ,$$  \hspace{1cm} (2.3b)

by the generalized GSO projection for the $Z_N \times Z_M$ orbifold models$^{3,5}$ in the similar way to the projection of the $Z_N$ orbifold models.$^{6,7}$ The conditions imply that the first three elements of $p^t$ and $P^t$ of the combined states are the same. Momenta $p^t = (1000)$ satisfy the massless condition (2.1), where the underline means all the possible permutations of the elements. Away from the enhancement point, the right-moving massless state with $p^t = (0001)$ is combined with the left-moving massless states whose momenta $P^t$ span the $E_6 \times U(1)^2 \times E_8$ root system and corresponds to the gauge bosons. The other states with $p^t = (1000)$ correspond to 27 representations of $E_8$ in the left states. Namely, the untwisted sector has three generations.

Further, the orbifold model has more massless gauge bosons at the specific radius of the $SO(4)$ torus so that $(P_i^t)^2 = 2$. The state with $(P_i^t)^2 = 2$ is denoted by a vertex operator $\exp iP_i^t X_i^t$. But each state is not physical, because the physical states should be invariant under the twists $\theta$ and $\omega$, i.e., the $Z_4$ twist on each 2-dim space $(X_2^{2j-1}, X_2^{2j})$. Therefore, for each 2-dim space, there is only one physical state denoted as

$$\frac{1}{2} (E_{i}(\sqrt{2}, 0) + E_{i}(0, \sqrt{2}) + E_{i}(-\sqrt{2}, 0) + E_{i}(0, -\sqrt{2})) ,$$  \hspace{1cm} (2.4)

where $E_{i}(a, b) = \exp i [a X_i^{2j-1} + b X_i^{2j}]$, and the state (2.4) corresponds to the enhanced $U(1)$ gauge boson. We can obtain at most $E_8 \times U(1)^2 \times E_8$ gauge group in the $Z_4 \times Z_4$ orbifold model with the $(2, 2)$ compactification and the gauge group coincides with one of the $2^6$ model.
On the other hand, the strings of the $\theta^a \omega^t$-twisted sector have the twisted boundary condition for the 6-dim compact string coordinates as

$$X(\sigma=2\pi)=\theta^a \omega^t X(\sigma=0)+e,$$

where $e$ is a vector on the $SO(4)^3$ root lattice. The zero-mode of this twisted string, i.e., the center of mass satisfies the same condition as Eq. (2.5). A point satisfying the condition is called a fixed point and is sometimes denoted by a space group element $(\theta^a \omega^t, e)$, where we call the vector $e$ the return vector. Independent fixed points are distinguished by the conjugacy class and classify the ground states of the twisted sectors. The twisted string of this sector has the $SO(8)$ momenta $\bar{p}^t=p^t+kv_1^t+ lv_2^t$ and $E_8 \times E_8$ momenta $P^t+kV_1^t+IV_2^t$, where $p^t$ and $P^t$ are vectors on $\Lambda SO(8)$ and $\Lambda E_8 \times E_8$, respectively. Massless conditions of the right-moving and left-moving twisted sectors are obtained as

$$\frac{1}{2} \sum_{i=1}^{28} (P^t+kV_1^t+IV_2^t)^2 + N_{R(h,i)} + c_{ht} = \frac{1}{2},$$

$$\frac{1}{2} \sum_{i=1}^{28} (P^t+kV_1^t+IV_2^t)^2 + N_{L(h,i)} + c_{lt} = 1,$$

where

$$c_{ht} = \frac{1}{2} \sum |kv_1^t+lv_2^t|-\text{Int}|kv_2^t+lv_2^t|\sum_{i=1}^{28} (P^t+kV_1^t+IV_2^t)^2 + N_{R(h,i)} + c_{ht} = \frac{1}{2},$$

and $N_{R(h,i)}$ is an oscillation number of the $\theta^a \omega^t$-twisted sector. Momenta satisfying Eq. (2.6) are explicitly obtained in Ref. 5). For the $SO(8)$ and $E_8 \times E_8$ parts, eigenvalues under shifts $v_1^t, V_1^t$ and $v_2^t, V_2^t$ are

$$\Delta_1 = \exp 2\pi i \sum v_1^t (p^t+kv_1^t+ lv_2^t) - \sum V_1^t (P^t+kV_1^t+IV_2^t) - \sum v_1^t (P^t+kV_1^t+IV_2^t)$$

$$\Delta_2 = \exp 2\pi i \sum v_2^t (p^t+kv_1^t+ lv_2^t) - \sum V_2^t (P^t+kV_1^t+IV_2^t).$$

The physical states should be invariant under a simultaneous transformation of the twist $\theta$ and shifts $v_1^t, V_1^t$ as well as a simultaneous transformation of the twist $\omega$ and shifts $v_2^t, V_2^t$. In the $Z_4 \times Z_4$ orbifold model with the (2, 2) compactification, all 27 physical massless states have the eigenvalues $\Delta_1=1$ and $\Delta_2=1$. Namely, the 27 physical states should be invariant under both twists $\theta$ and $\omega$ of the 6-dim space.

