Method for Solving Quantum Field Theory
in the Heisenberg Picture

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This paper is a review of the method for solving quantum field theory in the Heisenberg picture, developed by Abe and Nakanishi since 1991.

Starting from field equations and canonical (anti)commutation relations, one sets up a (q-number) Cauchy problem for the totality of d-dimensional (anti)commutators between the fundamental fields, where d is the number of spacetime dimensions. Solving this Cauchy problem, one obtains the operator solution of the theory. Then one calculates all multiple commutators. A representation of the operator solution is obtained by constructing the set of all Wightman functions for the fundamental fields; the truncated Wightman functions are constructed so as to be consistent with all vacuum expectation values of the multiple commutators mentioned above and with the energy-positivity condition.

By applying the method described above, exact solutions to various 2-dimensional gauge-theory and quantum-gravity models are found explicitly. The validity of these solutions is confirmed by comparing them with the conventional perturbation-theoretical results. However, a new anomalous feature, called the “field-equation anomaly”, is often found to appear, and its perturbation-theoretical counterpart, unnoticed previously, is discussed. The conventional notion of an anomaly with respect to symmetry is reconsidered on the basis of the field-equation anomaly, and the derivation of the critical dimension in the BRS-formulated bosonic string theory is criticized.

The method outlined above is applied to more realistic theories by expanding everything in powers of the relevant parameter, but this expansion is not equivalent to the conventional perturbative expansion. The new expansion is BRS-invariant at each order, in contrast to that in the conventional perturbation theory. Higher-order calculations are generally extremely laborious to perform explicitly.

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Quantum field theory is the most successful theory in the fundamental physics. In particular, quantum gauge theory provides us the standard theory of particle physics. Even the quantum field theory of gravitation (quantum gravity) has been formulated satisfactorily, especially, quite beautifully for Einstein gravity.\textsuperscript{1,2)}

The standard method for calculating the solution in quantum field theory is covariant perturbation theory. This method is applicable, at least formally, to any model of quantum field theory, and we can obtain quantitative results if the model is renormalizable. It should be noted, however, that perturbation theory is based on the interaction picture. To realize the interaction picture, we must split the Lagrangian density into the free part and the interaction part. This splitting is quite artificial because it is not invariant under nonlinear transformations of the fundamental fields. As a consequence, it is not consistent with BRS invariance in quantum gauge theory.\textsuperscript{*1)}

Unfortunately, many physicists do not regard quantum gravity as a physically sensible theory. The reason for this belief is that if perturbation theory is applied to it, we obtain unrenormalizable results. However, this reasoning is wrong, because perturbative unrenormalizablity does not imply that the theory inevitably involves divergence. Rather, the source of the problem here is the application of perturbation theory to quantum gravity.\textsuperscript{4)} This can be shown very simply as follows.

In order to apply perturbation theory to quantum gravity, we must construct the interaction picture. This is possible only if we introduce the \textit{ad hoc ansatz} that the fundamental gravitational field $g_{\mu \nu}$ consists of the classical spacetime metric, such as the Minkowski metric tensor $\eta_{\mu \nu}$, and a quantum gravitational field, which is usually written $\sqrt{\kappa} h_{\mu \nu}$. Here, the latter vanishes when the Einstein gravitational constant $\kappa$ goes to zero. However, this ansatz is wrong. In $d$-dimensional classical gravity, there are $d$ degrees of freedom of general coordinate transformations even in the Minkowski space. In the quantum theory, these degrees of freedom necessarily become \textit{q-number} quantities (unless they are eliminated at the Lagrangian level beforehand), because any local transformation must be replaced by the corresponding BRS transformation in the quantization. That is, there remain $d$ q-number degrees of freedom even if $\kappa$ goes to zero, in contradiction to the above ansatz. Thus, we should conclude that the perturbative approach is not applicable to quantum gravity, and therefore there are no grounds to claim that quantum gravity is not a physically sensible theory.

It is clear from the above consideration that we cannot treat quantum gravity in the interaction picture. We should therefore develop a new systematic approach to quantum field theory in the \textit{Heisenberg picture}. It is the purpose of the present paper to review a recently proposed method for solving quantum field theory in the Heisenberg picture. As explained in the subsequent sections, the solution is obtained by constructing the set of all (truncated) Wightman functions for the fundamental fields. (A Wightman function is the vacuum expectation value of a simple product of fields.) According to Wightman,\textsuperscript{5)} given the set of all Wightman functions, one can

\textsuperscript{*1)} Regarding the fundamental importance of BRS invariance, see Refs. 2) and 3).
reconstruct the operator formalism of quantum field theory (the GNS construction). The Wightman function is conceptually simpler than the Green function ($\tau$ function), though the former is less convenient in calculations, because it depends on the ordering of fields. However, we should emphasize that the Wightman function is generally less singular than the corresponding Green function because the latter contains discontinuous $\theta$ functions of time differences. More importantly, it should be noted that the Green function is not necessarily a vacuum expectation value of the genuine T-product, which is generally non-covariant. When transferring the Hamiltonian formalism to the Lagrangian formalism, the T-product is transformed into the $T^\ast$-product, which is always covariant. The price to be paid for this bonus is that the $T^\ast$-product is not necessarily consistent with field equations and therefore with the Noether theorem.\(^6\) For example, if the Lagrangian density is given by \(\varphi\bar{\varphi} + f(\varphi, \partial\varphi)\), we have a field equation \(\varphi = 0\), but the 2-point Green function for \(\varphi\bar{\varphi}\) is a $\delta$ function; that is, it is non-zero. Because only the Green functions are calculated in the perturbative and path-integral approaches, we must always pay attention to this kind of unphysical characters of those approaches. For example, according to the intermediate boson theory, a charged pion with spin 0 is supposed to decay into a muon and a neutrino through a weak boson with spin 1, but this process violates the law of angular momentum conservation, because the orbital angular momentum is zero in the rest frame. This incorrect result is due to the use of $T^\ast$-product.\(^7\) By contrast, the Wightman function is consistent with both covariance and field equations (except for the “field-equation anomaly”, which arises as a consequence of the singular nature of the nonlinearity, as is explained later).

\section{Characterization of the new approach}

Nowadays, the path-integral formalism has become the most popular approach to quantum field theory, but one should recognize its characteristics. (a) It is based on the action itself rather than the action principle. This implies that even a total-divergence term of the Lagrangian density is treated as a meaningful quantity. Furthermore, the presence of a nonlocal term in the Lagrangian density is not excluded in principle. (b) In its proper framework, the path integral directly gives the Green functions beyond the extent justifiable by the operator formalism. This makes the fundamental nature of the path-integral formalism ambiguous. Because the unitarity of the physical S-matrix cannot be proved within the proper framework of the path-integral formalism, the effects of a total-divergence term and a nonlocal term on the unitarity have never been investigated seriously. (c) The solution is obtained in an already-integrated form. Because it involves divergences explicitly, renormalization must be carried out afterwards.

The starting point of our approach consists of field equations and canonical (anti)commutation relations rather than the action itself. Hence, a total-divergence term in the Lagrangian density is meaningless. Furthermore, the existence of a

\(^{6}\) Actual pion decay is described within the covariant operator formalism of the electroweak theory so as to be consistent with the angular momentum conservation.\(^7\)
nonlocal term is forbidden; that is, we do not consider a nonlocal theory. We first solve the theory at the operator level, where we encounter no divergence problem. We then proceed to the construction of a representation of the operator solution in terms of Wightman functions. Because we find the solution by integrating the equations, any divergence is excluded at this stage in principle, because no divergent quantity should be involved in the correct solution.\(^1\)

Although our solution is obtained in terms of Wightman functions, our formalism is different from that of axiomatic quantum field theory.\(^9\) In the latter, products of field operators at the same spacetime point are forbidden, because such products are generally mathematically ill-defined. However, if they are forbidden, it is impossible to develop a canonical formalism in the Heisenberg picture, except in the case of a free field. We allow their existence because they cause no trouble at the operator level. We understand that their mathematical meaning is determined afterwards, by determining the Wightman functions involving them.

\section*{3. Operator solution}

Let \(d\) be the number of spacetime dimensions. Of course, \(d = 4\) for the actual world. The action \(S\) is a \(d\)-dimensional integral of the Lagrangian density \(\mathcal{L}(x)\), which is a local function of the fundamental fields and their first derivatives. We denote a fundamental field by \(\varphi_A(x)\) generically. As is well known, the action principle \(\delta S = 0\) yields field equations for \(\varphi_A(x)\). Defining the canonical conjugate for \(\varphi_A(x)\) by the partial derivative of \(\mathcal{L}(x)\) with respect to \(\partial_0 \varphi_A(x)\), we set up canonical (anti)commutation relations. From them, we can calculate equal-time (anti)commutation relations. Field equations and equal-time (anti)commutation relations constitute our starting point.

It is important to note that field equations are formulae for one spacetime point, while equal-time (anti)commutation relations depend on two spacetime points. Therefore, the latter cannot be regarded as the initial data for the former. In order to set up a (q-number) Cauchy problem, we first rewrite a system of field equations

\[ F_k(\varphi(x)) = 0 \quad (3.1) \]

into the form

\[ [F_k(\varphi(x)), \varphi_B(y)] = 0 \quad \text{for any} \quad \varphi_B(y). \quad (3.2) \]

Evidently, (3.2) is equivalent to (3.1) if the field algebra of \(\varphi_A(x)\) is irreducible and if the field equations have no constant term. Furthermore, we rewrite (3.2) into the form of a system of partial differential equations for the \(d\)-dimensional (anti)commutators \([\varphi_A(x), \varphi_B(y)]\) with respect to the variables \(x^\mu\), though this procedure is not necessarily trivial as discussed in §12.

We can now regard the equal-time (anti)commutation relations as the initial data for the above system of partial differential equations; that is, we have set up a (q-number) Cauchy problem for \([\varphi_A(x), \varphi_B(y)]\). We postulate that the unique

\(^1\) This viewpoint is similar to that of the Lehmann-Symanzik-Zimmermann formalism,\(^8\) which directly gives the already-renormalized perturbation theory on the basis of asymptotic fields.
solvability of the Cauchy problem remains valid even in the q-number case. We note that this postulate is very convenient for proving operator identities and for defining new operators from known ones. If we can solve our Cauchy problem for \([\varphi_A(x), \varphi_B(y)]\) explicitly, we can claim that the operator solution for the fundamental field algebra has been obtained. We can then calculate any multiple commutator for \(\varphi_A(x)\).

Of course, the exact operator solution can be obtained only for some simple models. In general, we must employ an approximation method to solve the Cauchy problem. The simplest way to do this is to expand everything in powers of some parameter, such as a coupling constant, and solve the system of equations order by order. This expansion is different from the conventional perturbative expansion in the order counting. For BRS-invariant theories, our new expansion is BRS invariant at each order. The expansion parameter in quantum gauge theory is not the gauge coupling \(g\) but its square \(g^2\), while that in quantum gravity is not \(\sqrt{\kappa}\) but \(\kappa\). Remember that the square root of \(\kappa\) has been introduced artificially in order to apply the conventional perturbation theory to quantum gravity.

§4. Representation in terms of Wightman functions

It is very difficult to construct a matrix representation of the operator solution. Therefore, we construct its representation in terms of Wightman functions, \(\langle 0|\varphi_{A_1}(x_1)\cdots\varphi_{A_n}(x_n)|0\rangle\), where the vacuum state \(|0\rangle\) is assumed to be unique. The truncated Wightman function is defined by subtracting the contributions from the vacuum intermediate states, and it is written \(\langle 0|\varphi_{A_1}(x_1)\cdots\varphi_{A_n}(x_n)|0\rangle_T\).

The problem is to construct a system of truncated Wightman functions that are consistent with the multiple commutators constructed by the operator solution. It should be noted that the vacuum expectation value of a multiple commutator is already truncated. Hence, the vacuum expectation value of an \(n-1\)-ple commutator can be expressed as a linear combination of \(n\)-point truncated Wightman functions. We, therefore, have simultaneous linear algebraic equations for the truncated Wightman functions. While the number of independent equations, which is equal to the number of independent \(n-1\)-ple commutators, is \((n-1)!\), the number of \(n\)-point truncated Wightman functions is \(n!\). The insufficiency of these conditions is made up for by the following “energy-positivity condition”.

By virtue of translational invariance, the \(n\)-point truncated Wightman function can be extended to an analytic function of \(x_1^\mu - x_2^\mu, \cdots, x_{n-1}^\mu - x_n^\mu\). If all intermediate states except the vacuum have positive energy, it is a boundary value of an analytic function from the lower half-planes of the time-difference variables \(x_1^0 - x_2^0, \cdots, x_{n-1}^0 - x_n^0\). We require the Wightman function to have this property. That is, we require every Wightman function to be a boundary value of an analytic function from the lower half-planes of \(x_1^0 - x_2^0, \cdots, x_{n-1}^0 - x_n^0\).

In contrast to the positivity of energy, we do not require the positivity of the

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\(^*\) If translational invariance does not hold, we have only to introduce one extra spacetime point to the above.
norm of states; that is, we are generally working in the framework of the indefinite-metric quantum field theory. Indeed, the requirement of metric positivity is generally too stringent for a representation to exist. Rather, it turns out that the indefinite-metric theory gives us the natural framework to our approach; more precisely, the presence of zero-norm fields is favorable, as explained below.

Now, we return to the construction of truncated Wightman functions. The 0-point function $\langle 0 | 0 \rangle$ and the 1-point functions (vacuum expectation values of field operators) are arbitrary in principle, because there are no equations to be satisfied, but they are determined by some other natural requirements. We set $\langle 0 | 0 \rangle = 1$ as the normalization condition of the vacuum state. The 1-point functions should be chosen so as to be consistent with field equations and various conservation laws. They must be constants if the translational invariance is required, and those constants are, in most cases, zero from the requirement of symmetries such as Lorentz invariance, BRS invariance, FP-ghost number conservation, etc. The nonvanishing of the 1-point Wightman function usually implies spontaneous breakdown of symmetry.

