Oscillatory Orbits in the Standard Mapping

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(Received February 4, 2004)

Oscillatory motion is unbounded but returns infinitely many times to finite positions. The existence of this motion has been predicted and proved to exist in the three-body problem. In the present paper, we prove its existence in the standard mapping, using orbits of the accelerator mode and non-Birkhoff periodic orbits. This motion appears as soon as the last Kolmogorov-Arnold-Moser curve disintegrates.

§1. Introduction

Chazy introduced oscillatory solutions of the gravitational three-body problem as a logical consequence in the classification of final motions. Sitnikov proved the existence of this type of motion in the three-dimensional restricted three-body problem with masses \( m_1 = m_2 \neq 0 \) and \( m_3 = 0 \). The demonstration of this existence was named the Sitnikov problem. Below we summarize Sitnikov’s work.

Sitnikov considered the situation in which two bodies with equal mass move in elliptic orbits in the \( x-y \) plane. The center of mass is located at the origin. The third body moves along the \( z \) axis. Sitnikov proved that there is a set of initial conditions whose orbits survive the first ejection after the first syzygy (the line connecting two primaries) crossing (that is, the orbits do not escape), there is a subset of initial conditions whose orbits survive the second ejection, and so on, and that there are orbits whose ejection distances grow indefinitely. He proved that there exist initial conditions such that the distance \( r(t) \) of the third body from the origin does not grow monotonically although its supremum increases without bound. These conditions are those of an oscillatory orbit (see Definition 1 below). The Sitnikov problem has been extended by Alekseev and Easton and McGehee. Moser precisely analyzed this problem with a mapping technique. Recently, Xia proved the existence of oscillatory solutions and Arnold diffusion for the planar case with \( m_1 \gg m_2 \gg m_3 \). The existence of oscillatory orbits in other three-body systems was verified with the aid of numerical integration.

The orbits of the accelerator mode in the standard mapping correspond to escape orbits in three-body problems. The accelerator mode appears as soon as the last Kolmogorov-Arnold-Moser (KAM) curve disintegrates. In the vicinity of an orbital point of the accelerator mode, there exist points of non-Birkhoff periodic orbits which, in the language of the three-body problem, are ejected to various distances. Therefore we hypothesize the existence of oscillatory orbits if orbits of the accelerator mode exist. The purpose of the present paper is to prove that this hypothesis is correct.
The standard mapping $T$ we consider is defined on a cylinder (i.e., $S^1 \times \mathbb{R}$ with $x \in S^1$ and $y \in \mathbb{R}$):

$$T : y_{n+1} = y_n + f(x_n), \ x_{n+1} = x_n + y_{n+1} \ (\text{mod } 2\pi),$$

where $f(x) = a \sin x$ ($a > 0$). This mapping has two fixed points, $P = (0, 0)$ and $Q = (\pi, 0)$. $P$ is a saddle, and $Q$ is either an elliptic point or a saddle with reflection, depending on the value of $a$. $T$ can be expressed as a product of two involutions, $G$ and $H$ or $g$ and $h$:

$$T = H \circ G = h \circ g,$$

$$G : x \leftarrow -x, \ y \leftarrow y + f(x),$$

$$H : x \leftarrow y - x, \ y \leftarrow y,$$

$$g : x \leftarrow x, \ y \leftarrow -y - f(x),$$

$$h : x \leftarrow x - y, \ y \leftarrow -y.$$ (1)

It is noted that the value of $x$ is taken modulo $2\pi$. From the functional forms of $G, H, g$ and $h$, we can roughly characterize the involutions $G$ and $H$ as representing the left-right symmetry of the orbits and $g$ and $h$ the up-down symmetry. Thus the standard mapping possesses double reversibilities. The sets of fixed points of the involutions are the symmetry axes and are obtained as

$$x = 0, \pi \ \text{for } G,$$

$$y = 2(x + n\pi) \ \text{for } H,$$

$$y = 2n\pi - f(x)/2 \ \text{for } g,$$

$$y = 2n\pi \ \text{for } h$$ (2-10)

with $n \in \mathbb{Z}$. In the following, we seek the initial point of an oscillatory orbit as a limit of symmetric periodic orbits that have points on the symmetry axes of $G$ and $h$ (see §3).

From this point, we study the problems under consideration in the universal cover ($\mathbb{R}^1 \times \mathbb{R}^1$) of the cylinder, unless otherwise stated. We use the same coordinates $x$ and $y$ in the universal cover. We now define an oscillatory orbit in the standard mapping.