We discuss on the twists and the structure of the twisted ground states in detail. All twisted states of the 6-dim $Z_2 \times Z_2$ orbifold are obtained by tensoring three copies of the 2-dim $Z_2$ orbifold so that a sum of the twists of each 2-dim sector is an integer. We treat the 2-dim string coordinates $(X^1, X^2)$ and a field $H$ which is the bosonized field corresponding to the superpartner of $(X^1, X^2)$. Hereafter, we treat the right-moving sector mainly, because the corresponding left-moving sector has the same quantum numbers as the right sector for the internal space represented in terms of the $c=9$ theory, although the enhanced $U(1)$ charges have concrete meaning only in the left sector. In the left-moving sector, $H$ corresponds to $X^1$. Here, momenta $\bar{p}^1$ of $H$ are called $SO(2)$ momenta and we call $X^1, X^2$ and $H$ the super-orbifold. It is
remarkable that the $i\partial H$ is the $U(1)$ current of the $N=2$ superconformal algebra. Further, $X^1$ and $X^2$ are transformed as $X^1 \to -X^2$ and $X_2 \to X_1$ under the $Z_4$ twist.

The 2-dim $Z_4$ orbifold has 1/4, 1/2 and 3/4-twisted sectors. Let $e_1$ and $e_2$ be two simple roots of the $SO(4)$ algebra. For the 1/4-twisted sector, there are two fixed points whose return vectors are $\beta_0=(0, 0)$ and $\beta_1=(1, 1)$, where $(a, b)\equiv ae_1+be_2$. Therefore, there are two ground states $|\beta_0\rangle$ and $|\beta_1\rangle$ corresponding to the two fixed points in the 1/4-twisted sector. The conformal dimension of the bosonic 1/4-twist field equal to 3/32. These ground states have the $SO(2)$ momentum $p^1=1/4$ and the $SO(2)$ momentum contributes to the conformal dimension as 1/2($p^1$)$^2$=1/32. The conformal dimension of these states of the super-orbifold is totally equal to 1/8 and the $U(1)$ charge of the $N=2$ SCA of these states is equal to 1/4. Therefore, these states are chiral primary states as we will discuss below.

For the 1/2-twisted sector, there are four fixed points, whose return vectors are $\gamma_0=(0, 0), \gamma_1=(1, 0), \gamma_2=(1, 1)$ and $\gamma_3=(0, 1)$ in the $e_1-e_2$ basis. It is remarkable that a conjugacy class corresponding to $\gamma_1$ is transformed into one corresponding to $\gamma_3$ under the $Z_4$ twist and vice versa. There are four ground states $|\gamma_0\rangle, |\gamma_1\rangle, |\gamma_2\rangle$ and $|\gamma_3\rangle$ for the 1/2-twisted sector. Among the four ground states, $|\gamma_1\rangle$ and $|\gamma_3\rangle$ are not eigenstates of the $Z_4$ twist, while two states $|\gamma_0\rangle$ and $|\gamma_2\rangle$ are eigenstates. Linear combinations $|\gamma_{\pm}\rangle=|\gamma_1\rangle \pm |\gamma_3\rangle$ are eigenstates of the $Z_4$ twist with eigenvalues $\pm 1$, respectively.\(^9\) The eigenstate with the eigenvalue $-1$ does not survive as the physical state in the $Z_4 \times Z_4$ orbifold with the (2, 2) compactification. The conformal dimension of the bosonic 1/2-twist field is equal to 1/8. These ground states have the $SO(2)$ momentum $p^1=1/2$ and their conformal dimension is totally equal to 1/4. These ground states are also the chiral primary states.

The 3/4-twisted sector has the same structures of the fixed points and the ground states as the 1/4-twisted sector. However, the 3/4-twisted sector has the $SO(2)$ momenta $p^1=3/4$. We can obtain the twisted ground states of the 6-dim $Z_4 \times Z_4$ orbifold by tensoring three copies of the 2-dim $Z_4$ orbifold and each physical twisted ground state of the 6-dim orbifold corresponds to the 27 matter field with respect to the $E_8$ gauge group. The twisted sectors have totally 87 generations.

