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Memory Function Approach to Chaos and Turbulence
and the Continued Fraction Expansion

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The chaotic orbits of dynamical systems are deterministic and predictable on short timescales $\tau_r$, but they become stochastic and random on long timescales $\tau_M (\gg \tau_r)$ due to the orbital instability of chaos. This randomization of chaotic orbits has been formulated recently by deriving a non-Markovian stochastic equation for macrovariables in terms of a fluctuating force and a memory function. In order to develop this memory function approach to chaos and turbulence, we explore the following problems by studying the Duffing oscillator and the Navier-Stokes equation for an incompressible fluid: 1) the physical meaning of the projection of macrovariables $A(t)$ onto $A(0)$; 2) the method of calculating the short-lived motion with short timescale $\tau_r$, which determines the memory functions and the macroscopic transport coefficients due to chaos and turbulence; 3) the continued fraction expansion of the memory function, and the order estimation of short timescales $\tau_r$ and long timescales $\tau_M$; 4) the relation between the memory function and the time correlation function of a nonlinear force, which gives computable theoretical expressions for the macroscopic transport coefficients.

§1. Introduction

The chaotic orbits of dynamical systems have positive Liapunov exponents and exhibit sensitivity to the initial conditions and a loss of memory with regard to initial states.1)–3) It is particularly noteworthy that the randomness of chaotic orbits is generated on long timescales by the orbital instability due to the positive Liapunov exponents. Therefore, chaotic orbits of dynamical systems are deterministic and predictable on short timescales, but become stochastic and random on long timescales. Such randomization of chaotic orbits gives rise to the dissipation of macroscopic kinetic energy into the random kinetic energy of chaos or turbulence, leading to macroscopic transport processes, such as the chaos-induced friction of the Duffing oscillator and the turbulent viscosity of incompressible fluids.

The randomization of chaotic orbits on long timescales has been formulated recently4),5) by deriving a non-Markovian stochastic equation for macrovariables in terms of a fluctuating force and a memory function. The method used is the projection operator method developed in the theory of generalized Brownian motion.6),7) The central quantity of this formalism is the memory function, which represents the

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loss of memory of initial states. Hence, this is called the memory function formal-
ism.\textsuperscript{8)}

The randomization of chaotic orbits also indicates that chaotic dynamics consist of two kinds of dynamic processes: one is short-lived motion characterized by a short timescale $\tau_r$ and the other is long-lived motion characterized by a long timescale $\tau_M$. It seems that the memory function and the chaotic or turbulent transport coefficients are usually determined by the short-lived motion with short timescale $\tau_r$, whereas the main features of the time correlation functions of macrovariables are determined by the long-lived motion with long timescale $\tau_M$. The principal purpose of the present paper is to investigate these two kinds of dynamic processes.

The present paper is organized as follows. In $\S$ 2, we explore the physical meaning of the projection of macrovariables $A(t)$ onto $A(0)$, $PA(t)$, and their fluctuations $A'(t) = Q A(t)$. Then, in $\S$ 3, we summarize the physical framework of the memory function approach to chaos and turbulence by investigating the Duffing oscillator and the Navier-Stokes equation for an incompressible fluid. In $\S$ 4, we develop the continued fraction expansion of the memory function, and the order estimation of short timescales $\tau_r$ and long timescales $\tau_M$. Then, in $\S$ 5, we treat a relation between the memory function and the time correlation function of the nonlinear force, which enables us to compute the memory function numerically. Mode coupling approximations are also discussed in that section. Section 6 is devoted to a short summary.

\section*{\S 2. Most probable path and fluctuating forces}

\subsection*{2.1. Time correlation functions}

In order to develop a statistical-mechanical approach to chaos and turbulence of steady systems far from thermal equilibrium, let us take dynamical systems, such as Duffing equations,\textsuperscript{2)} Hénon-Heiles systems,\textsuperscript{9)} and the Navier-Stokes equation for the fluid velocity of an incompressible fluid,\textsuperscript{1)} and study the fluctuations of their macrovariables $A(t) = \{A_l(t)\}$ ($l = 1, 2, \cdots, \delta$) in chaotic or turbulent states. We assume that chaos and turbulence of the systems we investigate are not too weak, so that their largest positive Liapunov exponents are not too small.

Let us denote their state variables in phase space by $X(t) = \{X_j(t)\}$ ($j = 1, 2, \cdots, d$) with $d \geq \delta$.\textsuperscript{5)} Then, the evolution equations of the macrovariables $A(t)$ are given by

$$dA(t)/dt = \dot{A}(t) = \Lambda A(t) \tag{2.1}$$

in terms of the evolution operator

$$\Lambda \equiv \sum_{j=1}^{d} \dot{X}_j \frac{\partial}{\partial X_j}, \tag{2.2}$$

so that we have $A(t) = \exp[t\Lambda] A$.

The chaotic orbits of dynamical systems become stochastic and random on long timescales, so that their macrovariables $A_l(t)$ fluctuate randomly along a chaotic orbit $X(t)$ in phase space. Then, let us define the correlation function between the
fluctuations of $A_l(t+s)$ and $A_m(s)$ by the long-time average along a chaotic orbit $X(s)$ as

$$
\langle A_l(t)A_m^\dagger(0) \rangle \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau ds \ A_l(t+s)A_m^\dagger(s),
$$

where the values $\langle A_l(t) \rangle$ are set to zero. We may assume that the theorem of Birkhoff\(^{10}\) is valid for this long-time average, ensuring that the long-time average (2.3) has a definite value that does not depend on the choice of the initial state $X(0)$ for a given orbit $X(s)$. We may further assume that the long-time average (2.3) takes an identical value for almost all chaotic orbits on the same chaotic attractor.\(^2\)

Time correlation functions like (2.3) have been found to play a central role in the statistical-mechanical approach to the study of transport processes in chaotic and turbulent systems. For example, the power spectrum of $A_l(t)$ which exhibits dynamical structures of chaos can be written

$$
I_l(\omega) \equiv \lim_{\tau \to \infty} \frac{\tau}{2\pi} \left\langle \left| \frac{1}{\tau} \int_0^\tau dt \ A_l(t)e^{-i\omega t} \right|^2 \right\rangle,
$$

$$
= \lim_{\tau \to \infty} \frac{1}{\pi} \int_0^\tau dt \left( 1 - \frac{t}{\tau} \right) C_l(t) \cos(\omega t)
$$

in terms of the time correlation function $C_l(t) \equiv \langle A_l(t)A_l^\dagger(0) \rangle$. This equation leads to the Wiener-Khinchine theorem for the relation between $I_l(\omega)$ and $C_l(t).\(^{11}\)$

Also in physical experiments, such as light scattering and neutron scattering by a chaotic or turbulent steady system, time correlation functions of the form (2.3) are observed, since the interaction between the probe and the system can be considered sufficiently weak so that it does not affect the intrinsic dynamics of the system.\(^8\) Here, the intrinsic dynamics are represented by the time correlation functions (2.3), as is discussed in the next subsection.

### 2.2. Most probable path $PA(t)$ and fluctuations $QA(t)$

In deriving the reciprocal relations in irreversible processes from microscopic reversibility in 1931,\(^{12}\) Onsager indicated that the average decay of the thermal fluctuations of macrovariables $A(t) = \{A_l(t)\}$ obeys the same laws as the corresponding macroscopic irreversible processes and is given by the time correlation functions (2.3). In constructing a statistical-mechanical theory of generalized Brownian motion in 1965,\(^6\) one of the present authors showed that the most probable path of the time evolution of $A(t)$ is given by the relaxation functions of $A(t)$, which become the time correlation functions (2.3) in the classical case.

Thus we find that the most probable path of the time evolution of macrovariables $A(t)$ is given by $\Xi(t)A$ with

$$
\Xi(t) \equiv \langle A(t)A^\dagger \rangle \langle AA^\dagger \rangle^{-1}
$$

in terms of the time correlation functions (2.3), where $A \equiv A(t=0)$, and $A$ and $A^\dagger$ are a column and row matrix, respectively, so that $\langle AA^\dagger \rangle$ is a square matrix $\{\langle A_lA_m^\dagger \rangle\}$. The function (2.6) is called the relaxation function. It is our main goal
to derive a macroscopic evolution equation for this most probable path and to clarify its main features.

Therefore, following the previous work,\(^6\) let us introduce the projection of a quantity \(G\) onto the macrovariables \(A\) as

\[
\mathcal{P}G(X) = \langle G(X)A\rangle \langle AA\rangle^{-1}A,
\]

where the operator \(\mathcal{P}\) satisfies \(\mathcal{P}A = A, \mathcal{P}^2 = \mathcal{P}\), and \(\mathcal{P}Q = Q\mathcal{P} = 0\) for \(Q = 1 - \mathcal{P}\). Then, since \(\mathcal{P}A(t)\) gives \(\Xi(t)A\), the time evolution of \(A(t)\) can be divided into the most probable path \(\mathcal{P}A(t)\) and the fluctuations \(A'(t) \equiv QA(t)\) as

\[
A(t) = \mathcal{P}A(t) + QA(t) = \Xi(t)A + A'(t),
\]

where, since \(\mathcal{P}Q = 0\), the two parts are orthogonal (i.e., \(\langle A'(t)A\rangle = 0\)).

In order to determine the fluctuations \(A'(t)\), let us multiply (2.1) by \(Q\) from the left and insert (2.8) into \(A(t)\) on the right-hand side. We thereby obtain

\[
dA'(t)/dt = QA\{A'(t) + \Xi(t)A\}. \quad (2.9)
\]

Since \(A'(t = 0) = QA = 0\), this is integrated to give

\[
A'(t) = \int_0^t ds \Xi(s)\tilde{R}(t - s), \quad (2.10)
\]

where we have defined the fluctuating force

\[
\tilde{R}(t) \equiv \exp[tQA]Q\dot{A} \quad (2.11)
\]

in terms of the unnatural propagator \(\exp[tQA]\). Since \(\mathcal{P}\tilde{R}(t) = 0\), the fluctuating force (2.11) is orthogonal to the most probable path \(\mathcal{P}A(t)\) of (2.8):

\[
\langle A'(t)A\rangle = \langle \tilde{R}(t)A\rangle = 0. \quad (2.12)
\]

Inserting (2.10) into (2.8) finally leads to

\[
A(t) = \Xi(t)A + \int_0^t ds \Xi(t - s)\tilde{R}(s). \quad (2.13)
\]

The first part of (2.13) represents the most probable path \(\mathcal{P}A(t)\), and the second part consists of the fluctuations \(QA(t)\) around it. The orthogonality of the two parts is ensured by the operator \(Q\) of the unnatural propagator of (2.11).