**Definition 1:** Let $O(z_0) = \{z_0, z_1, z_2, \cdots \}$ [$z_n = (x_n, y_n)$] be the forward orbit of an initial point $z_0$. The orbit $O(z_0)$ is called an oscillatory orbit in the standard mapping or simply an oscillatory orbit if it satisfies

$$\limsup_{n \to \infty} |y_n| = \infty \ \text{and } \liminf_{n \to \infty} |y_n| < \infty.$$ (11)

There exist different oscillatory orbits. In this paper, we prove the existence of the following type of oscillatory orbit:

$$\limsup_{n \to \infty} y_n = \infty \ \text{and } \liminf_{n \to \infty} y_n < \infty \text{ with } y_n > 0.$$ (12)
We need some additional effort to prove the existence of other types of oscillatory orbits. The details of such proofs will be reported elsewhere.

In §2, we present the tools used in the proof. The relation between the accelerator mode orbits and the non-Birkhoff periodic orbits used in the proof are obtained in §3. We give a proof of the existence of oscillatory orbits in §4. Several remarks are made in §5.

§2. Preliminaries

2.1. Non-Birkhoff periodic orbits

If we lift a periodic orbit in the cylinder with rotation number $p/q$ ($p \neq 0$) to the universal cover $R^1 \times R^1$, we obtain $|p|$ distinct orbits. The exception to this rule is the case $p = 0$, in which there are infinitely many distinct orbits. The orbit of $z$ is denoted $o(z) = \{T^k z : k \in \mathbb{Z}\}$. We call $eo(z) = \{T^k z + (2\pi l, 0) : k, l \in \mathbb{Z}\}$ the extended orbit of $z$. If for any two points $r$ and $s$ from $eo(z)$ of a periodic point $z$, the relation $\pi_1(r) < \pi_1(s) \Rightarrow \pi_1(Tr) \geq \pi_1(Ts)$

is satisfied, the orbit is said to be a $p/q$-monotone or Birkhoff periodic orbit or (a $p/q$-BO). Otherwise, the orbit is said to be a $p/q$-non-monotone or non-Birkhoff periodic orbit or (a $p/q$-NBO). Here, $\pi_1(r)$ represents the projection to the $x$ coordinate of $r$. If a periodic orbit has two points on the symmetry axes, it is called a symmetric periodic orbit. A symmetric non-Birkhoff periodic orbit is generally referred to by the abbreviation SNBO. If the rotation number is $p/q$, then it is referred to as a $p/q$-SNBO.

There are two types of $p/q$-SNBOs.\(^{15}\) The first has turning points in the extended orbit. The second has no turning points. The SNBO of the first type starting from $\{y > 0\}$ ($\{y < 0\}$) passes through $\{y < 0\}$ ($\{y > 0\}$). This fact can be derived using the second equation in Eq. (1). Contrastingly, the SNBO of the second type starting from $\{y > 0\}$ ($\{y < 0\}$) passes through only $\{y > 0\}$ ($\{y < 0\}$).

Definition 2. Suppose a $p/q$-periodic point $z \in R^2$ is given.

(1) If for some $r, s \in eo(z)$ with $r \in o(s)$ we have

$\pi_1(r) < \pi_1(s) \Rightarrow \pi_1(Tr) \geq \pi_1(Ts)$,

then point $z$ is called a non-Birkhoff periodic point of Type I or a non-Birkhoff periodic point with turning points.

(2) If for any $r, s \in eo(z)$ with $r \in o(s)$ we have

$\pi_1(r) < \pi_1(s) \Rightarrow \pi_1(Tr) < \pi_1(Ts)$,

and if for some $r', s' \in eo(z)$ with $r' \not\in o(s')$ we have

$\pi_1(r') < \pi_1(s') \Rightarrow \pi_1(Tr') \geq \pi_1(Ts')$,
then point $z$ is called a non-Birkhoff periodic point of Type II or a non-Birkhoff periodic point without turning points.

A $p/q$-SNBO appears through a saddle-node bifurcation.\textsuperscript{15) The orbital points of a $p/q$-SNBO of Types I and II do not lie on a Lipschitz curve. This fact implies the following.

**Proposition 1.** Suppose that the Aubry graph\textsuperscript{16),17) of a periodic orbit with rotation number $(np)/(nq)$ intersects that of a $p/q$-Birkhoff periodic orbit at no fewer than two points. Then the orbit is an $(np)/(nq)$-NBO.

### 2.2. Intervals in symmetry axes and homoclinic lobes

The original fixed points $P \equiv P_{0,0}$ and $Q \equiv Q_{0,0}$ are lifted to period-1 points $P_{i,j} = (2\pi i, 2\pi j)$ and $Q_{i,j} = (2\pi i + \pi, 2\pi j)$, where $i$ and $j$ are arbitrary integers. These points inherit the properties of $P$ and $Q$. Let us give names to the branches of the stable and unstable manifolds of the saddle $P_{i,j}$. The branch of the stable manifold coming into $P_{i,j}$ from the left (resp., right) is denoted $W^s_i(P_{i,j})$ [resp., $W^u_s(P_{i,j})$], and the branch of the unstable manifold going out of $P_{i,j}$ from the left (resp. right) is denoted $W^u_i(P_{i,j})$ [resp. $W^u_s(P_{i,j})$].