Next, we study on the $U(1)$ charge of each physical twisted ground state under the enhanced $U(1)$ current (2·4). But the physical state is transformed into other physical state under the operation of the $U(1)$ current (2·4), i.e., they do not have definite charges. Therefore, we must take linear combinations ($U(1)$-diagonal basis) of these states in order to obtain states with definite $U(1)$ charges. Instead of doing so, we construct 2-dim 'shifted' $Z_4$ orbifold equivalent to the 2-dim twisted orbifold in the next section because the 2-dim 'shifted' orbifold states have definite enhanced $U(1)$ charges naturally.

§ 3. 2-dim shifted $Z_4$ orbifold

In this section, we construct 2-dim shifted $Z_4$ orbifold model equivalent to the 2-dim twisted $Z_4$ orbifold model at the enhancement point, following the procedure to transform the Kac-Moody algebra basis.\(^9\)-\(^11\) Vertex operators $E(\pm \sqrt{2}, 0), E(0, \pm \sqrt{2})$ and two Cartan elements $i\partial X^1, i\partial X^2$ compose the $SO(4)$ Kac-Moody algebra.
We select the current of Eq. (2·4) as a new Cartan element \(i\partial X'\), i.e.,

\[
i\partial X' = \frac{1}{2} (E(\sqrt{2}, 0) + E(0, \sqrt{2}) + E(-\sqrt{2}, 0) + E(0, -\sqrt{2})).
\]  

(3·1)

Of course eigenvalues of this operator (i.e., momenta of \(X'\)) are the enhanced \(U(1)\) charges. Next, we choose the other new Cartan element \(i\partial X^2\) so that it is orthogonal to \(i\partial X'\), i.e.,

\[
i\partial X^2 \equiv \frac{1}{2} (E(\sqrt{2}, 0) - E(0, \sqrt{2}) + E(-\sqrt{2}, 0) - E(0, -\sqrt{2})),
\]

(3·2)

These elements \(i\partial X'\) and \(i\partial X^2\) have eigenvalues 1 and \(-1\) under the \(Z_4\) twist, respectively. We can easily obtain new \(SO(4)\) Kac-Moody elements corresponding to four non-zero roots under the new Cartan subalgebra (3·1) and (3·2) as

\[
E'(\pm 1, \mp 1) = \frac{1}{\sqrt{2}} e^{\pm i\theta} \left[ E(\sqrt{2}, 0) - E(-\sqrt{2}, 0) \right] / \sqrt{2} \mp i\partial X_1,
\]

\[
E'(\pm 1, \mp 1) = \mp \frac{1}{\sqrt{2}} e^{\pm i(\gamma + \delta)} \left[ E(0, \sqrt{2}) - E(0, -\sqrt{2}) \right] / \sqrt{2} \mp i\partial X_2,
\]

where

\[
i\partial X^2(z)E'(P^1, P^2)(\omega) \sim \frac{P^1}{z - \omega} E'(P^1, P^2)(\omega),
\]

\[
i\partial X^2(z)E'(P^1, P^2)(\omega) \sim \frac{P^2}{z - \omega} E'(P^1, P^2)(\omega).
\]

The two phases \(\gamma\) and \(\delta\) are ambiguous. It is remarkable that the eigenvalues \((P^1, P^2)\) of the roots in the new \(SO(4)\) Kac-Moody algebra are rotated by \(\pi/2\) relative to the original eigenvalues \((P_1, P_2)\).

Under the \(Z_4\) twist, each element of the new \(SO(4)\) Kac-Moody algebra is transformed as

\[
i\partial X_1 \rightarrow i\partial X_1, \quad i\partial X_2 \rightarrow i\partial X_2,
\]

\[
E'(\pm 1, \mp 1) \rightarrow \pm ie^{\mp i\delta} E'(\pm 1, \mp 1), \quad E'(\pm 1, \mp 1) \rightarrow \pm e^{\pm i\delta} iE'(\pm 1, \mp 1).
\]

(3·3)

From Eq. (3·3), it is clear that the phase \(\gamma\) is meaningless. Further, Eq. (3·3) implies that the new string coordinates \(X'\) and \(X^2\) are transformed under the \(Z_4\) twist as

\[
X' \rightarrow X' + \frac{2\pi}{4}, \quad X^2 \rightarrow -X^2 - \delta.
\]

(3·4)