2.3. Linear stochastic equation for \(A(t)\)

In this subsection, we derive a linear stochastic equation for \(A(t)\) with the fluctuating force (2.11).

Since \(A(t) = e^{tA}A\), the evolution equation (2.1) can be written

\[
dA(t)/dt = e^{tA}\{\mathcal{P} + Q\}\dot{A} = i\tilde{\Omega}A(t) + e^{tA}Q\dot{A}, \quad (2.14)
\]
where we have defined the square matrix

\[ i\tilde{\Omega} \equiv \langle \dot{A} A^\dagger \rangle \langle AA^\dagger \rangle^{-1}, \tag{2.15} \]

which is a frequency matrix if \( \dot{A} \) is invariant under time reversal. Therefore, with the aid of the operator identity

\[ e^{t\Lambda} = e^{tQ\Lambda} + \int_0^t ds \, e^{(t-s)\Lambda} \mathcal{P} \Lambda e^{sQ\Lambda} \] (2.16)

and the fluctuating force (2.11), Eq. (2.14) can be transformed into

\[ \frac{d}{dt} A(t) = i\tilde{\Omega}A(t) - \int_0^t ds \, M(s) A(t-s) + \tilde{R}(t), \tag{2.17} \]

where we have defined the memory function

\[ M(t) \equiv \langle \tilde{R}(t) \tilde{R}^\dagger \rangle \langle AA^\dagger \rangle^{-1} \tag{2.18} \]

using \( \langle \{ A\tilde{R}(t)\} A^\dagger \rangle = -\langle \tilde{R}(t) \tilde{R}^\dagger \rangle \) with \( \tilde{R} = Q\Lambda A \). It should be noted here that the average \( \langle \tilde{R}(t) \tilde{R}^\dagger \rangle \) is the long-time average over a chaotic orbit \( X(s) \) with \( 0 < s < \infty \):

\[ \langle \tilde{R}(t) \tilde{R}^\dagger \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau ds \, \{ e^{tQ\Lambda} e^{s\Lambda} \tilde{R} \} \{ e^{s\Lambda} \tilde{R} \}^\dagger. \tag{2.19} \]

Equation (2.17) is a linear non-Markovian stochastic equation with a fluctuating force \( \tilde{R}(t) \), which has the same form as that derived in a previous paper on generalized Brownian motion.\(^6\)

Since (2.12) gives \( \langle \tilde{R}(t) A^\dagger \rangle = 0 \), Eq. (2.17) leads to a linear equation for \( \Xi(t) \):

\[ \frac{d}{dt} \Xi(t) - i\tilde{\Omega}\Xi(t) + \int_0^t ds \, M(s) \Xi(t-s) = 0. \tag{2.20} \]

The formal solution of this equation can be found by Laplace transform. The result is

\[ \Xi(z) \equiv \int_0^\infty dt \, e^{-zt} \Xi(t) = \frac{1}{z - i\tilde{\Omega} + M(z)}, \tag{2.21} \]

where \( M(z) \) is the Laplace transform of the memory function (2.18).

Thus it turns out that the projection operator method of this section just extracts the linear long-lived motion (2.20) from the evolution equation (2.1).

### 2.4. Short-lived motion of the memory function \( M(t) \)

The time evolution of the memory function (2.18) is governed by the unnatural propagator \( \exp\{tQ\Lambda\} \) of (2.11), where the projection operator \( Q \) excludes the linear long-lived motion (2.20), so that the memory function \( M(t) \) does not include the linear long-lived motion \( \Xi(t) \). Therefore, the fluctuating force (2.11) consists of nonlinear terms of \( A(t) \) and coupling terms of \( A(t) \) with fast molecular variables \( X_m(t) \), both of which usually change much faster than the linear long-lived motion.
This indicates that the memory function $M(t)$ usually consists of short-lived motion with a short timescale $\tau_r$. We therefore write $M(t)$ as

$$M(t) = 2\Gamma \delta(t)$$

(2.22)
on a long timescale $\tau_M (\gg \tau_r)$, where $\delta(t)$ is the $\delta$-function. Then (2.20) becomes Markovian, leading to

$$\Xi(t) = \exp \left[(i\tilde{\Omega} - \Gamma)t\right],$$

(2.23)which represents the linear long-lived motion with long timescale $\tau_M \sim 1/\text{Tr} \Gamma$.

In order to explore the short-lived motion of the memory function $M(t)$ from a different viewpoint, let us consider the nonlinear force

$$\tilde{F}(t) \equiv e^{tA}Q\dot{A} = \tilde{R}(t) - \int_0^t ds M(t-s)A(s),$$

(2.24)where (2.16) has been used. Then, similarly to the memory function (2.18), let us define the time correlation of the nonlinear force $\tilde{F}(t)$ as

$$\tilde{\Phi}(t) \equiv \langle \tilde{F}(t)\tilde{F}^\dagger\rangle\langle AA^\dagger\rangle^{-1}.$$  

(2.25)

Since $\tilde{F}^\dagger = \tilde{R}^\dagger$, inserting (2.24) into (2.25) leads to

$$\tilde{\Phi}(t) = M(t) - \int_0^t ds M(t-s)\langle (A(s)\tilde{R}^\dagger)\langle AA^\dagger\rangle^{-1}\rangle.$$ 

(2.26)Inserting (2.13) into $A(s)$ of (2.26) and using $\langle \tilde{R}^\dagger p \rangle = 0$, we find that the Laplace transform of (2.26) leads to

$$\tilde{\Phi}(z) = M(z) - M(z)\Xi(z)M(z).$$

(2.27)This is the relation between $\tilde{\Phi}(t)$ and $M(t)$ that is noted in a previous paper. The second term on the right-hand side of (2.27) contains the linear long-lived motion $\Xi(t)$.

As a simple example, let us consider Brownian motion of a massive particle in an equilibrium fluid and consider the time evolution of its momentum $p(t)$ in the $x$-direction. Then, we have $A(t) = p(t)$,

$$\mathcal{P}G(X) = \{\langle G(X)p \rangle/\langle p^2 \rangle\}p,$$

(2.28)and $i\tilde{\Omega} = 0$, so that Eq. (2.17) takes the form

$$\frac{d}{dt}p(t) = -\int_0^t ds M(s)p(t-s) + \tilde{R}(t)$$

(2.29)with $\langle \tilde{R}(t)p \rangle = 0$. The theory of Brownian motion assumes that the memory function $M(t)$ has the $\delta$-function form (2.22) on long timescale $\tau_M \sim 1/\Gamma$. Then, (2.29) becomes Markovian, leading to

$$\Xi(t) = \langle p(t)p \rangle/\langle p^2 \rangle = \exp[-\Gamma t].$$

(2.30)
Since (2.22) gives $M(z) = \Gamma$ and (2.30) leads to $\Xi(z) = 1/(z + \Gamma)$, Eq. (2.27) takes the form

$$\tilde{\Phi}(z) = \Gamma - \frac{\Gamma^2}{z + \Gamma}, \quad (2.31)$$

which is equivalent to

$$\tilde{\Phi}(t) = 2\Gamma\delta(t) - \Gamma^2 \exp[-\Gamma t]. \quad (2.32)$$

The second term in (2.32) is the long-time tail with long timescale $\tau_M \sim 1/\Gamma$.

Thus it turns out that the memory function $M(t)$ usually represents short-lived motion with a short timescale $\tau_r$, whereas the force correlation $\tilde{\Phi}(t)$ consists of both short-lived motion with a short timescale $\tau_r$ and long-lived motion with a long timescale $\tau_M (\gg \tau_r)$.

§3. Memory function approach to chaos and turbulence

3.1. Two examples of dynamical systems

In this section, we extend the stochastic equation (2.17) so as to be useful for dealing with chaos and turbulence of steady dynamical systems far from thermal equilibrium.

A) Duffing oscillator

The first example of such dynamical systems is a chaotic Duffing oscillator, whose coordinate $q$ and velocity $p$ obey

$$\left( \begin{array}{c} \dot{q} \\ \dot{p} \end{array} \right) = \left( \begin{array}{cc} p \\ -\gamma^0 p \\ v(q) \\ b \cos(\omega_0 t + \phi_0) \end{array} \right), \quad (3.1)$$

with $v(q) = -q^3 + q$, where $\gamma^0$ is the molecular friction coefficient, and $b$ is the amplitude of an external force with phase $\phi = \omega_0 t + \phi_0$. Then we have $X = \{q, p, \phi\}$, so that the phase space is three dimensional, and the evolution operator (2.2) takes the form

$$\Lambda = p \frac{\partial}{\partial q} + \{-\gamma^0 p + v(q) + b \cos \phi\} \frac{\partial}{\partial p} + \omega_0 \frac{\partial}{\partial \phi}. \quad (3.2)$$

The nonlinear part of $v(q)$ causes chaos for $\gamma^0 = 0.25$, $\omega_0 = 1$ and $b = 0.4$. In this case, the numerical computation of the long-time averages $\langle G \rangle$ gives

$$\langle q \rangle = \langle p \rangle = \langle qp \rangle = 0, \quad \langle q^2 \rangle = 0.774, \quad \langle p^2 \rangle = 0.302, \quad \Omega_0^2 = -\langle v(q)q \rangle/\langle q^2 \rangle = 0.316. \quad (3.3a, 3.3b)$$

In accordance with Eq. (3.1), however, we take $A = \{q, p\}$. Then the projection operator (2.7) takes the form

$$PG = \{\langle Gq \rangle/\langle q^2 \rangle\}q + \{\langle Gp \rangle/\langle p^2 \rangle\}p. \quad (3.4)$$

B) Navier-Stokes equation

The second example is the Navier-Stokes equation for an incompressible fluid, which is written in terms of the Fourier components $u = \{u_{\alpha k}\}$ of the fluid velocity $\{u_{\alpha}(r)\}$ with wavenumbers $0 < k < k_c (\sim 10^3 \text{cm}^{-1})$. Let us assume that...
turbulence develops in an inertial subrange \( k_L < k < k_L \) with \( k_L \ll k_c \). Then the Navier-Stokes equation takes the form

\[
du_{\alpha k}/dt = V^0_{\alpha k}(u) + V_{\alpha k}(u) + K_{\alpha k}.
\]

Here \( V_{\alpha k}(u) \) is the nonlinear inertial term responsible for turbulence and is given by

\[
V_{\alpha k}(u) = \sum_{\beta, \gamma} \sum_p' V_{\alpha \beta \gamma}(k) u_{\beta p} u_{\gamma k-p},
\]

with \( \Delta_{\alpha \gamma}(k) \equiv \delta_{\alpha \gamma} - (k_\alpha k_\gamma/k^2) \), where \( \sum_p' \) is the sum over wavevectors \( p \) satisfying \( k_L < p < k_c \). \( K_{\alpha k} \) is an external force that acts on the components \( u_{\alpha k} \) in the small wavenumber range \( k < k_L \), and \( V^0_{\alpha k}(u) \) is given by

\[
V^0_{\alpha k}(u) = v^0_{\alpha k}(u) - \nu^0 k^2 u_{\alpha k},
\]

(3.8)

where \( \nu^0 \) is the kinematic molecular viscosity and \( \sum_p'' \) is the sum over wavevectors \( p \) satisfying \( 0 < p < k_L \). Thus, \( v^0_{\alpha k}(u) \) is the nonlinear inertial term in the small wavenumber range \( k < k_L \), and the second term of (3.8) is the molecular viscosity term.