We determine $n$-point functions ($n \geq 2$) successively by solving the simultaneous linear algebraic equations for truncated Wightman functions under the energy-positivity condition. This procedure may well be straightforward if the expression obtained for the (anti)commutator $[\varphi_A(x), \varphi_B(y)]$ is linear with respect to the fundamental fields. If it is nonlinear, however, we encounter a problem in calculating the vacuum expectation values of multiple commutators, because we encounter such a singular product as $\varphi_A(x)\varphi_B(x)$. In general, we call a product of field operators at the same spacetime point a “composite field”. We define the Wightman function involving composite fields in the following way.

We first consider a higher-point untruncated Wightman function involving the relevant product of field operators consecutively; of course, the spacetime points involved in it are all distinct at this stage. Next, the spacetime points corresponding to the composite field are simply set equal. The infinity that arises as a consequence of this procedure is deleted in such a way that the resultant is invariant under any permutation of the constituent fields contained in the relevant composite field (of course, the statistical sign factor must be taken into account). This procedure is nothing but a generalization of the procedure of taking the normal product. In this way, we can construct the untruncated Wightman function involving composite fields.

If we can calculate beforehand all Wightman functions involving composite fields necessary for calculating the vacuum expectation values of the relevant multiple commutators, we can set up our simultaneous linear algebraic equations in almost the same way as in the linear case. If this cannot be done, unfortunately, we must employ a further approximation method, such as the expansion method.

As a remark, we point out the special role played by the zero-norm field. In the indefinite-metric quantum field theory, it is possible for a quantum field $\varphi_A(x)$ to satisfy $d$-dimensional self-commutativity

$$[\varphi_A(x), \varphi_A(y)] = 0. \quad (4\cdot1)$$

Then, multiple commutators for $\varphi_A(x)$ are, of course, zero, and hence all $n$-point
truncated Wightman functions vanish for \( n \geq 2 \). Accordingly, for any function \( f \), we find
\[
\langle 0 | f(\varphi_A(x)) | 0 \rangle = f(\langle 0 | \varphi_A(x) | 0 \rangle).
\]
(4.2)
Thus it behaves like a c-number, though it is an operator that does not necessarily commute with other fields. This property is very important for treating the classical spacetime metric in the framework of quantum gravity and for introducing an external electromagnetic field into quantum electrodynamics.

As is well known, the Feynman-diagrammatic method is very powerful and convenient for carrying out perturbative calculations. If a similar diagrammatic method could be invented in our new approach, it would be very helpful for writing down the solution. In the Heisenberg picture, however, because the interaction Lagrangian density is not separated from the free part, it is impossible to set up a simple rule for the vertex. Therefore, we introduce the new concept of a “building block”, which is a generalization of the concept of a line in the Feynman diagram. Each of its endpoints contains information about how this building block should be connected with other building blocks at this point. This new diagrammatic method is very successful in some models, but it becomes complicated in other models. Presently, we have no general prescription applicable to all models.

§5. Free field theory

It is, of course, trivial to solve the free field theory. Nevertheless, it is instructive to apply our new approach to the free field theory. For simplicity, we consider a neutral scalar field \( \phi(x) \) of mass \( m \). Then, the field equation is
\[
(\partial^2 + m^2)\phi(x) = 0.
\]
(5.1)
The equal-time commutation relations are
\[
[\phi(x), \phi(y)]_0 = 0,
\]
(5.2)
\[
[\partial_0 \phi(x), \phi(y)]_0 = -i\delta^{d-1}(x - y),
\]
(5.3)
where the subscript 0 of the commutator indicates that we set \( x^0 = y^0 \).

We first rewrite (5.1) into the form
\[
(\partial^2 + m^2)x[\phi(x), \phi(y)] = 0.
\]
(5.4)
Then (5.4) together with (5.2) and (5.3) constitutes a (c-number) Cauchy problem for \( [\phi(x), \phi(y)] \). Its solution is given by
\[
[\phi(x), \phi(y)] = i\Delta(x - y).
\]
(5.5)
This gives the operator solution of the free field theory. Note that our analysis does not use the Fourier transform of \( \phi(x) \).

Next, we construct the representation of the operator solution (5.5). The 1-point function \( \langle 0 | \phi(x) | 0 \rangle \) is equal to zero as long as \( m \neq 0 \), owing to (5.1). From (5.5), the 2-point truncated Wightman functions satisfy the linear algebraic equation
\[
\langle 0 | \phi(x_1)\phi(x_2) | 0 \rangle_T - \langle 0 | \phi(x_2)\phi(x_1) | 0 \rangle_T = i\Delta(x_1 - x_2).
\]
(5.6)
The solution to (5.6) satisfying the energy-positivity condition is
\[
\langle 0 | \phi(x_1)\phi(x_2) | 0 \rangle_T = \Delta^{(+)}(x_1 - x_2).
\] (5.7)

Of course, (5.7) is consistent with the hermiticity of \( \phi(x) \). Because all multiple (except for single) commutators vanish, all \( n \)-point truncated Wightman functions for \( n \geq 3 \) vanish.

Even in the free field theory, nontrivial graphs are encountered if we consider composite fields. For example, according to the prescription for the Wightman function involving composite fields, the 2-point Wightman function for \( [\phi(x)]^2 \) is given by
\[
\langle 0 | [\phi(x_1)]^2[\phi(x_2)]^2 | 0 \rangle = 2\langle 0 | \phi(x_1)\phi(x_2) | 0 \rangle^2 = 2[\Delta^{(+)}(x_1 - x_2)]^2.
\] (5.8)

This corresponds to a 1-loop graph. Of course, (5.8) is the conventional result for the 2-point function of the normal product \( :[\phi(x)]^2: \). Thus our prescription for composite fields is a generalization of the normal product.

§6. List of the models studied to this time

In the preceding sections, we have developed a method for solving quantum field theory in the Heisenberg picture. This method has been applied to various models since 1991. In the following, we list those models in roughly chronological order of their investigation.

1. The BRS-formulated 2-dimensional quantum gravity in the de Donder gauge has been studied extensively.\(^{10} - ^{19}\) All Wightman functions have been explicitly obtained. The validity of our results in this model was explicitly confirmed by comparing them with the 2-point and 3-point results of the conventional perturbation theory.\(^*\) The existence of a field-equation anomaly was discovered. A new diagrammatic method for writing down the solution was proposed; it works very satisfactorily for this model. (See §9 for more details.)

2. The BRS-formulated \( d \)-dimensional quantum Einstein gravity for \( d \geq 3 \) in the de Donder gauge has been studied through an expansion with respect to \( \kappa \).\(^{20}\) Its zeroth order, apart from its dimensionality, is essentially the same as the 2-dimensional case discussed in Item 1 above, except for the field-equation anomaly; that is, the zeroth order was exactly solved. (See §12 for more details.)

3. Quantum electrodynamics has been investigated through an expansion with respect to \( e^2 \).\(^{21},^{22}\) The resulting formulae almost reproduce the expressions obtained with the conventional perturbation theory, though the Ward-Takahashi identities are built-in. The new diagrammatic method was applied, but its achievement was not as effective as the model discussed in Item 1. (See §11 for more details.)

4. The analysis of the model mentioned in Item 1 has been extended to a covariantized Liouville-like theory by adding a cosmological term.\(^{23}\)

\(^*\) This model contains no Einstein gravitational constant; therefore, the trouble encountered in the perturbative approach pointed out in §1 is not relevant.
5. The BRS-formulated 2-dimensional nonabelian BF theory in the Landau gauge has been partially solved.\textsuperscript{24} The BF theory turns out to be the zeroth order of the Yang-Mills theory.

6. The \(d\)-dimensional scalar field theory has been studied through an expansion with respect to the coupling constant, and it was shown to reproduce the Ostendorf rules for the perturbation theory of the Wightman function.\textsuperscript{25}

7. The (indefinite-metric) one-loop model has been solved exactly.\textsuperscript{25, 22} The new diagrammatic method has also been applied to this model. (See §10 for more details).

8. 2-dimensional quantum gravity in the light-cone gauge has been solved exactly.\textsuperscript{26}–\textsuperscript{28} It was found that the field-equation anomaly exists. An extremely large supersymmetry was found.

9. 2-dimensional nonlinear abelian gauge theories, such as the Born-Infeld theory, have been solved.\textsuperscript{29} The solutions are not unique if translational invariance is not required.

10. The BRS-formulated 2-dimensional quantum gravity in the conformal gauge has been solved exactly.\textsuperscript{30}, 31) By adding string boundary conditions, the Kato-Ogawa bosonic string theory was obtained. A field-equation anomaly was found to exist. The Kato-Ogawa derivation of the string critical dimension \(D = 26\) was criticized by pointing out that the nilpotency of the BRS generator depends on the treatment of field-equation anomaly. (See §8 for more details.)

11. The BRS-formulated 2-dimensional BF and Yang-Mills theories in the light-cone gauge have been solved exactly.\textsuperscript{32} A field-equation anomaly was found to exist. This model is the simplest nontrivial model treated with our approach. (See §7 for more details.)

\section*{§ 7. 2-dimensional BF theory in the light-cone gauge}

The BRS-formulated 2-dimensional nonabelian BF theory in the light-cone gauge is convenient for explaining the main features of our approach.\textsuperscript{32}

In this model, the fundamental fields are the nonabelian gauge field \(A^a_\mu\), the conjugate field \(\bar{B}^a\), the \(B\) field \(B^a\), the FP ghost \(C^a\), the FP antighost \(\bar{C}^a\), and the \(D\) chiral Dirac fields \(\psi_M (M = 1, \cdots, D)\). With light-cone coordinates \(x^\pm = (x^0 \pm x^1)/\sqrt{2}\), the Lagrangian density is given by

\[
\mathcal{L} = \bar{B}^a(\partial_- A^a_+ - \partial_+ A^a_- - f^{abc} A^b_+ A^c_-) + B^a A^a_- + i\bar{C}^a(\delta^{ab} \partial_- + f^{abc} A^c_-)C^b + i\psi_M^\dagger(\partial_- - iA^a_+ T^a)\psi_M,
\]

where \(f^{abc}\) and \(T^a\) (normalized such that \(\text{tr}(T^a T^b) = \frac{1}{2} \delta^{ab}\)) are the structure constant and the representation matrix of a compact Lie algebra, respectively.

The field equations derived from (7.1) are

\[
A^a_- = 0,
\]

(7.2)

\[
\partial_- \Phi = 0 \quad \text{for} \quad \Phi = A^a_+, \bar{B}^a, C^a, \bar{C}^a, \psi_M,
\]

(7.3)

\[
B^a + \partial_+ \bar{B}^a + f^{abc}(A^b_+ \bar{B}^c - i\bar{C}^b C^c) + \psi_M^\dagger T^a \psi_M = 0.
\]

(7.4)
From (7.3) and (7.4), we obtain $\partial_- B^a = 0$. Thus, all fields are functions of $x^+$ alone. Note that (7.4) is the only nonlinear equation. Because it can be regarded as a defining equation of the B field, the theory is essentially trivial. Non-triviality arises from the incoherence between the BRS invariance and the elimination of the B field.

Canonical quantization is carried out by taking $A^a_+, C^a, \bar{C}^a$ and $\psi_M$ as canonical variables.\(^\dagger\) We then obtain
\[
[\tilde{B}^a(x), A^b_+(y)] = -i\delta^{ab}\delta(x^+ - y^+), \tag{7.5}
\]
\[
\{\tilde{C}^a(x), C^b(y)\} = \delta^{ab}\delta(x^+ - y^+), \tag{7.6}
\]
\[
\{\psi_M(x), \psi_N^\dagger(y)\} = \delta_{MN}\delta(x^+ - y^+). \tag{7.7}
\]
Because of the independence of $x^-$, (7.5)–(7.7) are already 2-dimensional (anti)commutation relations. All other (anti)commutators vanish if the B field is not involved. At the operator level, we can calculate the commutators involving $B^a$ by using (7.4). The result is simplified by introducing the field
\[
B'^a \equiv B^a + \partial_+ \tilde{B}^a. \tag{7.8}
\]
We then find the following:
\[
[\Phi^a(x), B'^b(y)] = -if^{abc}\Phi^c(y)\delta(x^+ - y^+) \quad \text{for} \quad \Phi^a = A^a_+, \tilde{B}^a, C^a, \bar{C}^a, B'^a, \tag{7.9}
\]
\[
[\psi_M(x), B'^b(y)] = -T^b\psi_M(y)\delta(x^+ - y^+). \tag{7.10}
\]
We thus find that all 2-dimensional (anti)commutators are linear with respect to the fundamental fields. Therefore, it is easy to calculate all multiple commutators. For example, we have
\[
[[A^a_+(x), B'^b(y)], \tilde{B}^c(z)] = f^{abc}\delta(x^+ - y^+)\delta(y^+ - z^+). \tag{7.11}
\]
Now, we proceed to the construction of the Wightman functions. Owing to the requirements of Lorentz invariance, BRS invariance, and FP-ghost number conservation, we set all 1-point functions equal to zero. Then, as far as 2- and 3-point functions are concerned, we need not distinguish truncated from untruncated.

From (7.5)–(7.7), the nonvanishing 2-point Wightman functions are easily seen to be given by\(^\star\star\)
\[
\langle A^a_+(x_1), \tilde{B}^b(x_2) \rangle = \frac{1}{2\pi}\delta^{ab}\frac{1}{x_1^+ - x_2^+ - i0}, \tag{7.12}
\]
\[
\langle C^a(x_1)\tilde{C}^b(x_2) \rangle = -\frac{i}{2\pi}\delta^{ab}\frac{1}{x_1^+ - x_2^+ - i0}, \tag{7.13}
\]
\[
\langle \psi_M(x_1)\psi_N^\dagger(x_2) \rangle = -\frac{i}{2\pi}\delta_{MN}\frac{1}{x_1^+ - x_2^+ - i0}. \tag{7.14}
\]
\[^\dagger\) Canonical quantization with respect to $x^-$ can be shown to be equivalent to that with respect to $x^0$.
\[^\star\star\) Hereafter, we omit writing the vacuum bra and ket in the vacuum expectation value.
The corresponding reversed-order (but keeping the order of \( x_1 \) and \( x_2 \)) functions are the same apart from the overall sign factors. Indeed, then, they are consistent with the energy-positivity condition and with the vacuum expectation values of the commutators.