The relative phase \(\delta\) shifts locations of fixed points and has no physical effect. In this section we choose the vanishing phase \(\delta = 0\), so that the \(Z_4\) twist is represented as the reflection of \(X^2\), i.e., \(X^2 \rightarrow -X^2\).
Under the above transformation \((3\cdot4)\), we consider twisted (or 'shifted') states. First of all, we discuss on the reflection of the \(X^2\). When we consider the ordinary 1-dim \(Z_2\) orbifold (i.e., a quotient of the circle by the \(Z_2\) reflection), the \(Z_2\) orbifold has two independent fixed points, i.e., the origin and the mid-point. Under the reflection of the \(X^2\) in the 2-dim space \((X'^1, X'^2)\), there are two fixed lines in a similar way to the 1-dim ordinary \(Z_2\) orbifold. One fixed line involves the origin and the other involves a point \((0, 1)\) in the \((e'_1, e'_2)\) basis. However these fixed lines belong to one conjugacy class. Thus we cannot distinguish twisted ground states by the center of mass.

For the 1/4-shifted sector, momenta \(\mathbf{P}'^1\) of the \(X'^1\) are shifted by 1/4 from the original values \(P'^1\) on the \(SO(4)\) weight lattice and they represent the enhanced \(U(1)\) charges under Eq. \((3\cdot1)\). The conformal dimensions of the ground states are obtained as

\[
h = \frac{1}{2} \left( P'^1 + \frac{1}{4} \right)^2 + \frac{1}{16}.
\]

The second term on the right-hand side is contributed by the \(Z_2\) twisted boundary condition of the coordinate \(X^2\). The conformal dimensions \(h\) take the minimum value \(h=3/32\), when \(P'^1=0\) and \(P'^1=-1/2\). Namely, the 1/4-shifted sector has two ground states and their conformal dimension is, of course, the same as one of the 1/4-twisted sector of the 2-dim twisted orbifold said in the previous section. Further, these two ground states have the enhanced \(U(1)\) charges \(\mathbf{P}'^1=\pm 1/4\), respectively.

For the 1/2-shifted sector, we can obtain the conformal dimensions of the ground states with the momenta \(P'^1\) and \(P'^2\) as

\[
h = \frac{1}{2} \left( P'^1 + \frac{1}{2} \right)^2 + \frac{1}{2} (P'^2)^2.
\]

The eigenvalues of \(h\) take the minimum value, when \((P'^1, P'^2)=(0, 0), (-1, 0), (-1/2, 1/2)\) and \((-1/2, -1/2)\). Therefore, the 1/2-shifted sector has four ground states \(|0, 0\rangle, |1, 0\rangle, |1/2, 1\rangle\) and \(|1/2, -1\rangle\), whose eigenvalue \(h=1/8\) is, of course, the same as one of the 1/2-twisted sector of the 2-dim \(Z_4\) twisted orbifold said in the previous section. But the states \(|1/2, 1\rangle\) and \(|1/2, -1\rangle\) are not physical, because they are transformed each other under the \(Z_4\) twist, i.e., the reflection of \(X^2\). Eigenstates are obtained in terms of linear combinations of these states. Thus eigenstates of the 1/2-shifted sector are \(|0, 0\rangle, |1, 0\rangle, |1/2, 1\rangle, |1/2, -1\rangle\) and \(|1/2, 1\rangle, |1/2, -1\rangle\). The eigenvalue of the last state under the \(Z_4\) twist is equal to \(-1\) and eigenvalues of the other states are equal to \(1\). The eigenstate with the eigenvalue \(-1\) does not appear in the massless spectrum for the \(Z_4\times Z_4\) orbifold model with the \((2, 2)\) compactification, as said in the previous section. Furthermore, the eigenstates with the eigenvalue \(1\) have the enhanced \(U(1)\) charges 1/2, \(-1/2\) and 0, respectively.

We can obtain the ground states of the 3/4-shifted sector in the similar way to those of the 1/4-shifted sector. As a result, there are two ground states in the 3/4-shifted sector and these states have conformal dimension \(h=3/32\), which is the same as one of the 3/4-twisted sector in the 2-dim twisted \(Z_4\) orbifold. Further, these shifted states have the enhanced \(U(1)\) charges \(\pm 1/4\).
Thus, we have constructed the shifted ground states on the 2-dim shifted $Z_4$ orbifold equivalent to the twisted ground states on the 2-dim twisted $Z_4$ orbifold and investigated the enhanced $U(1)$ charges of the shifted states. For the superpartner, the $k/4$-shifted sector in the 2-dim $Z_4$ shifted orbifold corresponds to the same $SO(2)$ momentum $\tilde{p}^1$ as the $k/4$-twisted sector in the 2-dim $Z_4$ twisted orbifold.