Hereafter, we use the notation $(A, B)_C$ to represent an open arc or interval and $[A, B)_C$ to represent a closed arc or interval between the points $A$ and $B$ along a one-dimensional manifold $C$. When $C$ has a complex expression, we use $(A, B) \subset C$ or $[A, B) \subset C$. Let $u$ be the first [along $W^u_i(P_{0,0})$ starting from $P_{0,0}$] intersection point between $W^s_i(P_{0,0})$ and $\{x = \pi, y > 0\}$ and $v$ be the first intersection point of $W^u_i(P_{0,0})$ and $\{y = 2(x - \pi), y > 0\}$. We know that these points exist for all $a > 0$.\textsuperscript{18) At these points, $W^u_i(P_{1,0})$ intersects $W^s_i(P_{0,0})$ transversely.\textsuperscript{19) Let $V^u_0, V^d_0$ be the open region bounded by $[u, v]_{W^s_i(P_{0,0})}$ and $[u, v]_{W^u_i(P_{1,0})}$, and let $U^u_0, U^d_0$ be the open region bounded by $[v, Tu]_{W^u_i(P_{0,0})}$ (identical to $\Gamma_u$) and $[v, Tu]_{W^u_i(P_{1,0})}$ (identical to $\Gamma_u$). Next, let $u'$ be the first intersection point of $W^u_i(P_{1,0})$ and $\{x = \pi, y < 0\}$, and $v'$ be that of $W^u_i(P_{1,0})$ and $\{y = 2(x - \pi), y < 0\}$. Let $V^u_0, V^d_0$ be the open region bounded by $[u', v']_{W^u_i(P_{1,0})}$ and $[u', v']_{W^u_i(P_{0,0})}$, and let $U^u_0, U^d_0$ be the open region bounded by $[v', Tu']_{W^u_i(P_{1,0})}$ and $[v', Tu']_{W^u_i(P_{0,0})}$. The existence of $u'$, $v'$, $U^u_0, U^d_0$ and $V^u_0, V^d_0$ is demonstrated by use of the left-right and up-down symmetries from $u$, $v$, $V^u_0$ and $U^u_0$.

The structure containing the homoclinic lobes $V^u_0, V^u_0, U^d_0, U^d_0$ and their boundary curves is called a **turnstile**.\textsuperscript{20) We denote by $Z_{0,0}$ the open region bounded by $[P_{0,0}, v]_{W^s_i(P_{0,0})}$, $[v, P_{1,0}]_{W^s_i(P_{1,0})}$, $[P_{1,0}, v']_{W^u_i(P_{1,0})}$ and $[v', P_{1,0}]_{W^u_i(P_{0,0})}$.

Now we introduce open intervals $I_i$ ($i \geq 1$) in $\{x = 0, 0 < y < 2\pi\}$:\textsuperscript{14),15) $I_i = T^{-i}V^u_0 \cap \{x = 0, 0 < y < 2\pi\}$.\textsuperscript{17) It is easy to show that every $I_i$ is a single connected interval. If an interval $I_i$ exists, then all intervals $I_j$ with $j > i$ exist. This is a simple consequence of the Lambda lemma.\textsuperscript{21) Let $t_i$ be the lower end of $I_i$ and $t_{i+1}$ the upper end of $I_{i+1}$ in the case that $I_i$ for all $i \geq 1$ exist. We define an open interval $L_i$ in $\{x = 0, 0 < y < 2\pi\}$ with upper and lower ends $t_i$ and $t_{i+1}$ (see Fig. 1). Let $\Omega_j$ be the open region bounded by $[t_j, t_{j+1}]_{W^u_i(P_{0,0})}$ and $[t_j, t_{j+1}]_{W^u_i(P_{1,0})}$.
by $L_j$ and $[t_j, t_{j+1}]_{W_j(P_1,0)}$. By definition, we have

$$T^{j+1}\Omega_j \supset U_{0,0}^u.$$  \hfill (18)

Next, we define the open intervals $M_i$ ($i \geq 1$) in \{\(y = 0, 0 < x < \pi\)\} by

$$M_i = T^{-i}U_{0,0}^u \cap \{y = 0, 0 < x < \pi\}.$$  \hfill (19)

If $M_i$ exists, then $M_j$ for all $j > i$ exist.