Similarly, we can construct the 3-point Wightman functions. The nonvanishing ones, apart from those obtained by field permutation, are as follows:

\[
\langle A_+^a(x_1)B^b(x_2)\bar{C}^c(x_3) \rangle = -f^{abc}\varphi_3(x_1^+, x_2^+, x_3^+),
\]

\[
\langle C^a(x_1)B^b(x_2)\bar{C}^c(x_3) \rangle = if^{abc}\varphi_3(x_1^+, x_2^+, x_3^+),
\]

\[
\langle \psi_M(x_1)B^b(x_2)\psi_N^\dagger(x_3) \rangle = \delta_{MN}f^{b}\varphi_3(x_1^+, x_2^+, x_3^+)
\]

with

\[
\varphi_3(x_1^+, x_2^+, x_3^+) = \frac{1}{(2\pi)^2} \frac{1}{(x_1^+ - x_2^+ - i0)(x_2^+ - x_3^+ - i0)}.
\]

Generally, nonvanishing \( n \)-point truncated Wightman functions are those which consist of \((n - 2)B^l\)-fields and one of the pairs \((A_+^a, \bar{B}^a), (C^a, \bar{C}^a)\) and \((\psi_M, \psi_N^\dagger)\). Their explicit expressions, which consist of \((n - 2)!\) terms, are easily written down.

The Lagrangian density (7.1) is invariant under the BRS transformation \( \delta_* \) defined by

\[
\delta_* A_\pm^a = \partial_\pm A^a + f^{abc}A_\pm^b C^c,
\]

\[
\delta_* C^a = -\frac{1}{2}f^{abc}C^b C^c,
\]

\[
\delta_* \bar{B}^a = -f^{abc}\bar{C}^b \bar{B}^c,
\]

\[
\delta_* \bar{C}^a = \partial^a,
\]

\[
\delta_* \psi_M = iC^a T^a \psi_M,
\]

\[
\delta_* \psi_M^\dagger = -i\psi_M^\dagger C^a T^a.
\]

We can verify that our solution is consistent with BRS invariance. For example, it can be shown that

\[
\Gamma(x_1, x_2, x_3) \equiv \langle \delta_* (A_+^a(x_1)\bar{C}^b(x_2)\bar{B}^c(x_3)) \rangle = 0
\]

in the following way. Substituting (7.19) into (7.20), we obtain a sum of three Wightman functions involving a composite field. Hence, according to the general prescription, it becomes

\[
\Gamma(x_1, x_2, x_3) = f^{aed} \langle A_+^e(x_1)\bar{B}^c(x_3) \rangle \langle C^d(x_1)\bar{C}^b(x_2) \rangle
\]

\[
+ i \langle A_+^a(x_1)B^b(x_2)\bar{C}^c(x_3) \rangle
\]

\[
+ f^{fde} \langle A_+^a(x_1)\bar{B}^c(x_3) \rangle \langle C^b(x_2)C^d(x_3) \rangle,
\]

where (7.8) has been used. Substituting (7.12), (7.13) and (7.15) into (7.21), we find \( \Gamma(x_1, x_2, x_3) = 0 \).

There is no anomaly concerning BRS invariance and FP-ghost number conservation, but the solution does have a field-equation anomaly, as explained below. First, we rewrite the only nonlinear equation (7.4) as

\[
G^a \equiv B^a - F^a = 0.
\]

\(^\dagger\) Specifically, they are obtained by multiplying the above by \(-1, +1, +1\), respectively.
\[ F^a \equiv -f^{abc}(A^b_+ \tilde{B}^c - i\bar{C}^b C^c) - \psi_M^\dagger T^a \psi_M. \]  
(7.23)

We, of course, have
\[ \langle \tilde{B}^a(x_1) B^b(x_2) \rangle = 0. \]  
(7.24)

On the other hand, from (7.15)–(7.17), we find
\[ \langle \tilde{B}^a(x_1) F^b(x_2) \rangle = -\frac{D}{2(2\pi)^2} \cdot \frac{\delta^{ab}}{(x_1^+ - x_2^+ - i0)^2}. \]  
(7.25)

Moreover, the prescription for composite fields, together with (7.12)–(7.14), yields
\[ \langle F^a(x_1) F^b(x_2) \rangle = -\frac{D}{2(2\pi)^2} \cdot \frac{\delta^{ab}}{(x_1^+ - x_2^+ - i0)^2}. \]  
(7.26)

Evidently, unless \( D = 0 \), (7.24)–(7.26) are inconsistent with the field equation (7.4) 
(i.e., \( G^a = 0 \)). This is the field-equation anomaly. (Note, however, that the equation \( \partial_+ G^a = 0 \), which reduces to the linear equation \( \partial_+ B^a = 0 \), is free of any anomaly.)

One may suspect that the appearance of such an anomaly might be a defect of the solution found by the present approach. This is not the case, however. Indeed, even in the conventional covariant perturbation theory, this anomaly appears in a disguised form. In spite of (7.1), the self-energy graphs of the field, which are three 1-loop graphs of \((A_+, \tilde{B}), (C, \bar{C})\) and \((\psi_M, \psi_N^\dagger)\), are nonvanishing. The former two cancel, but the last one survives. Thus, unless \( D = 0 \), the B-field 2-point Green function does not vanish. Because the B field is a BRS daughter of the FP antighost, this fact implies the violation of BRS invariance; that is, we encounter the BRS anomaly in perturbation theory. The relation between the field-equation anomaly in the BRS-invariant solution and the BRS anomaly in the conventional perturbation theory is as follows.

From (7.1), the Noether BRS current \( j^\mu_B \) is given by \( j^+ B = 0 \) and
\[ j^-_B = \tilde{B}^a \partial_+ C^a + f^{abc} \tilde{B}^a A^b_+ C^b + \frac{1}{2} i f^{abc} C^a C^b C^c - i C^a \psi_M^\dagger T^a \psi_M. \]  
(7.27)

At the operator level, by using \( C^a = 0 \), we can rewrite this into the form
\[ \hat{j}^-_B \equiv \hat{j}^-_B + C^a C^a \]
\[ = B^a C^a - \frac{1}{2} i f^{abc} C^a C^b C^c + \partial_+ (\tilde{B}^a C^a). \]  
(7.28)

Of course, we have \( \hat{j}^-_B = j^-_B \) at the operator level, but, because of the field-equation anomaly for \( C^a = 0 \), these two quantities are different at the representation level.

The BRS generators \( Q_B \) and \( \hat{Q}_B \) are defined by the integrals of \( j^-_B \) and \( \hat{j}^-_B \), respectively, over \( x^+ \). By using (anti)commutation relations, but \textit{without using the field equation (7.4)}, we can confirm that
\[ i[\hat{Q}_B, \Phi_+] = \delta_x \Phi \]  
(7.29)
for any fundamental field $\Phi$. Similarly, we have
\[ [G^a(x), \Phi(y)] = 0 \quad \text{for} \quad \Phi = A^a_+, \tilde{B}^a, C^a, \bar{C}^a, \psi_M, \]  
(7.30)
but
\[ [G^a(x), B^{\prime b}(y)] = -if^{abc}G^c(x)\delta(x^+ - y^+). \]  
(7.31)
Hence, with the help of (7.9) and (7.6), we see that $C^a G^a$ (anti)commutes with any fundamental field, except for $\bar{C}^b$:
\[ \{C^a(x)G^a(x), \bar{C}^b(y)\} = G^b(x)\delta(x^+ - y^+). \]  
(7.32)
Integrating (7.32) over $x^+$, we find
\[ \{Q_B - \hat{Q}_B, \bar{C}^b(y)\} = -G^b(y). \]  
(7.33)
Because the vacuum is BRS invariant, $^s$ (7.29) yields
\[ \langle \bar{C}^a(x_1)\hat{Q}_B^2\bar{C}^b(x_2) \rangle = \langle \{\bar{C}^a(x_1), \hat{Q}_B\}\{\hat{Q}_B, \bar{C}^b(x_2)\} \rangle \\
= \langle B^a(x_1)B^b(x_2) \rangle = 0. \]  
(7.34)
On the other hand, owing to (7.33), we have
\[ \langle \bar{C}^a(x_1)Q_B^2\bar{C}^b(x_2) \rangle = \langle \{\bar{C}^a(x_1), Q_B\}\{Q_B, \bar{C}^b(x_2)\} \rangle \\
= \langle (\partial_+ \tilde{B}^a(x_1) - F^a(x_1))(\partial_+ \tilde{B}^b(x_2) - F^b(x_2)) \rangle \\
= -\frac{D}{2(2\pi)^2} \cdot \frac{\delta^{ab}}{(x_1^+ - x_2^+ - i0)^2}. \]  
(7.35)
From the above, we see that while $\hat{Q}_B$ is nilpotent, the Noether BRS generator $Q_B$ is not nilpotent unless $D = 0$. In spite of the fact that the theory is BRS invariant and the correct BRS generator is given by $\hat{Q}_B$, there appears an apparent BRS anomaly, owing to the fact that the Noether BRS generator contains the contribution from the field-equation anomaly. The situation encountered here can be understood in the following way.

If a quantum gauge theory is BRS-formulated without introducing the B field, the nilpotency of $\delta^*_s$ does not hold at the Lagrangian level. The nilpotency of the BRS generator can be shown only by using the gauge-fixing condition; that is, the theory is not manifestly BRS invariant. Indeed, if a BRS-invariant term is added to the original Lagrangian density, the nilpotency of the BRS generator no longer holds in general. Thus, the elimination of the B field generally makes BRS invariance unstable. In the model considered here, the elimination of the B field is equivalent to the use of the field equation $G^a = 0$. Therefore, the disturbance caused by the field-equation anomaly makes BRS invariance apparently anomalous.

Finally, we remark that the above consideration can very easily be extended to the case of the Yang-Mills theory, whose Lagrangian density is obtained by adding

$^s$ This can be verified by showing that if the BRS generator is inserted into an arbitrary Wightman function in the rightmost position, the resulting expression vanishes, owing to its analyticity in the upper-half plane of the integration variable.
the term \(-\frac{1}{2}g^2 \tilde{B}^a \tilde{B}^a\) to (7.1).\(^{32}\) The exact solution is obtained simply by adding \(g^2\)-order contributions. The qualitative properties of the model remain unchanged. In general, the BF theory is nothing but the zeroth-order approximation of the Yang-Mills theory.

§8. 2-dimensional quantum gravity in the conformal gauge

2-dimensional quantum gravity is a very simple model, because the Einstein-Hilbert action is trivial. It turns out that it is essentially the zeroth-order approximation of quantum Einstein gravity in the \(\kappa\) expansion. On the other hand, as is well known, 2-dimensional quantum gravity coupled with \(D\) scalar fields can be interpreted as a theory describing a bosonic string in a \(D\)-dimensional spacetime.

In this section, we discuss the BRS-formulated 2-dimensional quantum gravity in the conformal gauge.\(^{30,6}\) In 1983, Kato and Ogawa investigated this model under the open-string boundary conditions.\(^{33}\) According to their conclusion, the BRS generator is not nilpotent if \(D \neq 26\); that is, the critical dimension of the bosonic string is obtained from the requirement of BRS invariance. The existence of such a BRS anomaly is, however, quite unreasonable as the zeroth-order quantum Einstein gravity. Thus, the BRS-formulated 2-dimensional quantum gravity has seemingly mutually conflicting aspects. We show that this paradox can be resolved by taking account of the field-equation anomaly. Although we can consider the model under the closed or open string boundary conditions completely,\(^{31}\) we here describe the case without boundary conditions for simplicity.

The conformal degree of freedom is irrelevant and can be eliminated beforehand. The gravitational field is described by the contravariant vector density \(\tilde{\gamma}^{\mu\nu}\), whose determinant is \(-1\). The other fundamental fields are the B field \(B_{\mu\nu}\), the FP ghost \(c^\mu\), the FP antighost \(\bar{c}^{\mu\nu}\), and the \(D\) scalar fields \(\phi_M\) \((M = 1, 2, \cdots, D\) with target-space metric arbitrary).\(^{\ast}\) The BRS transformation \(\delta_\ast\) is defined by

\[
\begin{align*}
\delta_\ast \tilde{\gamma}^{\mu\nu} &= \tilde{\gamma}^{\mu\sigma} \partial_\sigma c^\nu + \tilde{\gamma}^{\sigma\nu} \partial_\sigma c^\mu - \partial_\sigma (\tilde{\gamma}^{\mu\nu} c^\sigma), \\
\delta_\ast c^\mu &= -c^\sigma \partial_\sigma c^\mu, \\
\delta_\ast \bar{c}^{\mu\nu} &= iB_{\mu\nu}, \\
\delta_\ast B_{\mu\nu} &= 0, \\
\delta_\ast \phi_M &= -c^\sigma \partial_\sigma \phi_M.
\end{align*}
\]

(8.1)

Because \(\tilde{\gamma}^{\mu\nu}\) has only two degrees of freedom, it can be parametrized as

\[
\tilde{\gamma}^{\mu\nu} = (\eta^{\mu\nu} + h^{\mu\nu})(1 - \det h^{\sigma\tau})^{-\frac{1}{2}},
\]

(8.2)

where \(h^{\mu\nu}\) is symmetric and traceless in the sense of \(\eta_{\mu\nu} h^{\mu\nu} = 0\). Correspondingly, \(\bar{c}^{\mu\nu}\) and \(B_{\mu\nu}\) are also symmetric and traceless. It is convenient to rewrite any traceless symmetric tensor \(X_{\mu\nu}\) into the form of a vector-like quantity \(X^\lambda\) according to

\[
\sqrt{2} X^\lambda \equiv \xi^{\lambda\mu\nu} X_{\mu\nu},
\]

(8.3)

where \(\xi^{\lambda\mu\nu}\) is 1 when \(\lambda + \mu + \nu\) is even, and 0 otherwise. We introduce \(h_\lambda\), \(\bar{c}^\lambda\) and \(B^\lambda\) according to (8.3).\(^{\ast\ast}\)

\(^{\ast}\) In what follows, the definitions of the B field and the FP antighost differ by a factor 2 from those in the original papers.