We can obtain the shifted sectors of the $Z_4 \times Z_4$ shifted orbifold model by tensoring the three copies of the shifted sectors on the 2-dim $Z_4$ shifted orbifold so that a sum of shifts in each 2-dim space is an integer, in other words a sum of the corresponding $SO(2)$ momenta $\tilde{p}^1 + \tilde{p}^2 + \tilde{p}^3$ is equal to one. That is the similar procedure to obtain the $Z_4 \times Z_4$ twisted orbifold as the tensor product of the three 2-dim $Z_4$ twisted orbifold. Each shifted ground state obtained by the tensor product corresponds to a 27 massless matter field with respect to the $E_6$ gauge group in the heterotic construction.

§ 4. $N=2$ superconformal algebra

In this section, we discuss on representations of the $N=2$ SCA for the $2^6$ and the $Z_4 \times Z_4$ orbifold models. First of all, we review the $2^6$ model in brief. The $N=2$ SCA of the level $k=2$ minimal model is represented by a free boson $\phi$ and a real fermion $\psi$. Its elements are

$$T = -\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} \psi \partial \psi,$$

$$G^\pm = \phi e^{\pm i \pi \tilde{p}},$$

$$J = \frac{i}{\sqrt{2}} \partial \phi.$$ (4.1)

The primary states of this algebra are characterized by their conformal dimensions $h$ and $U(1)$ charges $Q$ under the current $J$. They are obtained as

$$h = \frac{l(l+2) - q^2}{16} + \frac{1}{2} s^2, \quad Q = -\frac{q}{4} + s,$$ (4.2)

where $l, q =$ integer, $s = 0, \pm 1/2, 1$ and they satisfy $l \leq 2$, $|q - 2s| \leq l$ and $l + q + 2s$ = even. Chiral primary states are the primary states satisfying the condition $2h = Q$, that is, they satisfy $l = q$ and $s = 0$. Namely, chiral primary states of the $k=2$ minimal model have conformal dimensions $h = q/8$ ($q = 0, 1, 2$) and $U(1)$ charges $Q = q/4$.

The $c=9$ SCFT is obtained by tensoring six copies of the $k=2$ minimal models. Chiral primary states with a sum of $U(1)$ charges $\sum Q = 1$ correspond to the 27 massless matter fields in the heterotic construction derived from such a $c=9$ theory. The $2^6$ model has ninety 27 massless matter fields, which are represented by

$$(q_1, q_2, \cdots, q_6) = (2, 2, 0, 0, 0, 0), (2, 1, 1, 0, 0, 0), (1, 1, 1, 1, 0, 0).$$ (4.3)

A purpose of this paper is to relate each state of Eq. (4.3) to a state of the $Z_4 \times Z_4$ orbifold model.

Next, we consider the $N=2$ SCA of the orbifold model. For the 2-dim $Z_4$ twisted
super-orbifold, elements of the $N=2$ SCA are
\begin{align*}
T_i &= -\frac{1}{2}(\partial X^1)^2 - \frac{1}{2}(\partial X^2)^2 - \frac{1}{2}(\partial H)^2, \\
G_i^\pm &= (\partial X^1 \pm i\partial X^2)e^{\pm i\theta}, \\
J_i &= i\partial H. 
\end{align*}
Using the transformation of the $SO(4)$ Kac-Moody algebra basis, we can easily derive $N=2$ SCA elements of the 2-dim shifted $Z_4$ super-orbifold from the elements (4.4) as
\begin{align*}
T_s &= -\frac{1}{2}(\partial X'^1)^2 - \frac{1}{2}(\partial X'^2)^2 - \frac{1}{2}(\partial H)^2, \\
G_s^\pm &= \frac{1}{\sqrt{2}}e^{\pm im}\{e^{-i\gamma}E'(1, 1) - e^{i\gamma}E'(-1, 1) \\
&\quad \pm e^{i(r+s)}E'(-1, 1)\pm e^{-i(r+s)}E'(1, -1)}, \\
J_s &= i\partial H, \tag{4.5}
\end{align*}
where
\begin{align*}
E'(P^1, P^2) &= \exp[iP^1X'^1 + iP^2X'^2].
\end{align*}