Then we have \( X = A = \{u_{\alpha k}\} \) with \( 0 < k < k_c \), so that the evolution operator (2.2) takes the form

\[
A = \sum_\alpha \sum_k \left\{ v^0_{\alpha k}(u) - \nu^0 k^2 u_{\alpha k} + V_{\alpha k}(u) + K_{\alpha k} \right\} \frac{\partial}{\partial u_{\alpha k}}.
\]

(3.10)

It is now assumed that statistically homogeneous and isotropic turbulence develops due to the nonlinear inertial term (3.6) in the inertial subrange \( k_L < k < k_L \), so that the turbulent fluctuations of \( u_{\alpha k} \) satisfy \( \langle u_{\alpha k} \rangle = 0 \) and

\[
\langle u_{\alpha k} u_{\beta l}^\dagger \rangle = \langle |u_{\alpha k}|^2 \rangle \delta_{\alpha \beta} \delta_{kl}.
\]

(3.11)

Then the projection operator (2.7) takes the form

\[
\mathcal{P} G = \sum_\alpha \sum_k \{ \langle G u_{\alpha k}^\dagger \rangle / \langle |u_{\alpha k}|^2 \rangle \} u_{\alpha k},
\]

(3.12)

where \( \sum_k \) is the sum over all wavevectors \( k \) with \( 0 < k < k_c \).

3.2. Stochastic equations for chaos and turbulence

The above examples suggest that the evolution equations of macrovariables \( A(t) = \{A_l(t)\} \) in dynamical systems consist of three types of terms as

\[
dA(t)/dt = V^0(A(t)) + V(X(t)) + K(X(t)),
\]

(3.13)
where $V(X)$ is the nonlinear term that causes the chaos or turbulence of interest, representing the nonlinear force $V(X) = \{0,v(q)\}$ of Eq. (3.1) and the nonlinear inertial term $V(X) = \{V_{ak}(u)\}$ of Eq. (3.5), $K(X)$ is a periodic external force with frequency $\omega_0$, which supplies macroscopic energy and compensates for its dissipation, and $V^0(A)$ is the remaining term that is a nonlinear function of $A$ in many cases, representing the first term of Eq. (3.1) and the first term $V_{ak}^0(u)$ of Eq. (3.5).

The nonlinear term $V(X)$ may have more than two types of nonlinear terms that cause more than two types of chaos:

$$V(X) = V^{(1)}(X) + V^{(2)}(X) + \cdots. \quad (3.14)$$

If there exists a term $V'(X)$ that is not certain to be included in $V^0(A)$ and $K(X)$, then that term is included in this $V(X)$.

Similarly to Eq. (2.14), the second term of Eq. (3.13) can be written

$$V(X(t)) = e^{tA}\{P + Q\}V(X) = i\Omega A(t) + e^{tA}QV(X), \quad (3.15)$$

where we have defined the square matrix

$$i\Omega \equiv \langle V(X)A^\dagger\rangle(AA^\dagger)^{-1}. \quad (3.16)$$

The second term of (3.15) is the nonlinear part of $V(X(t))$. With the aid of the operator identity (2.16), this nonlinear force can be written

$$F(t) \equiv e^{tA}QV(X) = R(t) - \int_0^t ds \Gamma(s)A(t-s), \quad (3.17)$$

where the fluctuating force $R(t)$ and the memory function $\Gamma(s)$ have been defined, similarly to (2.11) and (2.18), as

$$R(t) \equiv \exp[tQA]QV(X), \quad (3.18)$$

$$\Gamma(t) \equiv \langle R(t)\tilde{R}^\dagger\rangle(AA^\dagger)^{-1}. \quad (3.19)$$

Inserting Eq. (3.17) into the last term of (3.15) and then substituting (3.15) into Eq. (3.13), we obtain

$$\frac{d}{dt}A(t) - V^0(A(t)) - i\Omega A(t) + \int_0^t ds \Gamma(s)A(t-s) = R(t) + K(X(t)), \quad (3.20)$$

where $\langle R(t)A^\dagger \rangle = 0$. This is a non-Markovian stochastic equation characterized by the memory function $\Gamma(t)$, and it is useful for dealing with chaos and turbulence.\(^5\),\(^13\),\(^14\)

As discussed in §2.4, the projection operator $Q$ of the unnatural propagator $e^{tQA}$ of $R(t)$ excludes the motion linear in $A(t)$ from the motion of $F(t)$, and produces a linear transport term, just as the second term in Eq. (3.17). Therefore, this transport term represents the renormalization by the nonlinear force $F(t)$. The fluctuating force $R(t)$ and the memory function $\Gamma(t)$ usually consist of short-lived motion with a short timescale $\tau_r$. Therefore, we can set $\Gamma(t) = 2\Gamma'\delta(t)$ on the long timescale
\( \tau_M (\gg \tau_r) \), and the memory term of (3.20) becomes a transport term \( \Gamma' A(t) \), leading to the Markovian stochastic equation

\[
dA(t)/dt - V^0(A(t)) - i\Omega A(t) + \Gamma' A(t) = R(t) + K(X(t)),
\]

where \( \Gamma' = \Gamma(i\omega = 0) \) is the \( \omega = 0 \) limit of the frequency-dependent transport coefficient

\[
\Gamma(i\omega) \equiv \int_0^\infty dt \, e^{-i\omega t} \Gamma(t) = \int_0^\infty dt \, e^{-i\omega t} \langle R(t)\tilde{R}^\dagger \rangle \langle AA^\dagger \rangle^{-1}.
\]

This transport term \( \Gamma' A(t) \) induces linear long-lived motion whose characteristic time is the long timescale \( \tau_M \sim 1/\text{Tr} \, \Gamma' \) with \( \tau_M \gg \tau_r \).

The most important difference between Eqs. (3.20) and (3.21) is the time-reversal property of the memory term. Namely, the memory term of (3.20) is reversible with respect to time, whereas the transport term \( \Gamma' A(t) \) of Eq. (3.21) is irreversible, leading to entropy production given by

\[
\bar{S} = k_B \sum_l \left| \frac{\partial A_l}{\partial A_i} \right| = k_B \text{Tr} \{ \Gamma^0 + \Gamma' \},
\]

if \( V^0(A) \) has the form \( V^0(A) = \{ i\Omega^0 - \Gamma^0 \} A \) with the molecular transport coefficient \( \Gamma^0 \). This macroscopic transport coefficient \( \Gamma' \) brings about the dissipation of the macroscopic kinetic energy supplied by the external force \( K(X) \) into the random kinetic energy of the chaotic or turbulent motion. It should be noted here that \( \bar{S}/k_B \) given by (3.23) is identically the contraction rate of the A-space volume.\(^4\)

### 3.3. Two examples of the stochastic equation (3.20)

#### A) Duffing oscillator

For the Duffing oscillator (3.1), we have \( A = \{ q, p \}, V(X) = \{ 0, v(q) \} \) with \( \langle v(q)p \rangle = 0 \), and \( R(t) = \{ 0, r(t) \} \), so that (3.16) and (3.19) take the forms

\[
i\Omega = \begin{pmatrix} 0 & -\Omega_0^2 \\ \Omega_0^2 & 0 \end{pmatrix} \quad \Gamma(t) = \begin{pmatrix} 0 & 0 \\ 0 & \gamma(t) \end{pmatrix},
\]

where \( \Omega_0^2 \) is given by (3.3b), and \( \gamma(t), r(t) \) and \( \tilde{r} \) are defined by

\[
\gamma(t) = \langle r(t)\tilde{r} \rangle / \langle p^2 \rangle,
\]

\[
r(t) \equiv \exp[t\Omega A]Qv(q),
\]

\[
\tilde{r} \equiv \Omega \hat{p} = Qv(q) + bQ \cos \phi_0
\]

with \( \langle r(t)q \rangle = \langle r(t)p \rangle = \langle \tilde{r}q \rangle = \langle \tilde{r}p \rangle = 0 \). Then, since \( V^0(A) = \{ p, -\gamma^0 p \} \), (3.20) reduces to

\[
\frac{d}{dt} p + \gamma^0 p + \Omega_0^2 q + \int_0^t ds \gamma(s)p(t-s) = r(t) + b \cos(\omega_0 t + \phi_0).
\]

This is a linear non-Markovian stochastic equation with fluctuating force \( r(t) \).
We assume that the memory function (3.25) decays to zero in a short time \( \tau_r \), so that we may set \( \gamma(t) = 2\gamma'(t) \) on the long timescale \( \tau_M \gg \tau_r \), where \( \tau_M \) is the decay time of the time correlation \( \langle p(t)p \rangle \). Then the memory term of Eq. (3.28) becomes a friction term \( \gamma \), leading to the linear Markovian stochastic equation

\[
dp/dt + \Omega_0^2 q + \{\gamma_0 + \gamma'\}p = r(t) + b\cos(\omega_0 t + \phi_0)
\]  

(3.29)
on the long timescale \( \tau_M \simeq 2/(\gamma_0 + \gamma') \), where \( \gamma' = \gamma(i\omega = 0) \) with the frequency-dependent chaos-induced friction coefficient

\[
\gamma(i\omega) \equiv \int_0^\infty dt e^{-i\omega t} \gamma(t) = \frac{1}{2} \int_0^\infty dt e^{-i\omega t} \langle r(t)r(t) \rangle.
\]  

(3.30)

This friction coefficient brings about the dissipation of the kinetic energy of the oscillator supplied by the external force into the random kinetic energy of the chaotic motion.