Finally, we introduce $V_{mx,my}^u, U_{mx,my}^u, Z_{mx,my}, \Gamma_i^{mx,my}, \Lambda_i^{mx,my}$ and $M_i^{mx,my}$ as $V_{0,0}^u, U_{0,0}^u, Z_{0,0}, \Gamma_i^{0,0} = I_i, \Lambda_i^{0,0} = L_i$ and $M_i^{0,0} = M_i$ shifted by $2m_x\pi$ in the $x$-direction and by $2m_y\pi$ in the $y$-direction.

2.3. Doubly reversible periodic orbits

In doubly reversible dynamical systems, there exist periodic orbits that do not exist in singly reversible systems. We introduce these orbits in this subsection.

There are ten kinds of symmetric periodic orbits in doubly reversible systems. These orbits are classified with respect to the pair of involutions on whose symmetry axes the orbital points sit. Thus, we have classes corresponding to the pairs $(G,G)$, $(G,H)$, $(G,g)$, $(G,h)$, $(H,H)$, $(H,g)$, $(H,h)$, $(g,g)$, $(g,h)$ and $(h,h)$. We know the properties of periodic orbits\textsuperscript{13} that have two points on the axes of the pairs $(G,G)$, $(G,H)$, $(H,H)$, $(g,g)$, $(g,h)$ and $(h,h)$. We attempt to determine the properties of periodic orbits that have two points on the symmetry axes of the mixed pairs $(G,g)$, $(G,h)$, $(H,g)$ and $(H,h)$. We already have a partial result.\textsuperscript{14} Here we extend this.
Let \( z_0 \) be a point and \( o(z_0) = \{ \ldots, z_{-1}, z_0, z_1, \ldots \} \) be its orbit with \( z_n = (x_n, y_n) \). We have the following.

**Proposition 2.** In the standard mapping, there are periodic points of the following kinds:

(i) If \( z_0 = Gz_0 \) and \( z_k = hz_k \), then \( o(z_0) \) has period \( 4k - 2 \), \( k = 2, 3, \ldots \)

(ii) If \( z_0 = Hz_0 \) and \( z_k = gz_k \), then \( o(z_0) \) has period \( 4k + 2 \), \( k = 1, 2, \ldots \)

(iii) If \( z_0 = Gz_0 \) and \( z_k = gz_k \), then \( o(z_0) \) has period \( 4k \), \( k = 1, 2, \ldots \)

(iv) If \( z_0 = Hz_0 \) and \( z_k = hz_k \), then \( o(z_0) \) has period \( 4k \), \( k = 1, 2, \ldots \)

**Proof.** We consider the mapping on the cylinder.

(i) The proof in this case is given in Ref. 14). However, for the sake of completeness, we give a proof here as well. From \( z_k = hz_k \), we have \( z_{k-i} = g(z_{k+i-1}) \) for any \( i \). Taking \( i = k \), we have \( z_{2k-1} = gz_0 \). Because the involution \( g \) does not change the \( x \)-coordinate of \( z_0 \), and because \( z_0 \) is on the symmetry axis of \( G \) by assumption, \( z_{2k-1} \) is also on the symmetry axis of \( G \). This implies that \( o(z_0) \) is periodic with period \( 4k - 2 \).

(ii) From \( z_0 = Hz_0 \) and \( z_k = gz_k \), we have \( z_{-k-1} = Hz_{k+1} = Gz_k = (-x_k, y_k + f(x_k)) \). Also, because \( z_k \) is on the symmetry axis of \( g \), we have \( y_k = 2m\pi - f(x_k)/2 \) for some \( m \). From this, we obtain \( z_{-k-1} = (-x_k, 2m\pi + f(x_k)/2) = (-x_k, 2m\pi - f(-x_k)/2) \); i.e., \( z_{-k-1} = g(z_{-k-1}) \). The orbit \( o(z_0) \) is periodic with period \( 4k + 2 \).

(iii) From \( z_0 = Gz_0 \) and \( z_k = gz_k \), we have \( z_{k+j} = hz_{k-j+1} \) for any \( j \in \mathbb{Z} \). Taking \( j = k \), we have \( z_{2k} = hTz_0 = gz_0 \). The point \( z_0 \) is on \( x = 0 \) or \( x = \pi \). The involution \( g \) does not change the \( y \)-coordinate of \( z_0 \). This implies that \( z_{2k} = Gz_{2k} \), that is, that \( z_0 = z_{4k} \).

(iv) The proof is similar to those above.

Q.E.D.

We conjecture that Proposition 2 holds in general for doubly reversible mappings. We leave the proof of this conjecture for a future study. We refer to the periodic orbits described in Proposition 2 as the **doubly reversible periodic orbits** (DRPOs). These orbits have four points on the symmetry axes. Because they have even periods, DRPOs have pairs of points on the same symmetry axes. If the period of a given DRPO is \( q \), we refer to it as a \( q \)-DRPO.