We consider tensoring two copies of the $k=2$ minimal models as a counterpart to the 2-dim super-orbifold, because the 2$^2$ theory has the central charge $c=3$. The 2$^2$ theory consists of two free bosons and two real fermions $(\phi_i, \psi_i)$ $(i=1, 2)$, where $\phi_i$ and $\psi_i$ represent the $N=2$ SCA elements $T_i(i), G_i^\pm(i)$ and $J_i(i)$ like Eq. (4.1). We bosonize the fields $\psi_1$ and $\psi_2$ as
\begin{align*}
(\phi_1 \pm i\psi_2) &= \sqrt{2}e^{\pm i\theta}, \tag{4.6}
\end{align*}
in order to express the $N=2$ SCA in terms of only bosonic fields. Further, we mix the bosons $\phi_1$ and $\phi_2$ to derive new bosons $\phi_\pm$ as
\begin{align*}
\phi_\pm &= \frac{1}{\sqrt{2}}(\phi_1 \pm \phi_2). \tag{4.7}
\end{align*}
Here, we can represent the $N=2$ SCA in terms of the new fields. For instance, a sum of the supercurrents $G_i^1(i)$ and $G_i^2(i)$ is described as
\begin{align*}
G_i^1(i) + G_i^2(i) &= \frac{1}{\sqrt{2}}e^{i\theta^*}(e^{i\theta}e^{-i\phi} + e^{i\phi}e^{-i\phi} - ie^{-i\phi}e^{i\phi} + ie^{-i\phi}e^{-i\phi}). \tag{4.8}
\end{align*}
That is very similar to the supercurrent of Eq. (4.5). In fact, the supercurrent $G_i^1(i) + G_i^2(i)$ is equivalent to the $G_s^+$ up to an overall phase, when we choose the adequate phases $\gamma$ and $\delta$ and identify the fields one another as follows,
\begin{align*}
\phi_+ &= H, \quad \phi_- = X'^1, \quad \Phi = X'^2. \tag{4.9}
\end{align*}
In the same way, we can recognize that the other supercurrent $G_i^3(i) + G_i^4(i)$ is equivalent to the supercurrent $G_s^-$ of Eq. (4.5) under the above indentification between the fields.
It is easy to relate the elements $T_{(1)} + T_{(2)}$ and $J_{(1)} + J_{(2)}$ to $T_s$ and $J_s$. Furthermore, the shifted super-orbifold has another $U(1)$ current, that is the enhanced $U(1)$ current $i\partial X^\mu$. The existence of the current $i\partial X^\mu$ indicates reducibility of the $N=2$ SCA of Eq. (4.5). The current $i\partial X^\mu$ corresponds to the current $J_{(1)} - J_{(2)}$ under the field identification (4.9) as a matter of course.

§ 5. $Z_4 \times Z_4$ orbifold and $2^2$ models

In the previous section, it is found that the $N=2$ SCA elements of $2^2$ theory are represented in the same way as those of the 2-dim $Z_4$ shifted super-orbifold wholly. Here, we consider correspondence between the chiral primary states of these $N=2$ superconformal algebras. Actually, we relate the shifted ground states of the 2-dim $Z_4$ shifted orbifold to the chiral primary states of the $2^2$ theory. The latter states are characterized by $(q_1, q_2)$ and the $2^2$ theory has nine chiral primary states:

$$(q_1, q_2) = (0, 0), (1, 0), (2, 0), (1, 1), (2, 1), (2, 2).$$

(5.1)

The identification between the fields, Eq. (4.9) leads to relations between the quantum numbers of the states. Namely, the quantum numbers $q_1$ and $q_2$ are related to the $SO(2)$ momentum $\vec{p}_1$ and the enhanced $U(1)$ charge $\vec{P}^\mu$ as

$$\frac{1}{4}(q_1 + q_2) = \vec{p}_1, \quad \frac{1}{4}(q_1 - q_2) = \vec{P}^\mu. \quad (5.2)$$

Then we relate the chiral primary states of the $2^2$ theory to the shifted ground states of the 2-dim super-orbifold. Using the relation (5.2), the states $(1, 0)$ and $(0, 1)$ among the states of Eq. (5.1) have the $SO(2)$ momentum $\vec{p}_1 = 1/4$, that is, these states correspond to the 1/4-shifted sector of the orbifold model. Moreover, these states have the enhanced $U(1)$ charges $\pm 1/4$. Thus the above states $(1, 0)$ and $(0, 1)$ correspond exactly to the two 1/4-shifted ground states said in § 3. Of course, all of these states have the same conformal dimension.

In a similar way, the states $(2, 0), (1, 1)$ and $(0, 2)$ among the states of Eq. (5.1) have the $SO(2)$ momentum $\vec{p}_1 = 1/2$ and the enhanced $U(1)$ charges $1/2, 0$ and $-1/2$, respectively. Therefore, these states correspond to the 1/2-shifted ground states which are invariant under the $Z_4$ twist. Furthermore, the states $(2, 1)$ and $(1, 2)$ correspond to the two 3/4-shifted ground states on the orbifold, exactly.