B) Navier-Stokes equation

For the Navier-Stokes equation (3.5), we assume that statistically homogeneous and isotropic turbulence develops in the inertial subrange \( k_L < k < k_I \), so that there is no direct correlation between different components of \( u = \{u_{ak}\} \) and \( R = \{r_{ak}\} \), as shown in (3.11). Then, since the inertial term \( (3.6) \) gives \( V(X) = \{V_{ak}(u)\} \), we have

\[
\Gamma_{ak,\beta l}(t) = \gamma_{ak}(t)\delta_{\alpha\beta}\delta_{kl},
\]  

(3.31)

\[
\gamma_{ak}(t) \equiv \langle r_{ak}(t) r_{ak}^\dagger \rangle/\langle |u_{ak}|^2 \rangle
\]  

(3.32)
in terms of the turbulent fluctuating force,

\[
r_{ak}(t) \equiv \exp[tQA]QV_{ak}(u).
\]  

(3.33)

We should have \( i\Omega = 0 \), because there is no collective oscillation for incompressible fluids. Then Eq. (3.20) reduces to the non-Markovian stochastic equation

\[
\frac{d}{dt} u_{ak} - v^0_{ak}(u) + \nu^0 k^2 u_{ak} + \int_0^t ds \gamma_{ak}(s)u_{ak}(t-s) = r_{ak}(t) + K_{ak},
\]  

(3.34)

where \( \langle r_{ak}(t) u_{ak}^\dagger \rangle = 0 \).

If the memory function (3.32) decays to zero in a short time \( \tau_r \) that is much shorter than the decay time \( \tau_M \) of the time correlation \( \langle u_{ak}(t) u_{ak}^\dagger \rangle \), then the memory term of Eq. (3.34) becomes a viscosity term \( \nu'(k) k^2 u_{ak} \), leading to the Markovian stochastic equation

\[
\frac{d}{dt} u_{ak} - v^0_{ak}(u) + \{\nu^0 + \nu'(k)\} k^2 u_{ak} = r_{ak}(t) + K_{ak}
\]  

(3.35)
on the long timescale \( \tau_M \simeq 1/\{\nu^0 + \nu'(k)\} k^2 (\gg \tau_r) \), where \( \nu'(k) = \nu'(k, \omega = 0) \) with the frequency-dependent turbulent viscosity

\[
\nu'(k, \omega) \equiv \frac{1}{k^2} \int_0^\infty dt e^{-i\omega t} \gamma_{ak}(t) = \frac{1}{k^2 \langle |u_{ak}|^2 \rangle} \int_0^\infty dt e^{-i\omega t} \langle r_{ak}(t) r_{ak}^\dagger \rangle.
\]  

(3.36)

This turbulent viscosity brings about the dissipation of the macroscopic kinetic energy supplied by the external force \( K_{ak} \) into the random kinetic energy of turbulence in the inertial subrange.
3.4. Long-lived motion of the memory functions

As discussed in §2.4, the memory functions \( M(t) \) and \( \Gamma(t) \) do not include the linear long-lived motion with long timescale \( \tau_M \). The fluctuating forces (2.11) and (3.18), however, contain nonlinear combinations of the slow variables \( A \), such as \( AA \), \( AAA \) etc., as in the cases of (3.26) and (3.33), where \( \langle R(t)\{AA\}^{\dagger}\rangle \neq 0, \langle R(t)\{AAA\}^{\dagger}\rangle \neq 0 \).

In highly-correlated systems, such as thermodynamic systems in the critical region of continuous phase transitions, these nonlinear terms of the fluctuating forces can bring about long-lived processes with long timescales, and therefore the memory function consists of two parts, a short-lived part and a long-lived part:

\[
\Gamma(t) = \Gamma_0(t) + \Gamma_1(t),
\]

where \( \Gamma_0(t) \) denotes the contributions associated with short-lived motion characterized by a short timescale \( \tau_r \), and \( \Gamma_1(t) \) is the part due to a long-lived motion characterized by a long timescale \( \tau_M (\gg \tau_r) \).

It seems, however, that in many cases the long-lived part \( \Gamma_1(t) \) can be ignored in comparison with the short-lived part \( \Gamma_0(t) \), in which case the Markovian stochastic equations (3.21), (3.29) and (3.35) are valid.

§4. Continued fraction expansion of the memory functions

It has turned out that the memory function (3.19) is the central quantity for formulating the randomization of chaotic orbits, and it usually consists of short-lived motion with a short timescale \( \tau_r \), as discussed in §§2.4 and 3.4. Equations (3.25) and (3.32) provide two examples of the memory functions \( \gamma(t) \), which have the form

\[
\gamma(t) = \langle r(t)\tilde{r}^{\dagger}/\langle aa^{\dagger}\rangle \rangle
\]

in terms of the fluctuating force

\[
r(t) = \exp[tA_1]r
\]

with \( A_1 \equiv QA \). For the Duffing oscillator (3.1), we have \( r = Qv(q) \), \( \tilde{r} = Q\dot{p} \) and \( a = p \), and for the Navier-Stokes equation (3.5), we have \( r = \tilde{r} = r_{\alpha k} = QV_{\alpha k}(u) \) and \( a = u_{\alpha k} \).

In order to explore the decay of the memory functions on short timescales, let us construct a continued fraction expansion of the Laplace transform of \( \gamma(t) \).

4.1. Linear evolution equation for \( \gamma(t) \)

In analogy to the projection (2.7), let us introduce the projection of a quantity \( G \) onto the fluctuating force \( r \),

\[
P_1G(X) = \{(G(X)\tilde{r}^{\dagger})/\langle r\tilde{r}^{\dagger}\rangle\}r,
\]

and define \( Q_1 \equiv 1 - P_1 \) with \( P_1^2 = P_1, P_1Q_1 = Q_1P_1 = 0 \). Then, taking the time derivative of (4.2) and proceeding analogously to (3.15) and (3.17), we obtain

\[
dr(t)/dt = e^{tA_1}\{P_1 + Q_1\}A_1r,
\]
where we have defined $i\Omega_1 \equiv \langle \{A_1 r \} \tilde{r}^\dagger \rangle/\langle r \tilde{r}^\dagger \rangle$,

$$
\gamma_1(t) \equiv \langle r_1(t) \tilde{r}_1^\dagger \rangle/\langle r \tilde{r}^\dagger \rangle,
$$

(4.6)

$$
r_1(t) \equiv \exp[tQ_1 A_1]Q_1 A_1 r,
$$

(4.7)

and $\tilde{r}_1 \equiv Q_1 A_1 \tilde{r}$. Equation (4.5) is a linear stochastic equation of the same type as Eq. (2.17).

Since $P_1 Q_1 = 0$, we have $\langle r_1(t) \tilde{r}_1^\dagger \rangle = 0$, so that Eq. (4.5) leads to a linear evolution equation for the memory function $\gamma(t)$:

$$
\frac{d}{dt} \gamma(t) = i\Omega_1 \gamma(t) - \int_0^t ds \gamma_1(s) \gamma(t - s).
$$

(4.8)

Taking its Laplace transform leads to

$$
\gamma(z) \equiv \int_0^\infty dt e^{-zt} \gamma(t) = \frac{\Delta_0^2}{z - i\Omega_1 + \gamma_1(z)},
$$

(4.9)

where $\gamma_1(z)$ is the Laplace transform of (4.6), and we have defined

$$
\Delta_0^2 \equiv \gamma(t = 0) = \frac{\langle r \tilde{r}^\dagger \rangle}{\langle a a^\dagger \rangle}.
$$

(4.10)

4.2. Continued fraction expansion of $\gamma(z)$

Similarly to $\gamma(z)$ in (4.9), $\gamma_1(z)$ can be written in a fractional form by introducing $\gamma_2(z)$. Thus, for $j = 1, 2, \cdots$, let us successively introduce the projection onto the $(j - 1)$-th fluctuating force $r_{j-1}$,

$$
P_j G(X) = \{\langle G(X) \tilde{r}_{j-1}^\dagger \rangle/\langle r_{j-1} \tilde{r}_{j-1}^\dagger \rangle\} r_{j-1},
$$

(4.11)

where $r_0 = r$, $\tilde{r}_0 = \tilde{r}$. The time evolution of the $(j - 1)$-th fluctuating force is given by

$$
r_{j-1}(t) = \exp[tA_j] r_{j-1}
$$

(4.12)

in terms of the $j$-th evolution operator

$$
A_j \equiv Q_{j-1} A_{j-1} = \{1 - P_{j-1} - \cdots - P_1 - P_0\} A,
$$

(4.13)

where $Q_{j-1} = 1 - P_{j-1}$, $P_0 = P$, $A_0 = A$.

Therefore, taking the time derivative of (4.12) and proceeding as in the cases of (4.4) and (4.5), we obtain

$$
dr_{j-1}(t)/dt = e^{tA_j} \{P_j + Q_j\} A_j r_{j-1},
$$

(4.14)

$$
= i\Omega_j r_{j-1}(t) - \int_0^t ds \gamma_j(s) r_{j-1}(t - s) + r_j(t),
$$

(4.15)
where we have defined
\[ i\Omega_j \equiv \langle \{\Lambda_j r_{j-1}\} \hat{r}_{j-1}^{\dagger} / \langle r_{j-1} \hat{r}_{j-1}^{\dagger} \rangle, \quad (4.16) \]
\[ \gamma_j(t) \equiv \langle r_j(t) \hat{r}_{j-1}^{\dagger} / \langle r_{j-1} \hat{r}_{j-1}^{\dagger} \rangle, \quad (4.17) \]
\[ r_j(t) \equiv \exp[tQ_j \Lambda_j]Q_j \Lambda_j r_{j-1}, \quad (4.18) \]
and \( \hat{r}_j \equiv Q_j \Lambda_j \hat{r}_{j-1} \). Equation (4.15) is just a generalization of (4.5).
Since \( P_j Q_j = 0 \), we have \( \langle r_j(t) \hat{r}_{j-1}^{\dagger} \rangle = 0 \), and therefore (4.15) leads to
\[ \frac{d}{dt} \gamma_{j-1}(t) = i\Omega_j \gamma_{j-1}(t) - \int_0^t ds \gamma_j(s) \gamma_{j-1}(t - s), \quad (4.19) \]
whose Laplace transform yields
\[ \gamma_{j-1}(z) \equiv \int_0^\infty dt e^{-zt} \gamma_{j-1}(t) = \frac{\Delta_{j-1}^2}{z - i\Omega_j + \gamma_j(z)}, \quad (4.20) \]
where \( \gamma_j(z) \) is the Laplace transform of (4.17) and we have defined
\[ \Delta_{j-1}^2 \equiv \gamma_{j-1}(t = 0) = \langle r_{j-1} \hat{r}_{j-1}^{\dagger} / \langle r_{j-2} \hat{r}_{j-2}^{\dagger} \rangle, \quad (4.21) \]
with the aid of \( r_{-1} = \hat{r}_{-1} = a \).
Using (4.20) successively, therefore, we obtain the continued fraction expansion of (4.9):
\[ \gamma(z) = \frac{\Delta_0^2}{z - i\Omega_1 + \frac{\Delta_1^2}{z - i\Omega_2 + \frac{\Delta_2^2}{\cdots + \frac{\Delta_{n-1}^2}{z - i\Omega_n + \gamma_n(z)}}}}, \quad (4.22) \]
where, in addition to (4.10), we have defined
\[ \Delta_1^2 = \langle r_1 \hat{r}_1^{\dagger} \rangle / \langle r \hat{r}^{\dagger} \rangle, \quad \Delta_2^2 = \langle r_2 \hat{r}_2^{\dagger} \rangle / \langle r_1 \hat{r}_1^{\dagger} \rangle, \quad (4.23) \]
in terms of \( r_j = Q_j \Lambda_j r_{j-1}, \hat{r}_j = Q_j \Lambda_j \hat{r}_{j-1} \). This is consistent with the continued fraction expansion of the time correlation function \( \Xi(z) \) derived previously.\(^7\)