An orbit is uniquely defined by its initial position on \( x = 0 \) and its motion in the \( x \)-direction during the first \( n \) iterations. This implies that the rotation number is determined. The following specifies the relation between the period and the rotation number.

**Proposition 3.** The rotation numbers of the DRPOs in Proposition 2 are as follows:

(i) \( m(4k - 2)/(4k - 2) \), with \( y_k = 2m\pi \).

(ii) \( m(4k + 2)/(4k + 2) \), with \( y_k = 2m\pi - f(x_k)/2 \).

(iii) \( m(4k)/(4k) \), with \( y_k = 2m\pi - f(x_k)/2 \).

(iv) \( m(4k)/(4k) \), with \( y_k = 2m\pi \).

**Proof:** (i) Because \( hz_k = z_k \), we have the relations \( hz_i = z_{2k-i} \) (\( 1 \leq i \leq 2k-1 \)) and \( y_i + y_{2k-i} = 4m\pi \). As \( z_{2k-1} \) is located on the symmetry axis of \( G \), we have the shift
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in the $x$-direction given by

$$x_{4k-2} - x_0 = 2(x_{2k-1} - x_0) = 2 \sum_{i=1}^{2k-1} y_i = 2(4(k-1)m + 2m)\pi = 2\pi m q. \quad (20)$$

(ii) From $h_i = z_{2k+1-i} (0 \leq i \leq 2k+1)$, we obtain $y_i + y_{2k+1-i} = 4m\pi$. Then, the shift in the $x$-direction is evaluated as

$$x_{2k+1} - x_0 = \sum_{i=0}^{2k+1} y_i - y_0 \quad \text{and} \quad x_{4k+2} - x_{2k+1} = \sum_{i=2k+1}^{4k+2} y_i - y_{2k+1}. \quad (21)$$

Then we have $x_{4k+2} - x_0 = 2m\pi(4k+2) = 2\pi mq$.

(iii) The proof is similar to that of (i).

(iv) The proof is similar to that of (ii). Q.E.D.

The DRPOs we use in the present work are those possessing two points on the symmetry axes of the pair $(G, h)$. We take the initial position to be $x = 0$ and the symmetry axis at the $k$-th iterate to be $y = 2m\pi$. Then, the expression for the symmetry axes of $G$ is obtained by using the fact that the orbit moves an amount $2m\pi(2k-1)$ in the $x$-direction during a half period. The expression for $H$ is obtained from the relation $H = TG$. The facts that the initial symmetry axis of $h$ is given by $y = 2m\pi$ and that $h_{z_{k-n}} = z_{k+n}$ give us the expression for $h$. Finally, the expression for $g$ is obtained from $g = hT$. The following summarizes these results:

$$G : x_{4k-n-2} = -x_n + 2\pi m(4k-2), \quad (22)$$

$$y_{4k-n-2} = y_n + f(x_n), \quad (23)$$

$$H : x_{4k-n-1} = -x_n + y_n + 2\pi m(4k-2), \quad (24)$$

$$y_{4k-n-1} = y_n, \quad (25)$$

$$g : x_{2k-n-1} = x_n + 2\pi m(2k-2n-1), \quad (26)$$

$$y_{2k-n-1} = -y_n - f(x_n) + 4\pi m, \quad (27)$$

$$h : x_{2k-n} = x_n - y_n + 2\pi m(2k-2n+1), \quad (28)$$

$$y_{2k-n} = -y_n + 4\pi m, \quad (29)$$

where $0 \leq n \leq 4k-2$.

2.4. Accelerator mode

It is well known that there exists anomalous diffusion in the standard mapping defined on the cylinder.$^{9)-11)}$ This diffusion is caused by orbits of the so-called accelerator mode. An orbit of the accelerator mode corresponds to an escape orbit in the three-body problem. This is one of the key orbits necessary to prove the existence of oscillatory orbits in the standard mapping. We here briefly introduce the definition of the accelerator mode.