\(^{\ast\ast}\) Note that the 2-dimensional Lorentz group is abelian; therefore, spin is not intrinsic.
The BRS-invariant action is given by the Lagrangian density

\[ \mathcal{L} = -B^\lambda h_\lambda - i\bar{c}^\lambda \delta_\lambda h_\lambda + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi_M \cdot \partial_\nu \phi^M. \]  

(8.4)

Explicitly, we have

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I + (\text{a total divergence}), \]

with

\[ \mathcal{L}_0 = -B^\lambda h_\lambda - i\sqrt{2} \xi_{\lambda\mu\nu} \bar{c}^\lambda \partial^\mu c^\nu + \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi_M \cdot \partial_\nu \phi^M, \]

(8.5)

\[ \mathcal{L}_I = h_\lambda (i\xi_{\lambda\mu\nu} \xi_{\rho\sigma\mu} \bar{c}^\rho \partial_\nu c^\sigma - i\partial_\sigma \bar{c}^\lambda c^\sigma + 2 - \frac{3}{2} \xi_{\lambda\mu\nu} \partial_\mu \phi_M \cdot \partial_\nu \phi^M) + O(h^2). \]

(8.6)

It is not necessary to specify the higher-order terms \( O(h^2) \), because they contribute neither to field equations nor to the canonical (anti)commutation relations. Owing to the use of \( \bar{c}^\lambda \) and \( B^\lambda \), it becomes manifest that there exists FP-ghost conjugation invariance of the action, as in the covariant gauge.

The analysis can be greatly simplified by introducing the light-cone coordinates \( x^\pm = (x^0 \pm x^1)/\sqrt{2} \), because then \( \xi_{\lambda\mu\nu} = 0 \) except for \( \xi_{+++} \) and \( \xi_{---} \) (\( \xi_{+++} = \xi_{---} = \sqrt{2} \)). From (8.5) and (8.6), we have

\[ \mathcal{L}_0 = [-B^+ h_+ - 2i\bar{c}^- \partial^- c^+ + (+ \leftrightarrow -)] + \partial_+ \phi_M \cdot \partial_- \phi^M, \]

(8.7)

\[ \mathcal{L}_I = h_+ \left[ -2i\bar{c}^- \partial_+ c^+ - i(\partial_+ \bar{c}^- \cdot c^+ + \partial_- \bar{c}^+ \cdot c^-) + \frac{1}{2} \partial_+ \phi_M \cdot \partial_+ \phi^M \right] + (+ \leftrightarrow -) + O(h^2), \]

(8.8)

where \((+ \leftrightarrow -)\) stands for the expression obtained from the preceding terms by interchanging \(+\) and \(-\) for superscripts and subscripts only.

The field equations that follow from the above Lagrangian density are

\[ h_\pm = 0, \]

(8.9)

\[ G^\pm \equiv B^\pm - F^\pm = 0, \quad F^\pm \equiv \delta \mathcal{L}_I/\delta h_\pm = \delta \mathcal{L}_I/\delta h_\pm |_{h=0}, \]

(8.10)

\[ \partial_\mp \bar{c}^\pm = 0, \quad \partial_\mp c^\pm = 0, \]

(8.11)

\[ \partial_+ \phi_M = 0. \]

(8.12)

Here, \( O(h^2) \) does not contribute to the right-hand side of (8.10), because \( h_\pm = 0 \) from (8.9). Also, the terms involving \( \partial_\mp \bar{c}^\pm \) in (8.10) do not contribute to it because of (8.11). Furthermore, differentiating (8.10), one finds

\[ \partial_\mp B^\pm = 0. \]

(8.13)

Therefore each of \( B^\pm, c^\pm, \bar{c}^\pm, \partial_\pm \phi_M \) is a function of \( x^\pm \) alone. It should be emphasized that this remarkable simplicity is realized only in the operator formalism, not in the conventional perturbation theory. In the latter, all terms, including those of \( O(h^2) \), present in the Lagrangian density must be taken into account, because the \( T^* \)-product does not generally preserve the validity of field equations.
Canonical quantization is carried out by taking \( c^\pm \) and \( \phi_M \) only as canonical variables. We then obtain

\[
\{ c^\pm (x), c^\pm (y) \} = -\frac{1}{2} \delta(x^\pm - y^\pm), \tag{8.14}
\]

\[
[\partial_\pm \phi_M (x), \phi^N (y)] = -\frac{1}{2} i \delta_M^N \delta(x^\pm - y^\pm) \tag{8.15}
\]
as the 2-dimensional (anti)commutation relations. At the operator level, we can use (8\cdot10) as the defining equation of the B field. From (8\cdot14) and (8\cdot15), we obtain

\[
[B^\pm (x), c^\pm (y)] = -\frac{1}{2} i c^\pm (x) \delta'(x^\pm - y^\pm) - i \partial_\pm c^\pm (x) \cdot \delta(x^\pm - y^\pm), \tag{8.16}
\]

\[
[B^\pm (x), \bar{c}^\pm (y)] = i \bar{c}^\pm (x) \delta'(x^\pm - y^\pm) + \frac{1}{2} i \partial_\pm \bar{c}^\pm (x) \cdot \delta(x^\pm - y^\pm), \tag{8.17}
\]

\[
[B^\pm (x), B^\pm (y)] = \frac{1}{2} i (B^\pm (x) + B^\pm (y)) \delta'(x^\pm - y^\pm), \tag{8.18}
\]

\[
[B^\pm (x), \phi_M (y)] = -\frac{1}{2} i \partial_\pm \phi_M (x) \cdot \delta(x^\pm - y^\pm). \tag{8.19}
\]

Evidently, (8\cdot18) is the BRS transform of (8\cdot17). Thus, we have found the operator solution. Because all of the right-hand sides are linear with respect to the fundamental fields, it is easy to calculate all multiple commutators.

Now, we can construct the representation of the operator solution in terms of Wightman functions. For simplicity, we set all 1-point functions equal to zero. The nonvanishing truncated Wightman functions are the 2-point and 3-point functions (apart from field permutations) only:

\[
\langle c^\pm (x_1) \bar{c}^\pm (x_2) \rangle = -i \partial_\pm D^{(+)}(x_1 - x_2), \tag{8.20}
\]

\[
\langle \phi_M (x_1) \phi^N (x_2) \rangle = \delta_M^N D^{(+)}(x_1 - x_2), \tag{8.21}
\]

\[
\langle c^\pm (x_1) B^\pm (x_2) \bar{c}^\pm (x_3) \rangle = 2i \partial_\pm \bar{D}^{(+)}(x_1 - x_2) \cdot \partial_\pm D^{(+)}(x_2 - x_3)
- i \partial_\pm^2 D^{(+)}(x_1 - x_2) \cdot \partial_\pm D^{(+)}(x_2 - x_3), \tag{8.22}
\]

\[
\langle \phi_M (x_1) B^\pm (x_2) \phi^N (x_3) \rangle = -\delta_M^N \partial_\pm D^{(+)}(x_1 - x_2) \cdot \partial_\pm D^{(+)}(x_2 - x_3), \tag{8.23}
\]

where

\[
\partial_\pm D^{(+)}(x) \equiv -\frac{1}{4\pi} \frac{1}{x^\pm - i0}. \tag{8.24}
\]

As is well known, the 2-dimensional massless \( D^{(+)\prime} \) function needs an infrared subtraction term\(^{34},35\) but an undifferentiated \( D^{(+)\prime} \) appears only in (8\cdot21); therefore we ignore the infrared problem.

As in §7, we can confirm that the solution is BRS invariant, but we find that there appears a field-equation anomaly for (8\cdot10) [but not for (8\cdot13), of course]. By using the prescription for the Wightman function involving a composite field, we
obtain
\[
\langle F^\pm(x_1)B^\pm(x_2) \rangle = \langle F^\pm(x_1)F^\pm(x_2) \rangle = \frac{1}{2}(D - 26)[\partial^\pm_2 D^{(\pm)}(x_1 - x_2)]^2,
\]
while we, of course, have
\[
\langle B^\pm(x_1)B^\pm(x_2) \rangle = 0.
\]

Hence, the field-equation anomaly for (8.10) is present for \(D \neq 26\).

If the same model is calculated using the conventional perturbation theory, the results exhibit unusual complications, owing to the fact that \(T^\ast\)-product does not preserve the validity of field equations.\(^6\)

First, in addition to the natural expressions for the free propagators, \(\langle T^\ast c^\pm(x_1) \bar{c}^\pm(x_2) \rangle\) and \(\langle T^\ast \phi_M(x_1)\phi^N(x_2) \rangle\), there is a nonvanishing 2-point Green function
\[
\langle T^\ast B^\pm(x_1)h^\pm(x_2) \rangle = -i\delta^2(x_1 - x_2).
\]
Because it is not consistent with the field equation (8.9), it induces many pathological effects, which are absent in the operator formalism. Owing to (8.27), the Feynman diagrammatic calculation based on (8.8) implies the existence of 1-loop graphs for the \(n\)-point Green functions consisting of \(B\)-fields only; for example, we find
\[
\langle T^\ast B^\pm(x_1)B^\pm(x_2) \rangle = \frac{1}{2}(D - 26)[\partial^\pm_2 D_F(x_1 - x_2)]^2.
\]

Because \(B^\pm\) is the BRS daughter of \(\bar{c}^\pm\), (8.28) shows that the BRS anomaly appears for \(D \neq 26\). This is the perturbation-theory counterpart of the field-equation anomaly.

Second, as remarked above, the \(T^\ast\)-product does not preserve the fact that each of \(\tilde{b}^\pm, c^\pm, \bar{c}^\pm, \partial^\pm_\phi M\) is a function of \(x^\pm\) alone. Therefore, the Green function consisting of both \(+\)-components and \(\text{--}\)-components can remain nonvanishing, in contradiction to the common sense. For example, we have
\[
\langle T^\ast B^+(x_1)B^-(x_2) \rangle = \frac{1}{8}(D - 2)[\delta^2(x_1 - x_2)]^2,
\]
\[
\langle T^\ast c^+(x_1)B^-(x_2)\bar{c}^-(x_3) \rangle = -\frac{1}{2}\delta^2(x_1 - x_2)\partial_- D_F(x_2 - x_3).
\]

We now return to the operator formalism. The Noether BRS current is given by
\[
\hat{j}_B^\pm = -2i\bar{c}^\pm c^\pm \partial^\pm_\phi c^\pm - c^\pm \partial^\pm_\phi M \cdot \partial^\pm_\phi M
\]
\[
= \hat{j}_B^\mp + 2G^\pm c^\pm, \quad (8.31)
\]
with
\[
\hat{j}_B^\pm \equiv -2B^\pm c^\pm + 2i\bar{c}^\pm c^\pm \partial^\pm_\phi c^\pm. \quad (8.32)
\]
At the operator level, \(\hat{j}_B^\pm\) strictly equals \(j_B^\pm\). But this equality no longer holds at the representation level, because of the appearance of the field-equation anomaly.
Let $Q_B$ and $\hat{Q}_B$ be the BRS generators corresponding to $j^\pm_B$ and $\hat{j}^\pm_B$, respectively. Then, $Q_B$ is not nilpotent if $D \neq 26$, while $\hat{Q}_B$ is always nilpotent. That is, the latter is the genuine BRS generator. On the other hand, the Noether BRS generator $Q_B$ is the BRS generator adopted by Kato and Ogawa in their bosonic string theory. Indeed, from (8.25), we find
\[
\langle \bar{c}^\pm (x_1) Q_B^2 \bar{c}^\pm (x_2) \rangle = \langle \{ \bar{c}^\pm (x_1), Q_B \} \{ Q_B, \bar{c}^\pm (x_2) \} \rangle = \langle F^\pm (x_1) F^\pm (x_2) \rangle = \frac{D - 26}{32\pi^2} \frac{1}{(x_1^\pm - x_2^\pm - i0)^4},
\]
(8.33)
If we take account of the open-string boundary conditions, we obtain
\[
\langle \bar{c}^\pm (x_1) Q_B^2 \bar{c}^\pm (x_2) \rangle = \frac{D - 26}{128\pi^2} \left( \frac{\sin x_1^\pm - x_2^\pm}{\sqrt{2}} \right)^{-4} + \frac{1 - \alpha_0}{8\pi^2} \left( \frac{\sin x_1^\pm - x_2^\pm}{\sqrt{2}} \right)^{-2},
\]
(8.34)
where $\alpha_0 = 0$. Kato and Ogawa claimed the derivation of the critical dimension $D = 26$ from the nilpotency requirement for $Q_B$, but the above analysis shows that they made this claim simply because they were unaware of the field-equation anomaly. One may suspect that the critical dimension can be derived by requiring the absence of the field-equation anomaly. However, as will be seen in §9, it is a specialty of the conformal-gauge case that the field-equation anomaly is absent for $D = 26$. Thus, it should be understood that what Kato and Ogawa did was not to derive $D = 26$ from the BRS nilpotency but to establish the BRS-formulated bosonic string theory under the assumption that $D = 26$ (and $\alpha_0 = 1$) in the conformal gauge.

Finally, we give a remark on the FP-ghost number conservation. The exact solution evidently remains nothing anomalous about the FP-ghost number conservation. Moreover, the Noether FP-ghost number current,
\[
j_\bar{c}^\mp = -2i\bar{c}^\pm c^\pm,
\]
(8.35)
is irrelevant to the field-equation anomaly. Nevertheless, it has been believed that the FP-ghost number current is anomalous, that is, its divergence is nonvanishing (in perturbation theory, this is a consequence of the $T^*$-product) and proportional to the scalar curvature. The main reason for this belief is the Riemann-Roch theorem, which implies that if the two-dimensional spacetime manifold is compact and not flat, then the FP-ghost number is proportional to the Euler characteristic.

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(8.35)
is irrelevant to the field-equation anomaly. Nevertheless, it has been believed that the FP-ghost number current is anomalous, that is, its divergence is nonvanishing (in perturbation theory, this is a consequence of the $T^*$-product) and proportional to the scalar curvature. The main reason for this belief is the Riemann-Roch theorem, which implies that if the two-dimensional spacetime manifold is compact and not flat, then the FP-ghost number is proportional to the Euler characteristic.