On the other hand, the state $(2, 2)$ of Eq. (5.1) has the $SO(2)$ momentum $\vec{p}_1 = 1$, that is, the state does not have fractional quantum number for the $SO(2)$ momentum. For that reason, the state $(2, 2)$ corresponds to the unshifted (untwisted) sector. Namely, the three states $(2, 2, 0, 0, 0, 0), (0, 0, 2, 2, 0, 0)$ and $(0, 0, 0, 0, 2, 2)$ in the $2^6$ model correspond to the three untwisted states with the $SO(8)$ momenta $\vec{p}_1 = (1, 0, 0, 0), (0, 1, 0, 0)$ and $(0, 0, 1, 0)$, respectively, in the $Z_4 \times Z_4$ orbifold model.

Through the above discussion, it becomes clear that both structures of the chiral primary states of the $2^6$ theory and the 2-dim $Z_4$ shifted super-orbifold are the same. Namely, the 2-dim shifted $Z_4$ super-orbifold represents two $N=2$ superconformal algebras with the level $k=2$. Finally, we relate the $2^6$ model to the $Z_4 \times Z_4$ orbifold model. Especially, we compare the $c=9$ superconformal field theories of the two
Correspondence of $Z_4 \times Z_4$ Orbifold Model and 2\textsuperscript{6} Model

models and their chiral primary states corresponding to 27 matter fields in the heterotic construction. For that purpose, we study tensoring conditions to derive chiral primary states corresponding to the 27 matter fields in the $c=9$ theories from the 2\textsuperscript{6} model. For the $2^6$ model, the chiral primary states (4·3) of the $c=9$ theory are obtained by tensoring the chiral primary states (5·1) of the 2\textsuperscript{2} theories so that a sum of $U(1)$ charge $Q$ is equal to one,

$$\sum Q = \frac{1}{4} \sum q_i = 1.$$  \hfill (5.3)

Making use of Eq. (5.2), this tensoring condition (5.3) leads to a tensoring condition for the momenta $\tilde{p}^t$ as

$$\tilde{p}^1 + \tilde{p}^2 + \tilde{p}^3 = 1.$$  \hfill (5.4)

This condition is nothing but the tensoring condition to derive states of the $Z_4 \times Z_4$ orbifold model from the 2-dim orbifold. Therefore, we can derive equivalence between the chiral primary states corresponding to the 27 matter fields in the $2^6$ model and the shifted $Z_4 \times Z_4$ orbifold model, from correspondence between the $2^6$ theory and the 2-dim shifted $Z_4$ super-orbifold and equivalence between the tensoring conditions of the chiral primary states. We can clarify correspondence between the $E_6$ singlet massless matter fields of the two models in the same way as the 27 massless matter contents.

The above discussion on the correspondence can be extended to the case of the 6-dim $Z_4$ orbifold model with the (2, 2) compactification and a twisted $2^6$ model,\textsuperscript{15} imposing a new tensoring condition. The 6-dim $Z_4$ orbifold is a quotient of the $SO(4)^3$ torus by only one twist $\Theta$ whose eigenvalue in each 2-dim space is $\exp(-2i\pi/4)$, $\exp(-2i\pi/2)$, $\exp(2i\pi/2)$. Namely, the twist $\Theta$ is a product of two twists $\theta$ and $\omega$, i.e., $\Theta = \theta \omega$. On this orbifold, there are only three twisted sectors, i.e., $\Theta$-twisted, $\theta^2$-twisted and $\theta^3$-twisted sectors. These twisted sectors are obtained by picking out the twisted sectors which are symmetric with respect to the two twists $\theta$ and $\omega$, from the $Z_4 \times Z_4$ orbifold model. This condition to pick out is represented in terms of the $SO(8)$ momenta as $\tilde{\mathbf{p}}^1 - \tilde{\mathbf{p}}^2 = \text{integer}$, or

$$\sum c^t \tilde{p}^t = \text{integer},$$  \hfill (5.5)

where $c^t = (1, -1, 0, 0)$. Through the relation (5.2), the condition (5.5) is explained in the $2^6$ model as among the states (4·3), we have to pick out states satisfying a condition,

$$\sum \gamma_i q_i = 0 \pmod{4}$$  \hfill (5.6)

where $\gamma_i = (1, 1, -1, -1, 0, 0)$. In Ref. 15, it has been shown that the 6-dim $Z_4$ orbifold model with the (2, 2) compactification and the $2^6$ model twisted by $\gamma_i$ have the same massless spectrum. For example, in the twisted sector of the twisted $2^6$ model, a right-moving state with all vanishing quantum numbers, $l_i = q_i = s_i = 0 (i = 1 \sim 6)$ corre-