An orbit of the accelerator mode is a periodic orbit moving by an amount $2\pi$ in the $y$-direction of the standard mapping defined on a torus. In order to systematically
obtain orbits of the accelerator mode, let us consider the standard mapping \( \hat{T} \) on a torus. This mapping is written as a product of the involutions \( \hat{g} \) and \( \hat{h} \); i.e., \( \hat{T} = \hat{h} \circ \hat{g} \), with

\[
\begin{align*}
\hat{g} : & \quad x \leftarrow x, \ y \leftarrow -y - f(x), \\
\hat{h} : & \quad x \leftarrow x - y, \ y \leftarrow -y.
\end{align*}
\]

Two symmetry axes are obtained as invariant sets of \( \hat{g} \):

\[
y = -f(x)/2, \ y = \pi - f(x)/2 \quad \text{for} \ \hat{g}.
\]

Similarly, one symmetry axis is obtained as an invariant set of \( \hat{h} \):

\[
y = 0 \quad \text{for} \ \hat{h}.
\]

If we lift the mapping to the cylinder and denote the lifted involutions by \( g \) and \( h \), the symmetry axes of these involutions are

\[
y = 2n\pi \quad \text{for} \ h,
\]

\[
y = n\pi - f(x)/2 \quad \text{for} \ g,
\]

with \( n \in \mathbb{Z} \). Periodic orbits of the accelerator mode are lifted to non-periodic orbits. This is because the shift in the \( y \)-direction is repeated, and the \( y \)-coordinate of the orbit changes monotonically, which implies escape. There are various kinds of the lift of orbits. In general, a shift by \( 2n\pi \ (n \in \mathbb{Z}) \) in one period is possible. It is to be noted that there are differences among the sets of symmetry axes on the universal cover for the standard mappings defined on a torus and that defined on a cylinder.

There are symmetric orbits in the accelerator mode. Let us start from the original mapping on the torus. Then, orbits possessing two points on either or both of the symmetry axes (32) and (33) are symmetric. On the cylinder, a symmetric orbit of the accelerator mode, which we will call a SAMO, has two points on the axes (32) and/or (33). It can be shown that SAMOs appear through a saddle-node bifurcation.

In order to investigate the properties of SAMOs, it is convenient to further lift the mapping to a plane. We consider SAMOs with starting positions in the set \( \{0 < x < \pi, y = 0\} \) of the plane. During the first period, a periodic orbit of period \( q \) that moves a distance \( 2m_x\pi \) in the \( x \)-direction and a distance \( 2m_y\pi \) in the \( y \)-direction \( (m_x, m_y > 0) \) is referred to as a \( q^{m_x,m_y} \)-SAMO. We consider only \( q^{1,1} \)-SAMOs in what follows. For notational convenience, we abbreviate these as \( q^{m_x} \)-SAMOs. The following dynamical ordering for SAMOs has been proved.\(^{12}\)

**Theorem 1.**\(^{12}\) The following dynamical order relations hold.

\[
\begin{array}{c|cccccccc}
 & 1^1 & 2^1 & 3^1 & 4^1 & 5^1 & 6^1 & \cdots \\
\hline
M_1: & 1 & 3 & 5 & 7 & 9 & 11 & \cdots \\
\hline
M_2: & 3 & 5 & 7 & 9 & 11 & \cdots \\
\hline
M_3: & 5 & 7 & 9 & 11 & \cdots \\
\hline
\end{array}
\]
Here, the right arrow and the down arrow represent the forcing relation between two SAMOs. For example, $1^1 \in M_1 \rightarrow 3^2 \in M_2$ means that if a point of a $1^1$-SAMO exists in $M_1$, then a point of a $3^2$-SAMO exists in $M_2$. $q_{i,j}^{(q+1)/2}$-SAMOs starting from $M_i$ appear through the first tangency of $T^{(q+1)/2}M_i$ and the symmetry axis $y = \pi - f(x)/2$ ($y > 0$).

The orbital points from $z_1$ to $z_{(q-1)/2}$ of a $q^{(q+1)/2}$-SAMO are located in $\{y > 0\}$. In the following, we show that $q^{(q+1)/2} \not\in M_i$ is not a SAMO that appears through the first tangency if its future orbit has a point in $\{y < 0\}$. Suppose that the orbital point $z_l$ ($i < l < k$) of $q^{(q+1)/2} \in M_i$ is located in $\{y < 0\}$ and $z_k$ is the first tangency point between $T^k M_i$ and the symmetry axis $y = \pi - f(x)/2$ ($y > 0$). Then, there exist two integers $l_1$ and $l_2$ ($l_1 < l < l_2$) such that $z_{l_1} \in V_{n,0}^{u}$ and $z_{l_2} \in U_{n,0}^{u}$ for some integer $n$. Because we have $T^{l_2} M_i \cap U_{n,0}^{u} \neq \emptyset$, $T^{l_2} M_i$ exists along the unstable manifold which comprises part of the boundaries of $U_{n,0}^{u}$. Let $\xi_{in}$ be the arc of $T^{l_2} M_i$ in $U_{n,0}^{u}$ and $\xi_{out}$ be the arc of $T^{l_2} M_i$ outside the unstable manifold and in the nearest neighborhood of $U_{n,0}^{u}$. The two endpoints of $\xi_{out}$ are located in the stable manifold $W_s^{l}(P_{n+1,0})$. Because $T^{l_3} \xi_{in}$ (where $l_3 = k$) is a tangency point in the symmetry axis $y = \pi - f(x)/2$ ($y > 0$), $T^{l_3} \xi_{out}$ possesses intersection points on the same symmetry axis. This is a contradiction. Combining this result with a well-known property of the accelerator mode and the left-right symmetry, Property 1 is obtained.