***) It should be noted, however, that the Euler characteristic vanishes for any even-dimensional compact manifold with the Lorentzian metric. We claim that the FP-ghost number current is always strictly conserved in 2-dimensional quantum gravity.

---

* Although Kato and Ogawa set $\alpha_0 = 1$ so as to eliminate the second term in (8.34), the “Regge intercept” $\alpha_0$ must be zero, because otherwise the simplest Ward-Takahashi identity, $\langle \{ Q_B, \bar{c}^\pm \} \rangle = 0$, is violated.

***) The Euler characteristic is equal to the invariant integral of the scalar curvature.
§9. 2-dimensional quantum gravity in the de Donder gauge

In this section, we study the BRS-formulated 2-dimensional quantum gravity in the de Donder gauge. Quite remarkably, our approach makes it possible to obtain explicit expressions for all truncated Wightman functions for the fundamental fields of this model in beautiful forms. It is probably impossible to obtain such a complete result by any other approach.

As in §8, because \( \det \tilde{g}^{\mu\nu} = -1 \), where \( \tilde{g}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} \) and \( g \equiv \det g_{\mu\nu} \), we may start with the Lagrangian density in which the conformal or Weyl degree of freedom is already eliminated. However, here we consider the formalism in which all three degrees of freedom of \( g_{\mu\nu} \) are taken into account, because in this case the correspondence with quantum Einstein gravity becomes more transparent. It is easy to separate the conformal degree of freedom from the results.

In quantum Einstein gravity, it is convenient to introduce the concept of the intrinsic BRS transformation,\(^{39),(1)–(3)}\) because general covariance or diffeomorphism is a spacetime symmetry: The conventional BRS is decomposed into a sum of the intrinsic BRS and the orbital part. The action can be written down under the requirement of intrinsic BRS invariance, just as we can write down a Lorentz-invariant action by taking account of intrinsic spins only. While the BRS daughter of the FP-antighost \( \bar{c}_\lambda \) is the conventional B field \( B_\lambda \), we introduce the intrinsic B field \( b_\lambda \) as the intrinsic BRS daughter of \( \bar{c}_\lambda \). Quantum Einstein gravity can be formulated most beautifully if we adopt \( b_\lambda \) as the fundamental field rather than \( B_\lambda \). Then the latter is expressed as

\[
B_\lambda = b_\lambda + ic^\sigma \partial_\sigma \bar{c}_\lambda, \tag{9.1}
\]

where \( c^\sigma \) denotes the FP ghost, of course.

We start with the Lagrangian density

\[
\mathcal{L} = -\tilde{g}^{\mu\nu}(\partial_\mu b_\nu + i\partial_\mu \bar{c}_\lambda \cdot \partial_\nu c^\lambda) + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu \phi_M \cdot \partial_\nu \phi^M + \sqrt{-\tilde{g}}\tilde{R}\tilde{b}, \tag{9.2}
\]

where \( \phi_M, R \) and \( \tilde{b} \) are \( D \) scalar fields (as in §8), the scalar curvature, and the conformal B field, respectively. Although the conformal FP ghost and antighost are not introduced, it is possible to make the action conformal BRS-invariant.

The field equations derived from (9.2) are as follows:

\[
G_{\mu\nu} \equiv -E_{\mu\nu} + \frac{1}{2}g_{\mu\nu}E + T_{\mu\nu} + 2(\nabla_\mu \nabla_\nu - g_{\mu\nu}g^{\sigma\tau}\nabla_\sigma \nabla_\tau)\tilde{b} = 0 \tag{9.3}
\]

with

\[
E_{\mu\nu} \equiv \partial_\mu b_\nu + i\partial_\mu \bar{c}_\lambda \cdot \partial_\nu c^\lambda + (\mu \leftrightarrow \nu), \quad E \equiv g^{\mu\nu}E_{\mu\nu}, \tag{9.4}
\]

\[
T_{\mu\nu} \equiv \partial_\mu \phi_M \cdot \partial_\nu \phi^M - \frac{1}{2}g_{\mu\nu}g^{\sigma\tau}\partial_\sigma \phi_M \cdot \partial_\tau \phi^M; \tag{9.5}
\]

\[
\partial_\mu \tilde{g}^{\mu\nu} = 0, \tag{9.6}
\]

\[
\partial_\mu(\tilde{g}^{\mu\nu} \partial_\nu \Phi) = 0 \quad \text{for} \quad \Phi = c^\lambda, \bar{c}_\lambda, \phi_M, \tag{9.7}
\]

\[
R = 0. \tag{9.8}
\]
The trace of (9.3) becomes
\[ \partial_\mu (\tilde{g}^{\mu \nu} \partial_\nu \tilde{b}) = 0. \] (9.9)

Then, taking the covariant derivative of (9.3), we obtain
\[ \partial_\mu (\tilde{g}^{\mu \nu} \partial_\nu b_\lambda) = 0. \] (9.10)

Because the degrees of freedom of (9.9) and (9.10) are the same as those of (9.3), we can replace (9.3) by (9.9) and (9.10) if we ignore “integration constant” of (9.3). It should be noted that (9.6), (9.9) and (9.10) are of the form of (9.7) with \( \Phi = x^\nu, \tilde{b}, b_\lambda \), respectively; that is, all the field equations, except for (9.8), are the covariantized d’Alembert equations.

Taking all the fundamental fields other than \( b_\lambda \) as canonical variables, we carry out the canonical quantization.\(^*\) After analysis based on the results obtained in quantum Einstein gravity,\(^2\) we find all 2-dimensional (anti)commutation relations explicitly.\(^{10}\) The nonvanishing relations are as follows:

\[ [g_{\mu \nu}(x), b_\lambda(y)] = i [g_{\lambda \nu} \partial_\mu + g_{\mu \lambda} \partial_\nu + (\partial_\lambda g_{\mu \nu})] x D(x,y), \] (9.11)

\[ [b_\rho(x), b_\lambda(y)] = i [\partial_\lambda b_\rho(x) + \partial_\rho b_\lambda(y)] \cdot D(x,y), \] (9.12)

\[ [\Phi(x), b_\lambda(y)] = i \partial_\lambda \Phi(x) \cdot D(x,y) \quad \text{for} \quad \Phi = c^\rho, \tilde{c}_\rho, \phi_M, \tilde{b}, \] (9.13)

\[ \{ c^\rho(x), \tilde{c}_\lambda(y) \} = - \delta^\lambda_\rho D(x,y), \] (9.14)

\[ [\phi_M(x), \phi^N(y)] = i \delta_M^N \cdot D(x,y), \] (9.15)

\[ [g_{\mu \nu}(x), \tilde{b}(y)] = - ig_{\mu \nu}(x) D(x,y). \] (9.16)

The other (anti)commutators vanish. In particular, \( g_{\mu \nu}(x) \) commutes with \( g_{\sigma \tau}(y), c^\rho(y), \tilde{c}_\lambda(y) \) and \( \phi_M(y) \). The self-commutativity of \( g_{\mu \nu} \) is the basis of the exact solvability of this model.

The q-number Pauli-Jordan \( D(x,y) \) function, is defined by the following Cauchy problem:

\[ (\partial_\mu \tilde{g}^{\mu \nu} \partial_\nu)^x D(x,y) = 0, \] (9.17)

\[ D(x,y)|_0 = 0, \] (9.18)

\[ \partial_0^x D(x,y)|_0 = -(\tilde{g}^{00})^{-1} \delta(x^1 - y^1). \] (9.19)

Here, \( |_0 \) indicates that we set \( x^0 = y^0 \). From (9.17)–(9.19), we can prove

\[ D(x,y) = -D(y,x) = D^\dagger(x,y) \] (9.20)

by using trivial-type Cauchy problems.\(^{40}\) Because \( D(x,y) \) is expressible in terms of \( g_{\mu \nu}, D(x,y) \) commutes with the latter, and hence with \( D(z,w) \). However, from (9.11), we have

\[ [D(x,y), b_\lambda(z)] = i [\partial_\lambda^x D(x,y) \cdot D(x,z) + \partial_\lambda^y D(x,y) \cdot D(y,z)]. \] (9.21)

\(^*\) Note that \( \tilde{b} \) is not a multiplier field, because the second derivatives involved in \( R \) must be partially integrated.
Then, we see that on the right-hand side of (9.12), the two factors commute, as is necessary to realize consistency with the hermitian conjugation. However, a single term in the square brackets does not commute with $D(x, y)$.

If we consider the formulation in which the conformal degree of freedom is not introduced, (9.8), (9.9) and (9.16)\(^*\) are withdrawn and we should set $\bar{b} = 0$ in (9.3).

We now construct a representation of the operator solution in terms of Wightman functions. The 1-point function of $g_{\mu\nu}(x)$ is identified with the c-number spacetime metric $g_{\mu\nu}(x)$; that is,

$$\langle g_{\mu\nu}(x) \rangle = g_{\mu\nu}(x).$$ \hspace{1cm} (9.22)

The 1-point functions of the other fields are taken to be zero.

Because $g_{\mu\nu}$ is self-commuting, (9.22) implies

$$\langle D(x, y) \rangle = D(x, y) = -D(y, x),$$ \hspace{1cm} (9.23)

where $D(x, y)$ denotes the D function in the 2-dimensional spacetime with the metric $g_{\mu\nu}(x)$. Of course, $D(x, y)$ is defined by the c-number version of the Cauchy problem (9.17)–(9.19). If $g_{\mu\nu}(x) = \eta_{\mu\nu}$, its explicit expression is

$$D(x, y) = -\frac{1}{2} \varepsilon(x^0 - y^0)\theta((x - y)^2).$$ \hspace{1cm} (9.24)

We decompose $iD(x, y)$ into its positive-energy part $D^{(+)}(x, y)$ and its negative-energy part $-D^{(+)}(y, x)$:

$$D^{(+)}(x, y) - D^{(+)}(y, x) = iD(x, y).$$ \hspace{1cm} (9.25)

If $g_{\mu\nu}(x) = \eta_{\mu\nu}$, the expression for $D^{(+)}(x, y)$ is given by

$$D^{(+)}(x, y) = -(4\pi)^{-1} \log(-\mu^2(x - y)^2 + i0(x^0 - y^0)),$$ \hspace{1cm} (9.26)

where $\mu$ is an infrared cutoff parameter.\(^{34,35}\)

The nonvanishing 2-point (truncated) Wightman functions are as follows:

$$\langle g_{\mu\nu}(x_1)b_\lambda(x_2) \rangle = [g_{\lambda\nu}\partial_\mu + g_{\mu\lambda}\partial_\nu + (\partial_\lambda g_{\mu\nu})]x_1 D^{(+)}(x_1, x_2),$$ \hspace{1cm} (9.27)

$$\langle b_\rho(x_1)b_\lambda(x_2) \rangle = \partial_\lambda^2 D^{(+)}(x_1, x_2) \cdot \partial_\rho^2 D^{(+)}(x_1, x_2),$$ \hspace{1cm} (9.28)

$$\langle c^\rho(x_1)c_\lambda(x_2) \rangle = i\delta^\rho_\lambda D^{(+)}(x_1, x_2),$$ \hspace{1cm} (9.29)

$$\langle \phi_M(x_1)\phi_N(x_2) \rangle = \delta_M^N D^{(+)}(x_1, x_2),$$ \hspace{1cm} (9.30)

$$\langle g_{\mu\nu}(x_1)b(x_2) \rangle = -g_{\mu\nu}(x_1)D^{(+)}(x_1, x_2).$$ \hspace{1cm} (9.31)

In general, the nonvanishing $n$-point truncated Wightman functions are those consisting of the following:

1. one $g_{\mu\nu}$ and $(n - 1)$ B fields (both $b_\lambda$-fields and $\bar{b}$-fields),
2. $n$ $b_\lambda$-fields,
3. one pair $(c^\rho, \bar{c}_\lambda)$ and $(n - 2)$ $b_\lambda$-fields,
4. two $\phi_M$ fields and $(n - 2)$ $b_\lambda$-fields.

\(^*\) Note that $\tilde{g}_{\mu\nu}$ commutes with $\bar{b}$.\(^{36}\)
The graphs for Case (2) are 1-loop graphs, while those for all other cases are tree graphs. No internal vertices appear in any graph.

*All truncated Wightman functions have been explicitly constructed and BRS invariance has been confirmed.*\(^{12,14,16,17}\)\(^{1)}\) It should be emphasized that our solution in terms of Wightman functions is free of ultraviolet divergence, though the corresponding Green functions for Case (2) contain logarithmic divergences.

All expressions have been obtained in closed form, but those in Cases (1) and (2) are quite lengthy. It is more convenient to express them diagrammatically. As is well known, the conventional perturbation-theoretical expressions can be easily written down using Feynman rules. We can invent analogous diagrammatic rules in the Heisenberg picture. In contrast to the situation in the interaction picture, however, we cannot determine the incidence relation by the vertex, because in the Heisenberg picture, there is no concept like the interaction Lagrangian density. A diagram is constructed by composing “building blocks”, each of which consists of a “rod” and “hooks”.\(^{\ast\ast}\) To each building block there corresponds a particular quantity (whose expression is suggested by the (anti)commutation relation) consisting of one \(D^{(+)}\) function, together with differential operators whose operands may be unspecified. Building blocks are connected by their hooks in the prescribed ways, and the hooks thus combined at a vertex determine on what partial product of the \(D^{(+)}\) functions (associated with the relevant building blocks) the differential operators (whose operands are unspecified beforehand as stated above) act. (For details, see the original paper.\(^{16}\))

When many hooks are placed at a vertex (their number is unlimited), the manner in which differential operators act becomes complicated. If we determine their action by using the Leibniz rule, we encounter many cancellations among the contributions from different diagrams. Avoiding this redundancy, we can directly determine the rules for obtaining the final results by using the graph-theoretical concept of a “rooted tree”.\(^{17}\) In this way, we can write down explicit expressions for all truncated Wightman functions quite elegantly.