4.3. Short timescale \( \tau_r \) and friction coefficient \( \gamma' \)

The memory functions \( \gamma(t) \) describe the loss of memory of the initial states due to orbital instability, and therefore it would seem that they decay to zero in a short timescale \( \tau_r \). If the decay time \( \tau_r \) is much shorter than the macroscopic decay time \( \tau_M \) of the time correlation \( \Xi(t) \), such as \( \langle p(t)p \rangle \) and \( \langle u_{ak}(t)u_{ak}^{\dagger} \rangle \), then the Markovian stochastic equations (3.29) and (3.35) hold for \( t \gtrsim \tau_M \). In this case we have
\[ \tau_M \simeq 2 / \{\gamma^0 + \gamma'\}, \quad 1 / \{\nu^0 + \nu'(k)\} k^2, \quad (4.24) \]
Then, in many cases we have \( \gamma_0 \ll \gamma', \nu_0 \ll \nu'(k) \), so that

\[
\tau_r \ll \tau_M \simeq 2/\gamma', \quad 1/\nu'(k)k^2.
\] (4.25)

In order to determine the magnitudes of \( \tau_r \) and \( \tau_M \) in terms of \( \Delta_0 \) and \( \Delta_1 \), in this subsection, we investigate approximations of the continued fraction expansion (4.22).

1) Exponential decay law for \( \gamma(t) \)

A simple approximation of the decay of the memory functions (4.1) is obtained by assuming \( \gamma_1(z) \simeq \Delta_1 \) for \( |z| \lesssim \Delta_1 \) in (4.9):

\[
\gamma(z) \simeq \frac{\Delta_0^2}{z + \Delta_1},
\] (4.26)

where we have assumed \( i\Omega_1 = 0 \). This is equivalent to the exponential decay law

\[
\gamma(t) \simeq \Delta_0^2 \exp[-\Delta_1 t]
\] (4.27)

for \( t \gtrsim 1/\Delta_1 \), which is an extension of the \( \delta \)-function law \( \gamma(t) = 2\gamma'\delta(t) \) used in (3.29). \(^8\) Then the decay time \( \tau_r \) of the memory functions is given by

\[
\tau_r \sim \frac{1}{\Delta_1} = \left\{ \frac{\langle r\tilde{r}\rangle}{\langle \tilde{r}_1\tilde{r}_1 \rangle} \right\}^{1/2}.
\] (4.28)

Hence, if (4.25) holds, then (3.30) and (4.27) lead to

\[
\frac{2}{\tau_M} \sim \gamma' = \int_0^\infty dt \gamma(t) \simeq \frac{\Delta_0^2}{\Delta_1}.
\] (4.29)

Equations (4.28) and (4.29) thus lead to

\[
\frac{\tau_r}{\tau_M} \sim \frac{\Delta_0^2}{\Delta_1^2} = \frac{\{\langle r\tilde{r}\rangle\}^2}{\langle \tilde{r}_1\tilde{r}_1\rangle},
\] (4.30)

where (4.10) has been used.

2) Gaussian decay law for \( \gamma(t) \)

If the coefficients of the continued fraction expansion (4.21) satisfy the simple relation

\[
\Delta_n^2 = n\Delta_1^2 \quad \text{for} \quad n = 1, 2, \cdots, \infty,
\] (4.31)

then Eq. (4.22) leads to a Gaussian decay law \(^8\),15),16)

\[
\gamma(t) = \Delta_0^2 \exp[-\Delta_1^2 t^2/2],
\] (4.32)

where we have assumed \( i\Omega_n = 0 \). The main advantage of this decay law is its consistency with the exact short time expansion \( \gamma(t)/\Delta_0^2 = 1 - \Delta_1^2 t^2/2 + \cdots \), while preserving the decay of \( \gamma(t) \) for \( t \to \infty \).
The Gaussian decay law (4.32) leads to
\[
\tau_r \sim 1/\Delta_1, \quad (4.33)
\]
\[
\frac{2}{\tau_M} \sim \gamma' = \int_0^\infty dt \gamma(t) = \sqrt{\frac{\pi}{2}} \frac{\Delta_0^2}{\Delta_1}. \quad (4.34)
\]
These results for \(\tau_r\) and \(\tau_M\) are consistent with (4.28) and (4.29). Therefore, we also obtain (4.30). The results of the numerical calculation of (4.28) and (4.30) will be reported elsewhere.

4.4. Short timescale \(\tau_{rk}\) and turbulent viscosity \(\nu'(k)\)

Let us assume that the Markovian stochastic equation (3.35) holds with \(\tau_{rk} \ll \tau_{MK} \simeq 1/\nu'(k)k^2\), where \(\nu'(k) \gg \nu_0\). Then, applying the exponential decay law (4.27) to the turbulent memory function (3.32), we obtain
\[
\gamma_{\alpha k}(t) \simeq \Delta_0^2 \Delta_1 k^2 \exp[-\Delta_1 k t] \quad (4.35)
\]
with the short timescale
\[
\tau_{rk} \sim \frac{1}{\Delta_1 k} = \left\{ \frac{\langle |r_{\alpha k}|^2 \rangle}{\langle |r_{1\alpha k}|^2 \rangle} \right\}^{1/2}. \quad (4.36)
\]
Then the turbulent viscosity (3.36) leads to
\[
\nu'(k)k^2 = \int_0^\infty dt \gamma_{\alpha k}(t) \simeq \frac{\Delta_0^2}{\Delta_1 k} = \frac{\{\langle |r_{\alpha k}|^2 \rangle\}^{3/2}}{\langle |u_{\alpha k}|^2 \rangle \{\langle |r_{1\alpha k}|^2 \rangle\}^{1/2}}, \quad (4.37)
\]
where \(r_{\alpha k}\) and \(r_{1\alpha k}\) are given by
\[
r_{\alpha k} = Q_0 V_{\alpha k}(u) = V_{\alpha k}(u), \quad (4.38)
\]
\[
r_{1\alpha k} = Q_1 A_1 r_{\alpha k} = (1 - P_1 - P_0) A V_{\alpha k}(u), \quad (4.39)
\]
\[
= AV_{\alpha k}(u) - \left[ \frac{\langle AV_{\alpha k}(u) \rangle}{\langle |r_{\alpha k}|^2 \rangle} \right] r_{\alpha k}
- \left[ \frac{\langle AV_{\alpha k}(u) \rangle}{\langle |u_{\alpha k}|^2 \rangle} \right] u_{\alpha k}, \quad (4.40)
\]
with \(V_{\alpha k}(u)\) being the inertial term (3.6).

Since \(\tau_{MK} \simeq 1/\nu'(k)k^2\), (4.36) and (4.37) lead to
\[
\frac{\tau_{rk}}{\tau_{MK}} \sim \frac{\Delta_0^2}{\Delta_1 k} \frac{\{\langle |r_{\alpha k}|^2 \rangle\}^2}{\langle |u_{\alpha k}|^2 \rangle \langle |r_{1\alpha k}|^2 \rangle}. \quad (4.41)
\]

§5. Time correlations \(\phi(t)\) of nonlinear forces \(f(t)\)

The fluctuating force \(R(t)\) is related to the nonlinear force \(F(t)\) by Eq. (3.17). Hence, similarly to the memory function (3.19), let us define the force correlation
\[
\phi(t) \equiv \langle F(t) \tilde{R}^\dagger \rangle \langle AA^\dagger \rangle^{-1}. \quad (5.1)
\]
Then, as shown in a previous paper,\textsuperscript{5}) the Laplace transform $\Gamma(z)$ is related to $\Phi(z)$ by

$$\frac{1}{\Gamma(z)} = \frac{1}{\Phi(z)} - \frac{1}{z - i(\Omega + \Omega^0) + \Gamma^0} \left\{ 1 + \mathcal{M}(z) \frac{1}{\Phi(z)} \right\}, \quad (5.2)$$

if $V_0(A) = \{i\Omega - \Gamma^0\}A$ for the evolution equation (3.13). Here, $\mathcal{M}(z)$ is the Laplace transform of

$$\mathcal{M}(t) \equiv \langle K(X(t))\tilde{R}^\dagger \rangle \langle AA^\dagger \rangle^{-1}. \quad (5.3)$$

The time evolution of the memory function $\Gamma(t)$ is governed by the unnatural propagator $\exp\left[tQ\Lambda(t)\right]$ of (3.18) and hence cannot be computed numerically. Equation (5.2), however, makes it possible to determine the memory function $\Gamma(z)$ numerically by computing $\Phi(z)$ and $M(z)$ numerically.

In this section, we discuss how to treat Eq. (5.2) for the Duffing oscillator (3.1) and the Navier-Stokes equation (3.5).

5.1. Duffing oscillator (3.1)

Let us consider the second equation of (3.1). Similarly to the nonlinear part of (3.15), the nonlinear force $v(q)$ can be written

$$v(q(t)) = e^{tA}\{P + Q\}v(q) = -\Omega_0^2q(t) + f(t), \quad (5.4)$$

where $\Omega_0^2$ is given by (3.3b), and according to (3.17) we have defined the nonlinear force

$$f(t) \equiv e^{tA}Qv(q) = r(t) - \int_0^t ds \gamma(s)p(t - s) \quad (5.5)$$

with the fluctuating force $r(t)$ and the memory function $\gamma(t)$ defined by (3.25) and (3.26). Then, inserting (5.4) into the $v(q)$ term of (3.1), we obtain

$$dp(t)/dt + \gamma^0 p(t) + \Omega_0^2q(t) = f(t) + b\cos(\omega_0t + \phi_0). \quad (5.6)$$

Inserting (5.5) into (5.6) leads to the linear non-Markovian stochastic equation (3.28) with fluctuating force $r(t)$.