**Property 1.** For a SAMO satisfying the initial condition $z_0 \in M_i$ in Theorem 1, we have $y_n \rightarrow \infty$ as $n \rightarrow \infty$ and $y_n \rightarrow -\infty$ as $n \rightarrow -\infty$ and the relations $y_n > 0$ ($n \geq 1$) and $y_n < 0$ ($n \leq -1$). For a SAMO satisfying the initial condition $z_0 \in HM_i$, we have $y_n \rightarrow -\infty$ as $n \rightarrow \infty$ and $y_n \rightarrow \infty$ as $n \rightarrow -\infty$ and the relations $y_n < 0$ ($n \geq 1$) and $y_n > 0$ ($n \leq -1$).

In order to specify the position of a SAMO in the table of Theorem 1, we use matrix notation. For example, the $(i,j)$th element corresponds to the $q_{i,j}^{(q_i,j+1)/2}$-SAMO with $q_{i,j} = 2i + 2j - 3$ ($i \geq 1$ and $j \geq 1$). Let $a_c(i,j)$ be the critical value at which the $q_{i,j}^{(q_i,j+1)/2}$-SAMO appears. For such critical values, the following relation holds:

$$\lim_{i,j \rightarrow \infty} a_c(i,j) = a_c,$$

where $a_c$ is the critical value at which the last KAM curve disintegrates.

The point $z_{mq_{i,j}}$ of the $q_{i,j}^{(q_{i,j}+1)/2}$-SAMO with $z_0 = (x_0,0)$ satisfies

$$z_{mq_{i,j}} = (m(mq_{i,j} + 1)\pi + x_0, 2m\pi) \in M_i^{mq_{i,j}+1/2,m},$$

where $m \geq 1$ is an integer.

**Proposition 4.** If a point of a $q_{i,j}^{(q_{i,j}+1)/2}$-SAMO for $q_{i,j} = 2i + 2j - 3$ with $j \geq i \geq 1$ exists in $M_i$, then the interval $I_j$ exists.

**Proof.** Let $z_j = (x_j, y_j)$ be the orbital points of a $q_{i,j}^{(q_{i,j}+1)/2}$-SAMO, where $z_k$ ($k = i + j - 2$) is located on the symmetry axis $y = \pi - f(x)/2$. Pairs of involutions
Suppose that the existence of this intersection. The result is the following.

\[ \hat{g} : x_{2k-n} = x_n + 2\pi(k - n), \quad y_{2k-n} = 2\pi - y_n - f(x_n), \]
\[ \hat{h} : x_{2k+1-n} = x_n - y_n + 2\pi(k - n + 1), \quad y_{2k-n} = 2\pi - y_n, \]

where \( 0 \leq n \leq 2k+1 \). We obtain that the x-coordinate increases as the y-coordinate increases. Using the equation for \( y \) in Eq. (38), we have the relation \( x_{k+1} = x_{k-1} + 2\pi \). The orbital point \( z_l \) (\( l \geq 1 \)) is located in \( \{ y > 0 \} \) (see Property 1). Thus we have the relations \( x_i < x_{k-1} < x_{k+1} \) and \( x_{k+1} - x_i > 2\pi \). Hence \( T^{k+1-i}U^u_{0,0} \) enters \( \{ x > 2\pi \} \), because of the relation \( \pi < x_i < 2\pi \). It follows that the set \( T^{k+1-i}U^u_{0,0} \) intersects \( \{ x = 2\pi, y > 0 \} \). The existence of \( I_j^{1,0} \) is thus proved. This implies the existence of \( I_j = I_j^{0,0} \). (Q.E.D.)

From the equation for \( y \) given in Eq. (39) and Property 1, we find that the orbital points from \( z_0 \) to \( z_{2k+1} \) are located in \( \{ 0 \leq y \leq 2\pi \} \).

\section{SAMO and DRPO}

In this section, deriving forcing relations between SAMOs and DRPOs, we obtain relative positions among DRPOs on the symmetry axis \( L_j \). An interval \( L_j \) (\( j \geq 1 \)) exists when there exists a point of the \( q_{i,j}^{(q_{i,j}+1)/2} \)-SAMO with \( q_{i,j} = 2i + 2j - 3 \) in \( M_i \) (see Proposition 4). For any integer \( m \geq 1 \), the intersection of \( T^kL_j \) (\( k \geq j + 1 \)) and the symmetry axis \( y = 2m\pi \) is a point of a \((4k-2)\)-DRPO with rotation number \((4k-2)m/(4k-2) \) (see Proposition 3). The existence of a certain SAMO ensures the existence of this intersection. The result is the following.