Our solution is consistent with the field equations (9.6)–(9.10), but not with (9.3); that is, we encounter a field-equation anomaly. Calculating \(\langle G_{\mu\nu}(x_1)b_\lambda(x_2)\rangle\), where \(G_{\mu\nu}\) denotes the left-hand side of (9.3), we find that it is nonvanishing and independent of \(D\). Its explicit expression is greatly simplified in the flat spacetime, where we have

\[
\langle G_{\mu\nu}(x_1)b_\lambda(x_2)\rangle = \partial_\nu[\partial_\mu D^{(+)}(x_1-x_2) \cdot \partial_\lambda D^{(+)}(x_1-x_2)] - \eta_{\mu\lambda}\partial_\sigma D^{(+)}(x_1-x_2) \cdot \partial_\nu \partial_\sigma D^{(+)}(x_1-x_2) + (\mu \leftrightarrow \nu). \tag{9.32}
\]

We encounter, however, a \(D\)-dependent field-equation anomaly for the 2-point function of \(G_{\mu\nu}\). For convenience, we give the expression for the corresponding T*-
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\[ \langle T^\ast G_{\mu\nu}(x_1)G_{\lambda\rho}(x_2) \rangle = (D + 10)\Phi_{\mu\nu\lambda\rho}(x_1 - x_2) + \cdots. \] (9.33)

Here

\[ \Phi_{\mu\nu\lambda\rho}(x) = \frac{i}{12\pi} \int \frac{d^2 p}{(2\pi)^2} \cdot \frac{p_\mu p_\nu p_\lambda p_\rho}{p^2 + i\epsilon} e^{-ipx}, \] (9.34)

which is called “nonlocal term” in the perturbation-theoretical analysis of the conformal anomaly,\(^{41}\) and the dots represent local terms, which are irrelevant to the conformal anomaly. We note

\[ \partial_\mu \partial_\nu D F(x) \cdot \partial_\lambda \partial_\rho D F(x) = \frac{1}{2} \Phi_{\mu\nu\lambda\rho}(x) + \cdots, \] (9.35)

\[ \partial_\mu \partial_\nu \partial_\lambda D F(x) \cdot \partial_\rho D F(x) = \Phi_{\mu\nu\lambda\rho}(x) + \cdots. \] (9.36)

In contrast to the conformal-gauge case, the field-equation anomaly given by (9.33) does not vanish for any \( D \geq 0. \)

In analogy to the conformal-gauge case, we introduce the following sum of \( G_{\mu\nu} \) and a BRS-exact quantity:\(^{42}\)

\[ F_{\mu\nu} \equiv G_{\mu\nu} + \partial_\mu B_\nu + \partial_\nu B_\mu, \] (9.37)

where \( B_\lambda \) is given by (9.1). Then because

\[ \langle T^\ast (\partial_\mu B_\nu(x_1) + \partial_\nu B_\mu(x_1))(\partial_\lambda B_\rho(x_2) + \partial_\rho B_\lambda(x_2)) \rangle = 0, \] (9.39)

we have

\[ \langle T^\ast F_{\mu\nu}(x_1)F_{\lambda\rho}(x_2) \rangle = (D - 26)\Phi_{\mu\nu\lambda\rho}(x_1 - x_2) + \cdots. \] (9.40)

The expression for \( F_{\mu\nu} \) is quite similar to that for the energy-momentum tensor defined in the perturbation-theoretical analysis for the conformal anomaly,\(^{\star} \) though these two quantities are defined in completely different ways.

The genuine BRS generator is given by

\[ \hat{Q}_B = \int dx^1 g^{0\nu}(b_\rho \partial_\nu c^\rho - \partial_\nu b_\rho \cdot c^\rho), \] (9.41)

which is, of course, nilpotent. On the other hand, the Noether BRS generator is given by

\[ Q_B = \hat{Q}_B - \int dx^1 g^{0\nu} c^\lambda G_{\nu\lambda}. \] (9.42)

\(^{\star} \)In the perturbative approach, one introduces a background metric so as to be generally covariant, and the energy-momentum tensor is defined by taking the Euler derivative with respect to this background metric.\(^{41}\) In order to have \( D = 26 \), it is essential to adopt \( B_\lambda \), but not \( b_\lambda \), as the B field.\(^{43}\) On the other hand, Takahashi calculated the 2-point function of \( B_\lambda \) and found the BRS anomaly to be proportional to \( D - 26. \)\(^{44}\) His result is due to his inadequate application of dimensional regularization.\(^{42}\)
Therefore, we expect that $Q_B$ exhibits the field-equation anomaly. In contrast to the conformal-gauge case, however, we have

$$\{Q_B, \bar{c}_\lambda\} = B_\lambda,$$

(9.43)

instead of encountering $F_{\mu\nu}$ on the right-hand side. Therefore, we do not find the relation between the nilpotency condition for $Q_B$ and the anomaly formula (9.40) as in §8. Thus, if Kato and Ogawa had considered the de Donder-gauge case, they could not have obtained $D = 26$ from the nilpotency condition for $Q_B$. The anomalous nature of $Q_B$ can be seen by checking that the vacuum expectation value of a product of operators involving $\partial_0 Q_B$ does not necessarily vanish, because the time derivative of the second term on the right-hand side of (9.42) does not vanish without using (9.3).

Finally, we give a historical review of the investigation of the BRS-formulated 2-dimensional quantum gravity in the de Donder gauge. This model was first investigated by Sato.\textsuperscript{45} He wished to extend Nakanishi’s formalism for quantum Einstein gravity to the 2-dimensional case. Because of the trivial nature of the Einstein-Hilbert action in 2-dimensional spacetime, Sato introduced a conformal or Weyl gauge-fixing term, as in (9.2), and the corresponding FP-ghost term. Unfortunately, he failed to obtain a unitary theory because of the ghost-counting mismatch. Abe and Nakanishi resolved this problem by making the general-coordinate FP (anti)ghost also play the role of the Weyl FP (anti)ghost.\textsuperscript{10} At the same time, they found that $g_{\mu\nu}$ is 2-dimensionally self-commuting. This fact implies that non-geometric terms are absent in Nakanishi’s geometric commutation relation established previously in quantum Einstein gravity.\textsuperscript{2} From this finding, it was straightforward to obtain all 2-dimensional (anti)commutation relations. Indeed, the right-hand sides of those relations are expressible in compact form in terms of the fundamental fields and the q-number Pauli-Jordan D-function $D(x, y)$. Those results were extended to the zweibein formalism,\textsuperscript{11} in which an explicit expression for $D(x, y)$ was found.

A representation of the above operator solution was constructed in terms of Wightman functions.\textsuperscript{12} At this stage, however, an erroneous ansatz was employed for the vacuum expectation value of $[b_\rho(x), b_\lambda(y)]$. In order to treat it correctly, it was necessary to invent a new prescription (as stated in §4) for the Wightman function involving composite fields.\textsuperscript{13} With this prescription, Abe and Nakanishi succeeded in explicitly constructing some of the truncated Wightman functions consisting of the $b_\lambda$-fields only.\textsuperscript{14} At the same time, the existence of the field-equation anomaly for the quantum Einstein-like equation (9.3) was discovered.

The correctness of the obtained solution was confirmed by checking that it reproduces the results of the conventional perturbation theory for all 2- and 3-point Green functions\textsuperscript{*} in the flat spacetime.\textsuperscript{15} It should be remarked that this comparison cannot be made in a manifestly covariant way; that is, we need some identities whose validity is due to the special nature of $d = 2$.

In analogy to the Feynman diagrammatic method in the interaction picture, Abe and Nakanishi proposed a new diagrammatic method in the Heisenberg picture,\textsuperscript{16,17}

\textsuperscript{*} The Feynman diagrammatic calculation of higher-point functions is too cumbersome to work out.
according to which all truncated Wightman functions for the fundamental fields can be written down directly, without a cumbersome calculation.

§10. One-loop model

In the framework of the indefinite-metric quantum field theory, it is easy to construct exactly solvable models even in an arbitrary $d$-dimensional spacetime. The “one-loop model” is a model whose exact solution is given by the contributions from tree and 1-loop graphs only. Here, we show how the one-loop model is solved using our approach.

The Lagrangian density of our model is

$$\mathcal{L} = \partial^\mu \phi \cdot \partial_\mu \tilde{\phi} - m^2 \phi \tilde{\phi} + \frac{1}{2} \phi^2 \tilde{\phi},$$  \hspace{1cm} (10.1)$$

where $\phi$ and $\tilde{\phi}$ are scalar fields possessing the same mass, $m$.

The field equations are

$$\left(\partial^2 + m^2\right)\phi - \frac{1}{2} \phi^2 = 0, \hspace{1cm} (10.2)$$
$$\left(\partial^2 + m^2\right)\tilde{\phi} - \phi \tilde{\phi} = 0. \hspace{1cm} (10.3)$$

The equal-time commutation relations are

$$[\phi(x), \phi(y)]_0 = [\partial_0 \phi(x), \phi(y)]_0 = 0, \hspace{1cm} (10.4)$$
$$[\phi(x), \tilde{\phi}(y)]_0 = -i\delta^{d-1}(x - y), \hspace{1cm} (10.5)$$
$$[\partial_0 \phi(x), \tilde{\phi}(y)]_0 = [\partial_0 \tilde{\phi}(x), \phi(y)]_0 = 0. \hspace{1cm} (10.6)$$

We rewrite the field equations as equations for $d$-dimensional commutators:

$$\left(\partial^2 + m^2\right)_{\xi}[\phi(x), \phi(y)] - \frac{1}{2} \{\phi(x), [\phi(x), \phi(y)]\} = 0, \hspace{1cm} (10.8)$$
$$\left(\partial^2 + m^2\right)_{\xi}[\phi(x), \tilde{\phi}(y)] - \frac{1}{2} \{\phi(x), [\phi(x), \tilde{\phi}(y)]\} = 0, \hspace{1cm} (10.9)$$
$$\left(\partial^2 + m^2\right)_{\xi}[\tilde{\phi}(x), \phi(y)] - [\phi(x) \tilde{\phi}(x), \tilde{\phi}(y)] = 0. \hspace{1cm} (10.10)$$

From the Cauchy problem, (10.8) with (10.4), implies

$$[\phi(x), \phi(y)] = 0. \hspace{1cm} (10.11)$$

We set

$$[\phi(x), \tilde{\phi}(y)] = i\Pi(x, y). \hspace{1cm} (10.12)$$

\footnote{If the last term of (10.1) is replaced by a function of $\phi$ only, we obtain a simpler model, called the “Glaser model”, which has no internal line.}

\footnote{The interaction term may contain an arbitrary function of $\phi$.}
Then, from (10.9) with (10.5) and (10.6), the q-number $\Delta$ function $\Delta(x, y)$ satisfies the following Cauchy problem:

$$\left(\partial^2 + m^2 - \phi\right)x \Delta(x, y) = 0, \quad (10.13)$$

$$\Delta(x, y)|_{0} = 0, \quad (10.14)$$

$$\partial_x^2 \Delta(x, y)|_{0} = -\delta^d(x - y). \quad (10.15)$$

Because this Cauchy problem involves $\phi$ only, $\Delta(x, y)$ is expressible in terms of $\phi$ alone. Moreover, since $\phi$ self-commutes, we see

$$\left[\Delta(x, y), \phi(z)\right] = 0. \quad (10.16)$$

We have used this fact in (10.13) beforehand in order to simplify its expression.

Of course, (10.16) implies that $\Delta$ self-commutes. As in §9, $\Delta(x, y)$ is shown to be antisymmetric with respect to $x$ and $y$.

We rewrite (10.13) into the form

$$(\partial^2 + m^2 - \phi)x \left[\Delta(x, y), \tilde{\phi}(z)\right] = i \Delta(x, y) \Delta(x, z). \quad (10.17)$$

The initial data of $\left[\Delta(x, y), \tilde{\phi}(z)\right]$ for $x^0 = y^0$ are evidently zero. Hence, solving the Cauchy problem, we obtain

$$\left[\Delta(x, y), \tilde{\phi}(z)\right] = -i \int d^d u \varepsilon(x^0 > u^0 > y^0) \Delta(x, u) \Delta(u, y) \Delta(u, z), \quad (10.18)$$

where $\varepsilon(x^0 > u^0 > y^0)$ is a sign function defined by

$$\varepsilon(x^0 > u^0 > y^0) \equiv \theta(x^0 - u^0) - \theta(y^0 - u^0)$$

$$= \begin{cases} 
+1 & \text{for } x^0 > u^0 > y^0 \\
-1 & \text{for } x^0 < u^0 < y^0 \\
0 & \text{otherwise.} 
\end{cases} \quad (10.19)$$

In order to set up the Cauchy problem, (10.10) should be rewritten as

$$(\partial^2 + m^2 - \phi)x \left[\tilde{\phi}(x), \tilde{\phi}(y)\right] = i \Delta(x, y) \tilde{\phi}(x). \quad (10.20)$$

However, because of the special nature of the model, it is more convenient to integrate (10.10) directly:

$$\left[\tilde{\phi}(x), \tilde{\phi}(y)\right] = - \int d^d u \varepsilon(x^0 > u^0 > y^0) \Delta(x - u) \left[\phi(u) \tilde{\phi}(u), \tilde{\phi}(y)\right]. \quad (10.21)$$

Now, we proceed to constructing the representation in terms of Wightman functions. For simplicity, we set all 1-point functions equal to zero:

$$\langle \phi(x) \rangle = \langle \tilde{\phi}(x) \rangle = 0. \quad (10.22)$$

Then, owing to the self-commutativity of $\phi(x)$, we have

$$\langle \Delta(x, y) \rangle = \Delta(x - y). \quad (10.23)$$
From the commutation relations, we can easily see that any truncated Wightman function containing two or more $\phi$-fields vanishes. Any truncated Wightman function containing only one $\phi$ and that containing no $\phi$ are expressible in the form of tree graphs and 1-loop graphs, respectively.

From (10.12) and (10.23) together with the energy-positivity condition, we obtain

$$\langle \phi(x_1)\phi(x_2) \rangle = \langle \phi(x_1)\phi(x_2) \rangle = \Delta^{(+)}(x_1 - x_2). \quad (10.24)$$

The graphs of any other nonvanishing truncated Wightman function contain internal vertices, each of which corresponds to a $d$-dimensional integral. In the following, we state the general rule for constructing a quantity satisfying the energy-positivity condition.