In analogy to the memory function (3.25), let us introduce the force correlation

$$\phi(t) \equiv \langle f(t)\tilde{r} \rangle / \langle p^2 \rangle = \gamma(t) - \int_0^t ds \gamma(s) \langle p(t - s)\tilde{r} \rangle / \langle p^2 \rangle, \quad (5.7)$$

whose Laplace transform takes the form

$$\phi(z) \equiv \int_0^\infty dt \exp(-zt) \phi(t) = \gamma(z) - \gamma(z) \langle p(z)\tilde{r} \rangle / \langle p^2 \rangle. \quad (5.8)$$

Since $\langle p\tilde{r} \rangle = \langle q\tilde{r} \rangle = 0$, we have

$$\langle \dot{p}(z)\tilde{r} \rangle = \int_0^\infty dt \left[ \frac{d}{dt} \langle e^{-zt}(p(t)\tilde{r}) \rangle + z e^{-zt} \langle p(t)\tilde{r} \rangle \right] = z \langle p(z)\tilde{r} \rangle, \quad (5.9a)$$

$$\langle q(z)\tilde{r} \rangle = \frac{1}{z} \langle \dot{p}(z)\tilde{r} \rangle = \frac{1}{z} \langle p(z)\tilde{r} \rangle, \quad (5.9b)$$
so that the Laplace transform of (5.6) leads to
\[
\langle p(z)\tilde{r} \rangle = \frac{1}{z + \gamma^0 + (\Omega_0^2/z)} \{ \langle f(z)\tilde{r} \rangle + \langle p^2 \rangle \mu(z) \},
\]
where we have defined
\[
\mu(z) \equiv \frac{b}{\langle p^2 \rangle} \int_0^\infty dt e^{-zt} \langle \tilde{r} \cos(\omega_0t + \phi_0) \rangle.
\]
Therefore, inserting (5.10) into (5.8) leads to
\[
\phi(z) = \gamma(z) - \gamma(z) \frac{z}{z^2 + \gamma^0 z + \Omega_0^2} \{ \phi(z) + \mu(z) \},
\]
which can be rewritten as
\[
\frac{1}{\gamma(z)} = \frac{1}{\phi(z)} - \frac{z}{z^2 + \gamma^0 z + \Omega_0^2} \left\{ 1 + \frac{\mu(z)}{\phi(z)} \right\}.
\]
This leads to \( \gamma(z) = \phi(z) \) for \( |z| \to \infty \). Equation (5.13) gives an interesting relation between the memory function \( \gamma(z) \) and the force correlation \( \phi(z) \). It seems that the \( \mu(z) \) term in (5.13) cancels out the forced oscillation term of \( 1/\phi(z) \).

It should be noted that (5.13) can also be derived from (5.2) directly for the Duffing oscillator (3.1), where \( A = \{q,p\} \), \( R(t) = \{0,r(t)\} \) and \( F(t) = \{0,f(t)\} \). Then we have
\[
\Gamma_{lm}(z) = \gamma(z)\delta_{l2}\delta_{m2}, \quad \Phi_{lm}(z) = \phi(z)\delta_{l2}\delta_{m2}
\]
and \( \mathcal{M}_{lm}(z) = \mu(z)\delta_{l2}\delta_{m2} \), and thus Eq. (5.2) reduces to Eq. (5.13).

The memory function \( \gamma(z) \) can be determined from (5.13) by computing the force correlation \( \phi(t) = \langle f(t)\tilde{r} \rangle/\langle p^2 \rangle \) numerically, where, using
\[
f(t) \equiv e^{tA}Qv(q) = -q^3(t) + \{\Omega_0^2 + 1\}q(t)
\]
and (3.27) for \( \tilde{r} \) with \( \Omega_0^2 + 1 = \langle q^4 \rangle/\langle q^2 \rangle \), we have
\[
\phi(t) = \frac{1}{\langle p^2 \rangle} \left[ \langle q^3(t)q^3 \rangle - \frac{\langle q^4 \rangle}{\langle q^2 \rangle} \left\{ \langle q^3(t)q \rangle + \langle q(t)q^3 \rangle - \frac{\langle q^4 \rangle}{\langle q^2 \rangle} \langle q(t)q \rangle \right\} \right.
\]
\[
\left. \quad -b(q^3(t)Q \cos \phi_0) + b\frac{\langle q^4 \rangle}{\langle q^2 \rangle} \langle q(t)Q \cos \phi_0 \rangle \right].
\]

The external force term (5.11) can be written
\[
\mu(z) = \frac{B_3}{\langle p^2 \rangle} \frac{z \cos \phi_3 - \omega_0 \sin \phi_3}{z^2 + \omega_0^2},
\]
where \( B_3 \) and \( \phi_3 \) are defined by \( B_3 \equiv b\sqrt{\zeta_3^2 + \eta_3^2} \),
\[
\zeta_3 \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau ds \{ e^{sA}\tilde{r} \} \cos(\omega_0s + \phi_0) = \sqrt{\zeta_3^2 + \eta_3^2} \cos \phi_3,
\]
\[
\eta_3 \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau ds \{ e^{sA}\tilde{r} \} \sin(\omega_0s + \phi_0) = \sqrt{\zeta_3^2 + \eta_3^2} \sin \phi_3.
\]
Equation (3.30) indicates that the friction coefficient \( \gamma' \) is given by
\[
\gamma' = \lim_{z \to 0^+} \gamma(z).
\] (5.19)

This limit of Eq. (5.13), however, depends on the magnitude of the natural frequency \( \Omega_0 \) in the manner described below.

\( \alpha \) Case of \( \Omega_0 \neq 0 \)
Here, in the limit \( z \to 0^+ \), the second term on the right-hand side of (5.13) vanishes, so that we have
\[
\frac{1}{\gamma'} = \frac{1}{\phi(z = 0)} = 1 \left\langle p^2 \right\rangle \int_0^\infty dt \left\langle f(t) \tilde{r} \right\rangle,
\] (5.20)
which is more useful than (3.30). Hence, the friction coefficient \( \gamma' \) is obtained by integrating (5.16).

\( \beta \) Case of \( \Omega_0 = 0 \)
Here, in the limit \( z \to 0^+ \), we have
\[
\frac{1}{\gamma'} = \frac{1}{\phi(z = 0)} \left\{ 1 + \frac{B_3 \sin \phi_3}{\left\langle p^2 \right\rangle \omega \gamma_0^0} \right\} - \frac{1}{\gamma_0^0}.
\] (5.21)

Finally, it should be noted that the memory function \( \gamma(z) \) and the friction coefficient \( \gamma' \) can be computed numerically by using (5.13), (5.20) and (5.21).

5.2. Navier-Stokes equation (3.5)
Let us investigate how to compute the memory function \( \gamma_{\alpha k}(t) \) of (3.32) and the turbulent viscosity \( \nu'(k, \omega) \) of (3.36) numerically by relating the memory function \( \gamma_{\alpha k}(t) \) to the time correlation function \( \phi_{\alpha k}(t) \) of the nonlinear inertial force
\[
f_{\alpha k}(t) \equiv e^{tA} QV_{\alpha k}(u) = r_{\alpha k}(t) - \int_0^t ds \gamma_{\alpha k}(s) u_{\alpha k}(t - s),
\] (5.22)
where \( r_{\alpha k}(t) \) is the turbulent fluctuating force (3.33).

Now, let us assume that the nonlinear inertial term \( v_{\alpha k}^0(u) \) of (3.8) also produce turbulence in the small wavenumber range \( k < k_L \) whose fluctuations are statistically homogeneous and isotropic, so that the projection operator also takes the form (3.12). Then, in correspondence to the three types of terms in (3.13), Eq. (3.5) can be written
\[
du_{\alpha k}/dt = -\nu^0 k^2 u_{\alpha k} + \hat{V}_{\alpha k}(u) + K_{\alpha k}
\] (5.23)
with \( \hat{V}_{\alpha k}(u) \equiv v_{\alpha k}^0(u) + V_{\alpha k}(u) \). Since \( \mathcal{P} \hat{V}_{\alpha k}(u) \simeq 0 \) for incompressible fluids, we have
\[
du_{\alpha k}(t)/dt + \nu^0 k^2 u_{\alpha k}(t) = \hat{f}_{\alpha k}(t) + K_{\alpha k}
\] (5.24)
in terms of the nonlinear inertial force
\[
\hat{f}_{\alpha k}(t) \equiv e^{tA} Q\hat{V}_{\alpha k}(u) = \hat{r}_{\alpha k}(t) - \int_0^t ds \hat{\gamma}_{\alpha k}(s) u_{\alpha k}(t - s),
\] (5.25)
where the fluctuating force (3.33) and the memory function (3.32) have been extended as
\[
\hat{r}_{\alpha k}(t) \equiv \exp[tQA] Q\hat{V}_{\alpha k}(u) = r_{\alpha k}^0(t) + r_{\alpha k}(t),
\] (5.26)
\[
\hat{\gamma}_{\alpha k}(t) \equiv \langle \hat{r}_{\alpha k}(t) \hat{r}_{\alpha k}^\dagger \rangle / \left\langle |u_{\alpha k}|^2 \right\rangle,
\] (5.27)
where \( r^0_{\alpha k}(t) \equiv \exp[tQ_A]Q_0^0u_{\alpha k}(u) \) and we have \( \langle r^0_{\alpha k}(t)u^\dagger_{\alpha k} \rangle = 0 \). Inserting (5.25) into (5.24) leads to the non-Markovian stochastic equation

\[
\frac{d}{dt}u_{\alpha k}(t) + \nu^0k^2u_{\alpha k}(t) + \int_0^t ds \hat{\gamma}_{\alpha k}(s)u_{\alpha k}(t-s) = \hat{r}_{\alpha k}(t) + K_{\alpha k}
\]

(5.28)

with \( \langle \hat{r}_{\alpha k}(t)u^\dagger_{\alpha k} \rangle = 0 \), which is a generalization of Eq. (3.34).