*) See Ref. 14) in detail.
sponds to left-moving states with $\bar{T}_i = 0, (\bar{q}_1, \cdots, \bar{q}_5) = (2, 2, -2, 0, 0)$ and $(\bar{s}_1, \cdots, \bar{s}_8) = (2, 2, 2, 2, 0, 0)$, as well as the state with all vanishing quantum numbers in the untwisted sector with respect of the twist $\gamma_i$. Through the relation (5·2), the left-moving states with the non-vanishing quantum numbers correspond to the left-moving untwisted (unshifted) states of the orbifold with the $E_8 \times E_8$ momenta $P^I = (1, -1, 0, \cdots, 0)$ ($I = 1 \sim 8$), which are non-zero roots of the $SU(2)$ algebra, exactly. Thus, the gauge group $E_6 \times U(1)^5 \times E_8$ is enlarged into the $E_6 \times SU(2) \times U(1)^4 \times E_8$. Similar analysis on the correspondence could be extended to the case of other twisted $2^6$ models and the $Z_4 \times Z_4$ orbifold and the $Z_4$ orbifold models with the $(0, 2)$ compactification.

§ 6. Yukawa coupling

In this section, we discuss the Yukawa coupling conditions for $(27)^3$ couplings, i.e., couplings of the chiral primary states and comment on phenomenological aspects connected with the Yukawa couplings. At first, we compare coupling conditions of the $2^2$ theory and the 2-dim $Z_4$ shifted super-orbifold. In the $2^2$ theory, three states denoted by $(q_1^{(i)}, q_2^{(i)})$ ($i = 1, 2, 3$) are allowed to couple, if their quantum numbers satisfy a condition,

$$\sum q_1^{(i)} = \sum q_2^{(i)} = 2.$$  \hspace{1cm} (6·1)

Through the relation (5·2), the coupling condition (6·1) leads to two conditions that a sum of enhanced $U(1)$ charges of coupling states should be equal to zero and that a sum of $SO(2)$ momenta of coupling states should be equal to one. These conditions mean the enhanced $U(1)$ invariance and the $SO(2)$ invariance, where the sum of the $SO(2)$ momenta is shifted from zero to one, because we consider couplings between one boson and two fermions. Thus the coupling condition of the $2^2$ theory is equivalent to one of the 2-dim $Z_4$ super-orbifold. Of course, that implies coincidence between coupling conditions of the $2^6$ model and the $Z_4 \times Z_4$ orbifold model at the enhancement point.

Next, we consider the twisted orbifold away from the enhancement point. Here, we treat the twisted states in the $U(1)$-diagonal basis, where the twisted states have the same enhanced $U(1)$ charges as the shifted ground states of the corresponding shifted sector. Away from the enhancement point, the state of Eq. (2·4) is no longer massless. That fact is explained from the point of view of the effective field theory that an untwisted singlet matter field with enhanced $U(1)$ charge equal to one has a non-vanishing vacuum expectation value. Such theory allows more Yukawa couplings than the theory at the enhancement point. Away from the enhancement point, states are allowed to couple, if a sum of the corresponding enhanced $U(1)$ charges is equal to an integer. Note that the general states have the fractional enhanced $U(1)$ charges. This selection rule suggests some symmetry, which is called "discrete" gauge symmetry. The selection rule due to this discrete gauge symmetry in the $U(1)$-diagonal basis corresponds to the selection rule due to the space group invariance in the basis corresponding to the fixed points. Recently, the discrete gauge symmetry
Correspondence of \( Z_4 \times Z_4 \) Orbifold Model and \( 2^6 \) Model attracts attentions.\(^{17,18} \) Originally, the discrete symmetries were introduced to suppress phenomenologically undesirable couplings. These discrete symmetries should be the gauge symmetries to survive under quantum gravity. The orbifold models derive the discrete gauge symmetry naturally, as discussed above.

§ 7. Conclusion and discussion

We have constructed the shifted \( Z_4 \times Z_4 \) orbifold model equivalent to the twisted \( Z_4 \times Z_4 \) orbifold model at the enhancement point and then shown correspondence between the \( 2^6 \) and the shifted \( Z_4 \times Z_4 \) orbifold models. The Yukawa couplings of these models have been studied, too.

We could discuss correspondence between other orbifold models and models by Gepner and Kazama-Suzuki constructions other than the coincidences of the massless spectra.

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References


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