Let $x_1, x_2, \ldots, x_n$ be the spacetime variables appearing in the truncated Wightman function in this ordering. Suppose that $u_1, u_2, \ldots, u_m$ are the integration variables corresponding to $m$ internal vertices. We divide the integration domain of the $m$ time-variables $u_1^0, u_2^0, \ldots, u_m^0$ into $(n + m)!/n!$ sectors, each of which is defined by one particular ordering with respect to the magnitudes of the $(n + m)$ variables $x_1^0, x_2^0, \ldots, x_n^0, u_1^0, u_2^0, \ldots, u_m^0$ consistent with $x_1^0 > x_2^0 > \cdots > x_n^0$. In each sector, the time-variable differences in all $\Delta^{(+)}$ functions should be positive under the ordering characterizing that sector.

From (10.18) with (10.12), we find

$$\langle [[\phi(x_1), \phi(x_2)], \phi(x_3)] \rangle = \int d^d u \varepsilon(x_1^0 > u^0 > x_2^0) \Delta(x_1 - u) \Delta(u - x_2) \Delta(u - x_3). \quad (10.25)$$

Therefore, the energy-positivity condition stated above yields

$$\langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle = \langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle = \langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle$$

$$= i \int d^d u \varepsilon(x_1^0 > u^0 > x_2^0) \Delta^{(+)}(u - x_1) \Delta^{(+)}(u - x_2) \Delta^{(+)}(u - x_3)$$

$$+ \varepsilon(x_1^0 > u^0 > x_2^0) \Delta^{(+)}(x_1 - u) \Delta^{(+)}(x_2 - u) \Delta^{(+)}(u - x_3)$$

$$+ \varepsilon(x_2^0 > u^0 > x_1^0) \Delta^{(+)}(x_1 - u) \Delta^{(+)}(x_2 - u) \Delta^{(+)}(u - x_3)$$

$$+ \theta(x_3^0 - u^0) \Delta^{(+)}(x_1 - u) \Delta^{(+)}(x_2 - u) \Delta^{(+)}(x_3 - u). \quad (10.26)$$

The graph corresponding to (10.26) is a “Y” shape tree.

Similarly, we have

$$\langle \phi(x_1)\phi(x_2) \rangle = i \int d^d u \varepsilon(x_1^0 > u^0 > x_2^0) \Delta^{(+)}(u - x_1) \langle \phi(u)\phi(u)\phi(u) \rangle$$

$$+ \varepsilon(x_1^0 > u^0 > x_2^0) \Delta^{(+)}(x_1 - u) \langle \phi(u)\phi(u)\phi(x_2) \rangle$$

$$+ \theta(x_2^0 - u^0) \Delta^{(+)}(x_1 - u) \langle \phi(x_2)\phi(u)\phi(u) \rangle. \quad (10.27)$$

According to the prescription for composite fields, we can immediately obtain the expressions for $\langle \phi(u)\phi(u)\phi(x_2) \rangle$ and $\langle \phi(x_2)\phi(u)\phi(u) \rangle$ from (10.26). The graph corresponding to (10.27) is a 1-loop graph of the self-energy type.
It is possible to carry out a diagrammatic construction of the truncated Wightman functions by introducing a building block of a composite type, in addition to that of the single-line type.\textsuperscript{22})

In 1984, Ostendorf proposed Feynman-like rules for writing down perturbation terms of the Wightman function.\textsuperscript{47}) They can be expressed in terms of $\Delta^{(+)}$ and $\Delta_F$ functions without using $\theta$ functions explicitly. It is possible to rewrite (10.26) and (10.27) into the Ostendorf form. Then, the expressions obtained coincide with those given by the Ostendorf rules.

The reproduction of the Ostendorf rules has been confirmed in the positive-norm scalar field theory with direct self-coupling.\textsuperscript{25}) This is because our solution satisfies the recursion formula from which the Ostendorf rules are derived.

\section*{11. Quantum electrodynamics}

As an example of realistic theories, we consider quantum electrodynamics in the Landau gauge.\textsuperscript{21}) Here, the fundamental fields are the electromagnetic field $A_\mu$, the $B$ field $B$ and the Dirac field $\psi$. The covariant derivative is defined in such a way that it does not depend on the coupling constant $e$. The Lagrangian density is given by

\begin{equation}
\mathcal{L} = -(4e^2)^{-1} F^{\mu\nu} F_{\mu\nu} + B \partial_\mu A^\mu + \bar{\psi}(i\gamma^\mu \partial_\mu + \gamma^\mu A_\mu - m) \psi, \tag{11.1}
\end{equation}

where $F^{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that $e$ is present only in the first term and only as $e^2$.

The field equations derived from (11.1) are

\begin{align}
\partial^2 A_\mu - \partial_\mu \partial_\nu A^\nu &= e^2 (\partial_\mu B + j_\mu), \quad j_\mu \equiv -\bar{\psi} \gamma^\mu \psi, \tag{11.2} \\
\partial_\mu A^\mu &= 0, \tag{11.3} \\
\partial^2 B &= 0, \tag{11.4} \\
(i\gamma^\mu \partial_\mu + \gamma^\mu A_\mu - m) \psi &= 0. \tag{11.5}
\end{align}

The second term on the left-hand side of (11.2) is eliminated by using (11.3).

Canonical quantization is carried out by taking $A_\mu$ and $\psi$ as canonical variables. Because $B$ is a free field, the $d$-dimensional commutation relations involving $B$ can be obtained exactly:

\begin{align}
[B(x), B(y)] &= 0, \tag{11.6} \\
[A_\mu(x), B(y)] &= -i \partial_\mu^x D(x - y), \tag{11.7} \\
[\psi(x), B(y)] &= \psi(x) D(x - y). \tag{11.8}
\end{align}

The other (anti)commutation relations can be obtained through an expansion in powers of $e^2$ (rather than $e$). Let

\begin{align}
A_\mu(x) &= \sum_{N=0}^{\infty} e^{2N} A^{(N)}_\mu(x), \tag{11.9} \\
\psi(x) &= \sum_{N=0}^{\infty} e^{2N} \psi^{(N)}(x). \tag{11.10}
\end{align}
respectively. Note that $\psi^{(0)}$ is not a free field.

The nonvanishing equal-time (anti)commutation relations (other than those involving the B field) are

\[
[A_\mu(x), A_\nu(y)]^{(1)}|_0 = -i(\eta_{\mu\nu} - \delta_\mu^{\ 0}\delta_\nu^{\ 0})\delta^{d-1}(x-y),
\]
(11.15)

\[
\{\psi(x), \bar{\psi}(y)\}^{(0)}|_0 = \gamma^0\delta^{d-1}(x-y).
\]
(11.16)

Cauchy problems are set up by rewriting (11.11)–(11.14) into the form of equations for the (anti)commutators.

The zeroth-order equations are as follows:

\[
(\partial^2)^x[A_\mu(x), A_\nu(y)]^{(0)} = 0,
\]
(11.17)

\[
(\partial^2)^x[A_\mu(x), \psi(y)]^{(0)} = 0,
\]
(11.18)

\[
(i\gamma^\mu\partial_\mu + \gamma^\mu A_\mu^{(0)} - m)^x\{\psi(x), \psi(y)\}^{(0)} = 0,
\]
(11.19)

\[
(i\gamma^\mu\partial_\mu + \gamma^\mu A_\mu^{(0)} - m)^x\{\psi(x), \bar{\psi}(y)\}^{(0)} = 0.
\]
(11.20)

Solving the zeroth-order Cauchy problems, we find

\[
[A_\mu(x), A_\nu(y)]^{(0)} = 0, \quad [A_\mu(x), \psi(y)]^{(0)} = 0,
\]
(11.21)

\[
\{\psi(x), \psi(y)\}^{(0)} = 0, \quad \{\psi(x), \bar{\psi}(y)\}^{(0)} = iS^{(0)}(x,y),
\]
where $S^{(0)}$ is the zeroth-order q-number S function defined by the Cauchy problem

\[
(i\gamma^\mu\partial_\mu + \gamma^\mu A_\mu^{(0)} - m)^xS^{(0)}(x,y) = 0,
\]
(11.22)

\[
S^{(0)}(x,y)|_0 = -i\gamma^0\delta^{d-1}(x-y).
\]
(11.23)

Because of (11.21), $S^{(0)}(x,y)$ commutes with $A^{(0)}_\mu(z)$.

We now proceed to first order. From (11.12), together with (11.7), (11.8) and (11.21), we have

\[
(\partial^2)^x[A_\mu(x), A_\nu(y)]^{(1)} = -i(\partial_\mu\partial_\nu)^x D(x-y),
\]
(11.24)

\[
(\partial^2)^x[A_\mu(x), \psi(y)]^{(1)} = \psi(y)\partial^\mu D(x-y) + [j_\mu(x), \psi(y)]^{(0)}.
\]
(11.25)

We refrain from writing the equations for $\{\psi(x), \psi(y)\}^{(1)}$ and $\{\psi(x), \bar{\psi}(y)\}^{(1)}$. In order to solve the Cauchy problem (11.24) with (11.15), we make use of the following
identity for an arbitrary function $f(x, y)$:

$$
 f(x, y) = -\int d^d u \varepsilon(x^0 > u^0 > y^0) D(x-u)(\partial^2 u f(u, y)
 - \int d^{d-1} u [D(x-u)\partial_0^u f(u, y) - \partial_0^u D(x-u) \cdot f(u, y)]_{u^0 = y^0},
$$

(11.26)

where $\varepsilon(x^0 > z^0 > y^0)$ is defined by (10.19). We obtain

$$
 [A_\mu(x), A_\nu(y)]^{(1)} = -i\eta_{\mu\nu} D(x-y) + i(\partial_\mu \partial_\nu) x E(x-y)
 \equiv iD_{\mu\nu}(x-y),
$$

(11.27)

where $E(x)$ is defined by

$$
 \partial^2 E(x) = D(x) \quad \text{with} \quad E(x) = \partial_0 E(x) = 0 \quad \text{at} \quad x^0 = 0;
$$

(11.28)

that is,

$$
 E(x) = -\int d^d z \varepsilon(x^0 > z^0 > 0) D(x-z) D(z).
$$

(11.29)

Similarly, (11.25) with trivial initial data yields

$$
 [A_\mu(x), \psi(y)]^{(1)} = \partial^2 x E(x-y) \cdot \psi^{(0)}(y)
 - \int d^d u \varepsilon(x^0 > u^0 > y^0) D(x-u) [j_\mu(u), \psi(y)]^{(0)}.
$$

(11.30)

We can also find expressions for $\{\psi(x), \psi(y)\}^{(1)}$ and $\{\psi(x), \bar{\psi}(y)\}^{(1)}$.

We can, further, derive the recurrence formula for $N$-th order (anti)commutators. Then, the expressions for multiple commutators can be calculated.

Now, we construct Wightman functions. First, we set all 1-point functions equal to zero. It should be noted that if $\langle A_\mu(x) \rangle \neq 0$, then $\langle A_\mu(x) \rangle$ is identified with the external electromagnetic field; thus we can take account of the external field, which is something under human control, without changing the Lagrangian density, which represents the fundamental law of physics. Taking the vacuum expectation value of (11.22) with (11.23), we find that

$$
 \langle S^{(0)}(0)(x, y) \rangle = S(x-y)
$$

(11.31)

is the S function of the Dirac field.

The nonvanishing zeroth-order 2-point truncated Wightman functions (excluding the reversed-order ones) are

$$
 \langle A_\mu(x_1)B(x_2) \rangle^{(0)} = -\partial^{x_1}_\mu D^{(+)}(x_1-x_2),
$$

(11.32)

$$
 \langle \psi(x_1)\bar{\psi}(x_2) \rangle^{(0)} = S^{(+)}(x_1-x_2),
$$

(11.33)

where the superscript $(+)$ indicates the positive-energy part. These forms follow from (11.7) and (11.21) with (11.31), respectively. The nonvanishing zeroth-order
$n$-point truncated Wightman functions are

$$
\langle \psi(x_1) \bar{\psi}(x_2) B(x_3) \cdots B(x_n) \rangle^{(0)}
= i^{2-n} S^+(x_1 - x_2) \prod_{k=3}^{n} [D^+(x_1 - x_k) - D^+(x_2 - x_k)],
$$

with similar expressions for the permutated ones. These forms follow from the multiple commutators derived from (11.8), together with (11.31).

For the first order, form (11.27), we obtain

$$
\langle A^\mu(x_1) A^\nu(x_2) \rangle^{(1)} = D^+_{\mu\nu}(x_1 - x_2).
$$

This is exactly the Landau-gauge Wightman function for the free electromagnetic field. The expressions for $\langle A^\mu(x_1) \psi(x_2) \bar{\psi}(x_3) \rangle^{(1)}$ and $\langle \psi(x_1) \bar{\psi}(x_2) \rangle^{(1)}$ are, apart from the differences due to spins, similar to those for $\langle \phi(x_1) \tilde{\phi}(x_2) \tilde{\phi}(x_3) \rangle$ and $\langle \tilde{\phi}(x_1) \tilde{\phi}(x_2) \rangle$ of the one-loop model, respectively. It is explicitly confirmed that the Green function corresponding to $\langle \psi(x_1) \bar{\psi}(x_2) \rangle^{(1)}$ is precisely the second-order self-energy part of the electron in the Landau gauge. Similarly, the Green function corresponding to $\langle A^\mu(x_1) A^\nu(x_2) \rangle^{(2)}$ is precisely the second-order self-energy part of the photon. In general, the order $N$ of the present approach is related to the perturbative order $n$ in the following way:

$$
2N = n + k,
$$

where $k$ denotes the number of the external photon lines.

The diagrammatic method can be applied to quantum electrodynamics. The building blocks in this case are derived from four zeroth-order commutators and from three first-order commutators, two of which are of a composite type. The building blocks involving no B fields precisely reproduce all physically relevant Feynman diagrams of quantum electrodynamics. Those involving B fields correspond to Feynman diagrams in such a way that the Ward-Takahashi identities are realized in a built-in form, as in (11.34).