In analogy to the memory function (5.27), let us introduce the force correlation

\[
\hat{\phi}_{\alpha k}(t) \equiv \frac{\langle f_{\alpha k}(t)f^\dagger_{\alpha k} \rangle}{\langle |u_{\alpha k}|^2 \rangle} = \hat{\gamma}_{\alpha k}(t) - \int_0^t ds \hat{\gamma}_{\alpha k}(s)\langle u_{\alpha k}(t-s)f^\dagger_{\alpha k} \rangle \langle |u_{\alpha k}|^2 \rangle
\]

(5.29)

with \( f^\dagger_{\alpha k} = \hat{r}^\dagger_{\alpha k} \). Taking the Laplace transform of (5.24), we obtain

\[
\langle u_{\alpha k}(z) \rangle f^\dagger_{\alpha k} = \frac{1}{z + \nu^0k^2} \langle f_{\alpha k}(z)f^\dagger_{\alpha k} \rangle,
\]

(5.30)

using \( \langle K_{\alpha k}f^\dagger_{\alpha k} \rangle = 0 \). Therefore, Eq. (5.29) leads to

\[
\hat{\phi}_{\alpha k}(z) = \hat{\gamma}_{\alpha k}(z) - \hat{\gamma}_{\alpha k}(z)\frac{1}{z + \nu^0k^2} \hat{\phi}_{\alpha k}(z),
\]

(5.31)

which can be rewritten as

\[
\frac{1}{\hat{\gamma}_{\alpha k}(z)} = \frac{1}{\hat{\phi}_{\alpha k}(z)} - \frac{1}{z + \nu^0k^2}.
\]

(5.32)

Next, let us assume that the two types of turbulence produced by \( v^0_{\alpha k}(u) \) and \( V_{\alpha k}(u) \) are sufficiently separated in wavenumber space, so that there are no correlations between \( r^0_{\alpha k}(t) \) and \( r_{\alpha k}(t) \) in (5.26): \( \langle r^0_{\alpha k}(t)r^\dagger_{\alpha k} \rangle = 0 \), \( \langle r_{\alpha k}(t)r^\dagger_{\alpha k} \rangle = 0 \). Then (5.27) leads to

\[
\hat{\gamma}_{\alpha k}(t) = \gamma^0_{\alpha k}(t) + \gamma_{\alpha k}(t)
\]

(5.33)

with \( \gamma^0_{\alpha k}(t) \equiv \langle r^0_{\alpha k}(t)r^\dagger_{\alpha k} \rangle / \langle |u_{\alpha k}|^2 \rangle \), where \( \gamma_{\alpha k}(t) \) is given by (3.32). Therefore, the inertial force (5.25) takes the form

\[
\hat{f}_{\alpha k}(t) = f^0_{\alpha k}(t) + f_{\alpha k}(t),
\]

(5.34)

where the second term \( f_{\alpha k}(t) \) is given by (5.22), and we have defined

\[
f^0_{\alpha k}(t) \equiv e^{tA}Qv^0_{\alpha k}(u) = r^0_{\alpha k}(t) - \int_0^t ds \gamma^0_{\alpha k}(s)u_{\alpha k}(t-s).
\]

(5.35)

It should be noted, however, that in contrast to (5.33), the force correlation \( \hat{\phi}_{\alpha k}(t) \) cannot be written as the sum of contributions of \( f^0_{\alpha k}(t) \) and \( f_{\alpha k}(t) \), since there exist correlations between them.

In the case of Eq. (3.34), it was assumed that, the inertial term \( v^0_{\alpha k}(u) \) produces no turbulence. Then \( \gamma^0_{\alpha k}(t) = 0 \) and \( r^0_{\alpha k}(t) = f^0_{\alpha k}(t) \), so that (5.33) leads to \( \hat{\gamma}_{\alpha k}(t) = \gamma_{\alpha k}(t) \) and \( \hat{\phi}_{\alpha k}(t) = \phi_{\alpha k}(t) \) with

\[
\phi_{\alpha k}(t) \equiv \langle f_{\alpha k}(t)f^\dagger_{\alpha k} \rangle / \langle |u_{\alpha k}|^2 \rangle.
\]

(5.36)
Then Eq. (5.32) reduces to
\[ \frac{1}{\gamma_{\alpha k}(z)} = \frac{1}{\phi_{\alpha k}(z)} - \frac{1}{z + \nu^0 k^2}. \] (5.37)

This leads to \( \gamma_{\alpha k}(z) = \phi_{\alpha k}(z) \) for \( |z| \to \infty \). It should be noted here that the second term of (5.37) results from the second term of (5.22), i.e., the renormalization of the linear transport term by the nonlinear inertial force \( f_{\alpha k}(t) \).

Equation (5.37) gives the memory function \( \gamma_{\alpha k}(z) \) in terms of the force correlation \( \phi_{\alpha k}(z) \). Since \( \mathcal{P}V_{\alpha k}(u) \simeq 0 \) for incompressible fluids, the inertial force (3.6) leads to
\[ f_{\alpha k}(t) = V_{\alpha k}(u(t)) = \sum_{\beta, \gamma} \sum_{p} V_{\alpha \beta \gamma}(k) u_{\beta p}(t) u_{\gamma k-p}(t). \] (5.38)

Inserting this into \( f_{\alpha k}(t) \) of (5.36) gives
\[ \phi_{\alpha k}(t) = \frac{2}{\langle |u_{\alpha k}|^2 \rangle} \sum_{\beta, \gamma} \sum_{p} \langle |V_{\alpha \beta \gamma}(k)|^2 \rangle (u_{\beta p}(t) u_{\gamma k-p}(t) u_{\beta p} u_{\gamma k-p}). \] (5.39)

Here, the time evolution of \( u_{\beta p}(t) \) is the natural motion \( e^{i\lambda t} \). In the next subsection, we discuss an approximation of (5.39).

Equation (3.35) indicates that the turbulent viscosity \( \nu'(k) \) is given by
\[ \nu'(k) k^2 = \gamma_{\alpha k}(z = 0) = \lim_{z \to 0^+} \gamma_{\alpha k}(z), \] (5.40)
so that Eq. (5.37) leads to
\[ \frac{1}{\nu'(k)} = \frac{k^2}{\phi_{\alpha k}(z = 0)} - \frac{1}{\nu^0}. \] (5.41)

This gives an interesting relation between the turbulent viscosity \( \nu'(k) \) and the force correlation \( \phi_{\alpha k}(z = 0) \).

The frequency-dependent turbulent viscosity is given by (3.36), i.e.,
\[ \nu'(k, \omega) \equiv \gamma_{\alpha k}(z = i\omega) k^2. \] (5.42a)

Indeed, since \( \langle \nu_{\alpha k}^0(u) u_{\alpha k}^\dagger \rangle \simeq 0 \) for incompressible fluids and \( \langle r_{\alpha k}(t) u_{\alpha k}^\dagger \rangle = \langle K_{\alpha k} u_{\alpha k}^\dagger \rangle = 0 \), the Laplace transform of the relaxation function \( \xi_{\alpha k}(t) \) of \( u_{\alpha k}(t) \) takes the form
\[ \xi_{\alpha k}(i\omega) \equiv \int_0^\infty dt e^{-i\omega t} \langle u_{\alpha k}(t) u_{\alpha k}^\dagger \rangle / \langle |u_{\alpha k}|^2 \rangle = \frac{1}{z + \nu^0 + \nu'(k, \omega)} k^2. \] (5.42b)

The turbulent viscosity \( \nu'(k, \omega) \) should be singular at \( k = 0 \) and \( \omega = 0 \), so that the order of limits \( k \to 0 \) and \( \omega \to 0 \) is important. Specifically, we must take \( k \to 0 \) first and \( \omega \to 0 \) second. Then, for \( z = i\omega \neq 0 \), Eq. (5.37) leads to
\[ \nu'(\omega) \equiv \lim_{k \to 0} \nu'(k, \omega) = \lim_{k \to 0} \frac{\phi_{\alpha k}(i\omega)}{k^2}, \] (5.43a)
\[ = \lim_{k \to 0} \frac{1}{k^2 \langle |u_{\alpha k}|^2 \rangle} \int_0^\infty dt e^{-i\omega t} \langle f_{\alpha k}(t) f_{\alpha k}^\dagger \rangle, \] (5.43b)
where \( f_{\alpha k}(t) \) is the inertial force (5.38). The limit
\[
\nu' \equiv \lim_{\omega \to 0} \nu'(\omega) = \lim_{\omega \to 0} \lim_{k \to 0} \frac{1}{k^2|u_{\alpha k}|^2} \int_0^\infty dt e^{-i\omega t} \langle f_{\alpha k}(t)f_{\alpha k}^\dagger \rangle
\]
gives the turbulent viscosity for large-scale flow with small wavenumbers \( k \ll k_L \). It should be noted that this viscosity \( \nu' \) cannot be obtained from \( \nu'(k) \) of (3.35) and (5.41).

Finally, it should be noted that the memory function \( \gamma_{\alpha k}(z) \) and the turbulent viscosity \( \nu'(k, \omega) \) can be computed numerically by using Eq. (5.37). This equation is also useful for clarifying the structure of the turbulent viscosity \( \nu'(k, \omega) \) around its singularity at \( k = \omega = 0 \). For this reason, we introduce the mode-coupling approximations.

5.3. Mode-coupling approximations

In order to simplify the force correlation functions (5.16) and (5.39), let us introduce the mode-coupling approximation that consists of factorizing the many-variable quantities into all possible products of two-variable correlation functions,\(^8\) such as
\[
\langle q(t)q^3 \rangle \simeq 3\langle q^2 \rangle \langle q(t)q \rangle, \quad (5.45a)
\]
\[
\langle u_{\alpha p}(t)u_{\alpha q}(t)u_{\alpha p}^\dagger u_{\alpha q}^\dagger \rangle \simeq \{\delta_{pp}\delta_{qq} + \delta_{pq}\delta_{qp}\}\langle u_{\alpha p}(t)u_{\alpha p}^\dagger \rangle \langle u_{\alpha q}(t)u_{\alpha q}^\dagger \rangle. \quad (5.45b)
\]

A) Force correlation \( \phi(t) \) of the Duffing oscillator (3.1)

In order to simplify (5.16), let us introduce the mode-coupling approximations
\[
\langle q^3(t)q^3 \rangle \simeq 3\langle q(t)q \rangle \langle q^2(t)q^2 \rangle \simeq 6\langle q(t)q \rangle^3, \quad (5.46a)
\]
\[
\langle q^4 \rangle \simeq 3\langle q^2 \rangle^2, \quad (5.46b)
\]
\[
\langle q^3(t)q \rangle \simeq 3\langle q^2 \rangle \langle q(t)q \rangle, \quad (5.46c)
\]
\[
\langle q^3(t)Q\cos \phi_0 \rangle \simeq 3\langle q^2 \rangle \langle q(t)Q\cos \phi_0 \rangle \quad (5.46d)
\]
in addition to (5.45a). Then, the force correlation (5.16) reduces to
\[
\phi(t) \simeq \frac{3}{(p^2)} \left[ 2\{\langle q(t)q \rangle\}^3 - 3\{\langle q^2 \rangle\}^2 \langle q(t)q \rangle \right], \quad (5.47)
\]
where the last two terms of (5.16) have been cancelled out under the approximations (5.46b) and (5.46d). Inserting the time evolution of the time correlation \( \langle q(t)q \rangle \) into (5.47), we obtain two parts
\[
\phi(t) = \phi^S(t) + \phi^L(t), \quad (5.48)
\]
where \( \phi^S(t) \) is the short-lived part with short timescale \( \tau_r \), and \( \phi^L(t) \) is the long-lived part with long timescale \( \tau_M \gg \tau_r \).