Finally, we make a remark on the renormalization constants. Renormalization must be performed in such a way that gauge invariance is not spoiled. Hence $A^\mu$ is not renormalized. Furthermore, because $B$ is a free field, it is not renormalized. What are renormalized are $\psi$ and the parameters $e$ and $m$, only. Thus, the renormalized Lagrangian density is

$$
\mathcal{L} = -Z_3 (4e_R^2)^{-1} F^\mu\nu F_{\mu\nu} + B \partial_\mu A^\mu
+ Z_2 \psi_R (i \gamma^\mu \partial_\mu + \gamma^\mu A^\mu - m_R + \delta m) \psi_R,
$$

where the subscript $R$ indicates a renormalized quantity. In (11.37), the Ward identity $Z_1 = Z_2$ is built in. The renormalization constants $Z_2$, $Z_3$ and $\delta m$, as well as $A^\mu$ and $\psi_R$, should be expanded in powers of $e_R^2$.

Similarly, the renormalization of the Landau-gauge pure Yang-Mills theory becomes as follows. In order not to spoil the nilpotency of the BRS transformation,

---

* In the non-Landau covariant gauge, therefore, the gauge parameter $\alpha$ is not renormalized.
the Yang-Mills field $A^a_\mu$ is not renormalized. Furthermore, because the Landau-gauge condition implies that
\[
\langle A^a_\mu(x_1)B^b(x_2) \rangle = -\delta^{ab}\partial^{x_1}_\mu D^{(+)}(x_1 - x_2),
\] (11-38)
the B field $B^a$ is not renormalized. Hence, the renormalized Lagrangian density is
\[
\mathcal{L} = -Z_3(4gR^2)^{-1}F_\mu^aF_\mu^a + B^a_\mu\partial_\mu A_\mu^a - i\tilde{Z}_3\partial^\mu\tilde{C}_R^a(D_\mu C_R)^a,
\] (11-39)
where $D_\mu$ denotes a covariant derivative.

§12. Quantum Einstein gravity

The BRS-formulated quantum Einstein gravity in the de Donder gauge is an outstandingly beautiful theory.\(^1,2\) There is a very large symmetry constituting the 4d-dimensional Poincaré-like superalgebra, which consists of $4d(2d + 1)$ generators. It contains $d$-dimensional general linear invariance, a tensorial extension of BRS invariance, etc. It is very important to solve the de Donder gauge quantum Einstein gravity in the Heisenberg picture,\(^20\) because, as pointed out in §1, the conventional perturbation theory is not applicable to quantum gravity.

As explained in §9, we should adopt the intrinsic B field $b_\lambda$ as a fundamental field, rather than the conventional B field $B_\lambda = b_\lambda + ie^\sigma\partial_\sigma\bar{c}_\lambda$. The Lagrangian density is given by
\[
\mathcal{L} = (2\kappa)^{-1}\sqrt{-g}R - \tilde{g}^{\mu\nu}(\partial_\mu b_\nu + i\partial_\mu\bar{c}_\lambda \cdot \partial_\nu c^\lambda) + \mathcal{L}_{\text{matter}},
\] (12-1)
where the notation is the same as in §9, though, of course, here all fields are $d$-dimensional quantities.

The field equations derived from (12-1) are as follows:
\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa \left( E_{\mu\nu} - \frac{1}{2}g_{\mu\nu}E - T_{\mu\nu} \right), \quad *)
\] (12-2)
where $E_{\mu\nu}$ and $E$ are defined by (9-4) and $R_{\mu\nu}$ and $T_{\mu\nu}$ denote the Ricci tensor and the matter energy-momentum tensor, respectively;
\[
\partial_\mu\tilde{g}^{\mu\nu} = 0,
\] (12-3)
\[
\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu\Phi) = 0 \quad \text{for} \quad \Phi = b_\lambda, c^\lambda, \bar{c}_\lambda,
\] (12-4)
and matter field equations. Of course, (12-4) for $\Phi = b_\lambda$ has been derived from (12-2) by taking the covariant derivative.

The second-order differential operator appearing in (12-2) is not invertible because of general covariance. In the case of quantum electrodynamics, the non-invertible differential operator $\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu$ appearing in (11-2) is replaced by the invertible one $\eta_{\mu\nu}\partial^2$ by using the gauge-fixing condition (11-3). Also, in quantum

\(^1\text{We call (12-2) the “quantum Einstein equation.”}\)
Einstein gravity, we should carry out a similar procedure. The most convenient way to do this is to replace $R_{\mu\nu}$ by

$$
\tilde{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}[g_{\nu\rho}\partial_\mu + g_{\mu\rho}\partial_\nu + (\partial_\rho g_{\mu\nu})](\sqrt{-g}^{-1}\partial_\lambda g^{\lambda\rho})
\approx \frac{1}{2}g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} - g^{\lambda\sigma}\partial_\lambda g_{\mu\sigma} \cdot \Gamma^\nu_{\nu\sigma}^\tau + \frac{1}{2}g_{\nu\sigma}\partial_\mu g^{\lambda\rho} \cdot \Gamma^\lambda_{\nu\sigma}^\rho, $$

(12.5)

where $\Gamma^\nu_{\nu\sigma}^\tau$ denotes the affine connection.

Canonical quantization is carried out by taking the fundamental fields other than $b_\lambda$ as canonical variables. Then, all equal-time (anti)commutation relations are explicitly obtained in closed form.\(^1\),\(^2\) In particular, we have

$$
[g_{\mu\nu}(x), g_{\sigma\tau}(y)]_0 = 0, \tag{12.6}
$$

which is a canonical commutation relation itself, and

$$
[\partial_0 g_{\mu\nu}(x), g_{\sigma\tau}(y)]_0 = \kappa f(x)\delta^{d-1}(x - y), \tag{12.7}
$$

where $f(x)$ is a certain function of $g_{\mu\nu}(x)$. Although its explicit expression is known, the appearance of the factor $\kappa$ on the right-hand side can be understood simply through dimensional analysis.

Now, we expand all fundamental fields in powers of $\kappa$:

$$
\varphi_A(y) = \varphi_A^{(0)}(y) + \kappa \varphi_A^{(1)}(y) + \kappa^2 \varphi_A^{(2)}(y) + \cdots. \tag{12.8}
$$

For simplicity, hereafter we omit the matter fields. At zeroth order, the field equations (12.2)–(12.4) reduce to

$$
\tilde{R}_{\mu\nu}^{(0)} = 0, \tag{12.9}
$$

$$
\partial_\mu \tilde{g}_{\mu\nu}^{(0)} = 0, \tag{12.10}
$$

$$
\partial_\mu (\tilde{g}_{\mu\nu}^{(0)} \partial_\nu \Phi) = 0 \quad \text{for} \quad \Phi = b_{\lambda}^{(0)}, c^{(0)}_{\lambda}, \bar{c}_{\lambda}^{(0)}. \tag{12.11}
$$

We rewrite these equations into the form of differential equations for the $d$-dimensional (anti)commutators. In particular, for $[g_{\mu\nu}^{(0)}(x), g_{\sigma\tau}^{(0)}(y)]$, we have trivial initial data from (12.6) and (12.7). Therefore, the Cauchy problem obtained from (12.9) implies

$$
[g_{\mu\nu}(x), g_{\sigma\tau}(y)]^{(0)} \equiv [g_{\mu\nu}^{(0)}(x), g_{\sigma\tau}^{(0)}(y)] = 0. \tag{12.12}
$$

All other (anti)commutators are the same as those of the 2-dimensional quantum gravity in the de Donder gauge apart from dimensionality. Thus the zeroth-order operator solution of the quantum Einstein gravity coincides with the exact operator solution of 2-dimensional quantum gravity with $\tilde{b} = 0$ (Because of manifest general linear invariance, this statement is meaningful.) apart from the quantum Einstein-like equation (9.3).

The zeroth-order truncated Wightman functions also coincide with the truncated Wightman functions of 2-dimensional quantum gravity with $\tilde{b} = 0$. However, because of the absence of the quantum Einstein-like equation, we encounter no field-equation
anomaly. This fact is very important for the quantum Einstein gravity to be a fundamental theory free from pathology.

We now proceed to first order. Without the matter fields, the first order of (12.2) is written

\[ \tilde{\mathcal{R}}^{(1)}_{\mu\nu} = E^{(0)}_{\mu\nu}, \tag{12.13} \]

We must rewrite (12.13) into the form of a differential equation for \([g_{\mu\nu}(x), \varphi_A(y)]^{(1)}\). Because \(\varphi_A(y)\) is a field operator at a different spacetime point, there is no reason to expect that \([g_{\mu\nu}(x), \varphi_A(y)]^{(1)}\) commutes with \(g^{(0)}_{\sigma\tau}(x)\). Hence, we cannot replace

\[ [\tilde{R}_{\mu\nu}(x), \varphi_A(y)]^{(1)} = [\tilde{R}^{(1)}_{\mu\nu}(x), \varphi_A^{(0)}(y)] \tag{12.14} \]

by

\[ (\partial \tilde{R}_{\mu\nu}/\partial g_{\sigma\tau})^{(0)}(x)[g_{\sigma\tau}(x), \varphi_A(y)]^{(1)}. \tag{12.15} \]

The mathematical problem that we encounter is as follows: Let \(A\) and \(\Phi\) be operators such that \(A\) does not commute with \([A, \Phi]\). Then for an arbitrary (analytic) function \(f\), express \([f(A), \Phi]\) in terms of \([A, \Phi]\) and \(A\). This problem is solved in the following way.\(^{48}\)\(^{,}\)\(^{,}\)\(^{,}\)\(^{,}\) We introduce two mutually commuting operators, \(A_R\) and \(A_L\), as copies of \(A\). Then, the solution to the problem is given by

\[ [f(A), \Phi] = \frac{f(A_R) - f(A_L)}{A_R - A_L} [A, \Phi], \tag{12.16} \]

where \(A_R\) and \(A_L\) are placed to the right of \([A, \Phi]\) and to the left of it, respectively, and then are identified with \(A\).

In essence, the coefficient differential operator in (12.15) should be written in left-right symmetric form. With this understanding, we may use (12.15).

We here refrain from reporting further analysis that has been made but is yet unpublished. Because of extreme complication, the first-order analysis has to this time not been completed.

Finally, we make a comment on the problem of the unrenormalizability of quantum gravity. Of course, we can say nothing about quantum Einstein gravity concretely, but we can construct a 2-dimensional quantum-gravity model involving a parameter \(\alpha\) which is such that its exact solution (solved using our method) is completely finite but its perturbative expansion with respect to \(\alpha\) is unrenormalizable.\(^{50}\) This fact shows that perturbative unrenormalizability does not necessarily imply any essential problem with the theory.

\[ \S 13. \text{ Discussion} \]

Discussing how to solve quantum field theory is a very important problem. Until the end of the 1940s, covariant perturbation theory together with renormalization theory was quite successfully established. Nevertheless, we must emphasize that it

\(^{49}\) This method can be applied for the purpose of extending the Heaviside-Mikusiński operational calculus for the differential equation to the case in which coefficients do not commute with the unknown function.
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is not perfectly free of pathological properties and that the interaction picture on which it is based is of quite an artificial nature.

Nowadays, the path-integral formalism, which directly reproduces the covariant perturbation theory, has become so popular that many physicists forget about the fundamental importance of the operator formalism. Of course, the path-integral formalism is a very convenient, handy approach, but this fact does not imply that it is more fundamental than the operator formalism. In order to study the most fundamental theoretical problems, we should always return to the operator formalism.

Various analyses in the Heisenberg picture, such as dispersion relations, the analyticity of various field-theoretical quantities, and axiomatic quantum field theory, have been carried out since the 1950s. Also, many exactly solvable models were found. Unfortunately, however, no systematic approach was proposed for solving quantum field theory in the Heisenberg picture until the 1990s.

Since 1991, we have developed a new systematic approach for solving quantum field theory in the Heisenberg picture. We believe that our approach is quite natural. We should emphasize, however, that the natural framework of our approach is indefinite-metric quantum field theory, as the positivity of the metric is quite foreign to our approach. Indeed, as emphasized in §4, a zero-norm or self-commuting field plays a very important role. It is rather difficult to treat such a simple theory as the $\phi^4$ theory in our approach. This is probably one of the main reasons why this approach was not discovered for quite a long time.

Our approach is most effective in quantum gauge theories and quantum gravity. In particular, it is most successful in 2-dimensional quantum gravity. The exact solutions to 2-dimensional quantum gravity in various gauges have provided for us many new insights such as the possible existence of a field-equation anomaly. The exact solutions have been found to be always invariant under the BRS invariance and the FP-ghost number conservation, in contradiction to the widely accepted assertions that there appear a BRS anomaly and an FP-ghost number current anomaly. The reasons for this discrepancy have been resolved: The BRS anomaly arose merely because the existence of a field-equation anomaly was unknown, while the FP-ghost current anomaly is due to the use of a Euclidean metric or the careless treatment of the $T^*$-product in the perturbative approach. We have criticized the well-known derivation of the bosonic string critical dimension $D = 26$ that is based on the BRS nilpotency in the conformal gauge. Our result strongly suggests that if one attempts to construct a BRS-formulated string theory in the de Donder gauge, it cannot be consistently formulated for any particular critical dimension.

For realistic theories, we have employed the method of expansion in powers of the coupling constant squared or the Einstein gravitational constant (without taking square root). This expansion is more natural than the conventional perturbative expansion, because the former is BRS invariant at each order. Unfortunately, however, the calculation that must be performed is much more complicated than that in the conventional perturbation theory. It is an important problem to find how we carry out the calculation more effectively.

In this connection, in analogy to the Feynman diagrammatic method in the interaction picture, we have proposed a new diagrammatic method for directly writing
down the expression for the Wightman functions in the Heisenberg picture. It is quite successful in the de Donder-gauge 2-dimensional quantum gravity. In other models, however, its success is not necessarily satisfactory. It is an important future problem to find the logical basis of this diagrammatic method in a general framework.

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