Let us consider the time correlation \( C_q(t) \equiv \langle q(t)q \rangle \). Since \( p = \dot{q} \) and \( \langle r(t)q \rangle = 0 \), Eq. (3.28) leads to
\[
\ddot{C}_q(t) + \gamma^0 \dot{C}_q(t) + \Omega_0^2 C_q(t) + \int_0^t ds \gamma(s)\dot{C}_q(t-s) = B_1 \cos(\omega_0 t + \phi_1), \quad (5.49)
\]
where $B_1$ and $\phi_1$ are defined by $B_1 \equiv b\sqrt{\zeta_1^2 + \eta_1^2}$,

$$
\zeta_1 \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau ds q(s) \cos(\omega_0 s + \phi_0) = \sqrt{\zeta_1^2 + \eta_1^2} \cos \phi_1,  
$$

(5.50a)

$$
\eta_1 \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau ds q(s) \sin(\omega_0 s + \phi_0) = \sqrt{\zeta_1^2 + \eta_1^2} \sin \phi_1.  
$$

(5.50b)

As discussed in (4.27) and (4.32), the memory function $\gamma(t)$ decays to zero in a short timescale $\tau_r$. Therefore the memory term of (5.49) becomes a friction term $\gamma' q(t)$. This leads to

$$
\dot{C}_q(t) + \{\gamma^0 + \gamma'\} \dot{C}_q(t) + \Omega_0^2 C_q(t) = B_1 \cos(\omega_0 t + \phi_1)  
$$

(5.51)

for $t \gg \tau_r$. Therefore, the general solution of (5.51) leads to

$$
C_q(t) = Ae^{-\gamma t/2} \cos(\omega_1 t + \alpha) + \frac{B_1 \cos(\omega_0 t + \phi_1 + \delta)}{\sqrt{(\omega_0^2 - \Omega_0^2)^2 + (\gamma_0)^2}}  
$$

(5.52)

for $t \gg \tau_r$, where $\gamma \equiv \gamma_0 + \gamma'$, $\omega_1 \equiv \sqrt{\Omega_0^2 - (\gamma/2)^2}$ and $\delta \equiv \tan^{-1}[\gamma_0/(\omega_0^2 - \Omega_0^2)]$, and $A$ and $\alpha$ are constants.

Inserting (5.52) into $q(t)q$ of (5.47) gives a theoretical approximation to the long-lived part $\phi^L(t)$ of (5.48). Then inserting (5.47) into (5.20) and (5.21) leads to equations from which we can determine the friction coefficient $\gamma'$.

B) Force correlation $\phi_{ak}(t)$ of the Navier-Stokes equation (3.5)

Introducing the mode-coupling approximation (5.45b), we rewrite (5.39) as

$$
\phi_{ak}(t) \approx \frac{2}{\langle |u_{ak}|^2 \rangle} \sum_{\beta,\gamma} \sum_p |V_{\alpha\beta\gamma}(k)|^2 \langle u_{\beta p}(t) u_{\beta p}^\dagger \rangle \langle u_{\gamma k-p}(t) u_{\gamma k-p}^\dagger \rangle.  
$$

(5.53)

The time evolution of $u_{ak}(t)$ is governed by Eq. (3.35) on long timescales $t \gg \tau_{rk}$. Since $\langle v_{0 k}^\dagger(u) u_{ak} \rangle \simeq 0$, $\langle r_{ak}(t) u_{ak}^\dagger \rangle = \langle K_{ak} u_{ak}^\dagger \rangle = 0$, therefore, the time correlation of $u_{ak}(t)$ takes the form

$$
\langle u_{ak}(t) u_{ak}^\dagger \rangle = \langle |u_{ak}|^2 \rangle \exp[-\gamma k t]  
$$

(5.54)

on long timescales $t \gg \tau_{rk}$, where $\gamma_k \equiv \{\nu^0 + \nu'(k)\}k^2$.

Similarly to (5.48), the force correlation $\phi_{ak}(t)$ consists of two parts, a short-lived part $\phi_{ak}^S(t)$ and a long-lived part $\phi_{ak}^L(t)$. This leads to

$$
\phi_{ak}(z) = \int_0^\infty dt e^{-zt} \phi_{ak}(t) = \phi_{ak}^S(z) + \phi_{ak}^L(z),  
$$

(5.55)

where $\phi_{ak}^S(z)$ is the time integral from $t = 0$ to $t = \tau_{rk}$, and $\phi_{ak}^L(z)$ is the time integral from $t = \tau_{rk}$ to $t = \infty$. The long-lived part $\phi_{ak}^L(z)$ can be obtained by inserting (5.54) into (5.53) and integrating from $t = \tau_{rk}$ to $t = \infty$:  

$$
\phi_{ak}^L(z) \simeq \frac{2}{\langle |u_{ak}|^2 \rangle} \sum_{\beta,\gamma} \sum_p |V_{\alpha\beta\gamma}(k)|^2 \langle |u_{\beta p}|^2 \rangle \langle |u_{\gamma k-p}|^2 \rangle \frac{z + \gamma\gamma + \gamma k-p}{z + \gamma + \gamma k-p}.  
$$

(5.56)
Here we have $\tau_r k \ll \tau_M k \simeq 1/\gamma_k$ and $\Re(\zeta) \lesssim 1/\tau_M k$.

If the force correlation $\phi_{\alpha k}(z)$ of Eq. (5.41) is approximated by this $\phi_{\alpha k}^L(z)$, then we have

$$1/\nu'(k) + 1/\nu^0 \simeq \frac{k^2 \langle |u_{\alpha k}|^2 \rangle}{2 \sum_{\beta,\gamma} \sum_p \langle |V_{\alpha \beta \gamma}(k)|^2 \rangle / \{ \nu^0 + \nu'(p) \} p^2 + \{ \nu^0 + \nu'(|k-p|) \} |k-p|^2}$$

which is the equation for determining the turbulent viscosity $\nu'(k)$. If we ignore the $1/\nu^0$ term, then (5.57) reduces to $\nu'(k) k^2 \simeq \phi_{\alpha k}^L(z = 0)$ which agrees with the result of the DIA for $\nu'(k)$ approximately.\(^{17}\) It is, however, difficult to ignore the $1/\nu^0$ term, and for this reason, our result (5.57) is different from that of the DIA. This implies that the renormalization by the nonlinear inertial force in the DIA theory is incomplete.

If $\phi_{\alpha k}(i\omega)$ of Eq. (5.43a) is approximated by $\phi_{\alpha k}^L(z = i\omega)$ of (5.56) and the limit $\omega \to 0$ is taken, then we obtain a theoretical expression for the turbulent viscosity (5.44) of the large-scale flow.

§6. Short summary

It turns out that the non-Markovian stochastic equations (2.17) and (3.20), characterized by the memory functions $M(t)$ and $\Gamma(t)$, are useful for formulating the randomization of chaotic orbits and describing macroscopic transport processes due to chaos and turbulence, as shown by (3.30), (3.36) and (5.44). The essential quantities of this memory function formalism are the fluctuating forces (2.11) and (3.18), whose time evolution is given by the unnatural propagator $\exp[tQ\Lambda]$. Here, the projection operator $Q$ excludes the linear long-lived motion (2.20), so that the fluctuating forces consist of nonlinear terms of $A(t)$ and coupling terms of $A(t)$ with fast molecular variables $X_m(t)$, both of which usually change much faster than the linear long-lived motion (2.20). Therefore, it seems that the time evolution of the memory functions (2.18) and (3.19) usually consists of short-lived motion with a short timescale $\tau_r (\ll \tau_M)$, as long as chaos and turbulence under investigation are not too weak.

It has been shown that the memory functions (4.1) can be investigated by use of the continued fraction expansion (4.22), which leads to Eqs. (4.28) and (4.36) for the short timescale $\tau_r$. Equations (4.29) and (4.37) give the transport coefficients $\gamma'$ and $\nu'(k)$ in terms of the expansion coefficients $\Delta_0$ and $\Delta_1$. It has also been shown that there exists a relation between the memory function $\gamma(t)$ and the time correlation function $\phi(t)$ of the nonlinear force $f(t)$, such as (5.13) and (5.37), which enables us to determine the memory function $\gamma(t)$ numerically by computing the force correlation $\phi(t)$. Such a memory function approach to chaos and turbulence has been recently applied to the Duffing oscillator and Hénon-Heiles systems. The numerical results of these studies for the memory function $\gamma(t)$, the macroscopic transport coefficient $\gamma'$ and the time correlation functions $\langle A_l(t)A_{l,n}^\dagger \rangle$ will be reported elsewhere.\(^{18}\),\(^{19}\)

Finally, it should be noted that there are two traditional types of approaches
to nonequilibrium statistical mechanics. One is that of the kinetic theory begun by Boltzmann (1872) and the dynamical systems theory begun by Poincaré (1899), and the other is that of the method of statistical ensembles begun by Gibbs (1902).\textsuperscript{20} The theory of reciprocal relations in irreversible processes constructed by Onsager (1931) belongs to the first type, whereas the linear response theory of Kubo (1957) belongs to the second type because it is based on the Liouville equation for a statistical ensemble. The present theory of chaos and turbulence belongs to the first type because it is based on dynamical systems theory and deals with the time evolution of macrovariables $A(t)$ and their most probable values $\langle A(t)A^\dagger \rangle$.

According to the Sinai-Ruelle-Bowen theorem of dynamical systems theory,\textsuperscript{2} there exists a unique measure $\mu(X)$ for a hyperbolic attractor $A$ such that, for almost all initial states $X(0) \in A$, the long time average (2.3) can be written\textsuperscript{2}:

$$\langle A_l(t)A_m^\dagger(0) \rangle = \int_A A_l(t, X)A_m^\dagger(0, X) \, d\mu(X). \quad (6.1)$$

Physical systems like the forced damped pendulum and the Duffing oscillator, however, are not hyperbolic, but have tangency structure where the unstable manifold $W^u$ is tangent to the stable manifold $W^s$.\textsuperscript{3} The strange attractor of such a system has a self-similar fractal structure of non-integer dimension, so that the probability measure $\mu(X)$ is singular (i.e., it does not have a density). It was recently shown that such a strange attractor is characterized by the singularity spectrum $f(\alpha)$ and the expansion rate spectrum $\psi(\Lambda)$.\textsuperscript{3,21}–\textsuperscript{23} Even for such a strange attractor, however, it would be reasonable to assume that there exists a unique probability measure $\mu(X)$ that gives the long time average (6.1). Therefore, it would be interesting to find a relation between the probability measure $\mu(X)$ and the stochastic equations (3.20) and (3.21) with a fluctuating force $R(t)$.

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