\textbf{Proposition 5.} Suppose that \( j \geq i \geq 1 \). If a \( q_{i,j}^{(q_{i,j}+1)/2} \)-SAMO exists for \( q_{i,j} = 2i + 2j - 3 \), then there exist, for any \( m \geq 1 \), at least two \((4k_{i,j,m}-2)\)-DRPOs, \( O(z^m_0) \) and \( O(z^m) \), with \( k_{i,j,m} = mq_{i,j} + j - i + 1 \). Let \( (0,A_m) \in L_j \) and \( (0,B_m) \in L_j \) \((A_m > B_m)\) be the initial points of \( O(z^m_0) \) and \( O(z^m) \), where \( m \) stands for the symmetry axis \( y = 2m\pi \) on which \( z^m_{k_{i,j,m}} \) and \( z^m_{k_{i,j,m}} \) remain. The following relations hold if an appropriate pair of \((4k_{i,j,m} - 2)\)-DRPOs is chosen for each \( m \geq 1 \):

\[ B_1 < B_2 < \cdots < B_m < \cdots < A_m < \cdots < A_2 < A_1. \]

\textbf{Proof.} First, we claim that \( \text{Arc} T^{k_{i,j,m}}L_j \) intersects \( M_i^{(mq_{i,j}+1)/2} \), and hence that at least two \((4k_{i,j,m} - 2)\)-DRPOs exist. This claim is demonstrated as follows. Note that \( z_0 \) is contained in \( T^{-i}U^u_{0,0} \). In view of (18), \( T^{j+1}L_j \) is not contained in \( U^u_{0,0} \). The point \( z_{mq_{i,j}} \) is in \( M_i^{(mq_{i,j}+1)/2} \), from Eq. (37). This implies \( T^{mq_{i,j}-1}U^u_{0,0} \cap M_i^{(mq_{i,j}+1)/2} \neq \emptyset \), which in turn implies that \( T^{k_{i,j,m}}L_j \) intersects \( M_i^{(mq_{i,j}+1)/2} \) at no fewer than two points. These are points of \((4k_{i,j,m} - 2)\)-DRPOs. Thus our claim is proven.

Now suppose that \( T^{k_{i,j,m}}L_j \cap M_i^{(mq_{i,j}+1)/2} \neq \emptyset \) for any given \( m \geq 1 \). We show that \( B_m < B_{m+1} < A_{m+1} < A_m \), which implies Eq. (40). The point \( z_{(m+1)q_{i,j}} \) of the SAMO is located in \( M_i^{(m+1)((m+1)q_{i,j}+1)/2,m+1} \subset \{ y = 2(m+1)\pi \} \). Let \( Z_1 = z^m_{k_{i,j,m}} \).
and $Z_2 = \tilde{z}_{k_{i,j,m}}$. Then, in view of Eq. (29), $T^{q_{i,j}} Z_1$ and $T^{q_{i,j}} Z_2$ remain below $y = 2(m+1)\pi$. The open arc $(Z_2, Z_1) \in T^{k_{i,j,m}} L_j$ has a subarc above $y = 2m\pi$. These facts imply that $T^{q_{i,j}} (Z_2, Z_1)$ intersects $M_{(m+1)(m+1)}^{(m+1)(m+1)}$. We refer to these intersection points as $Z_1'$ and $Z_2'$, where $\pi_1(Z_1') > \pi_1(Z_2')$. Then the points $Z_1'$ and $Z_2'$ are points of $(4k_{i,j,m} + 1 - 2)$-DRPOs with initial points in $L_j$. If we denote the initial points by $\tilde{z}_0^{m+1} = (0, A_m + 1)$ and $\tilde{z}_0^{m+1} = (0, B_{m+1})$, we obtain the relation $B_{m+1} < B_m < A_{m+1} < A_m$.

**Proposition 6.** For each $i, j$ and $m$, there exist $(4k_{i,j,m} + 1 - 2)$-DRPOs of Eq. (40) whose orbital points are all located in $\{y > 0\}$. These DRPOs are SNBOs without turning points.

**Proof.** First we prove the former statement. Let us consider the case in which $\tilde{z}_{k_{i,j,m}}$ and $\tilde{z}_{k_{i,j,m}}$ are unique. For brevity, we consider only the case for $\tilde{z}_{k_{i,j,m}}$. We assume that $O(\tilde{z}_0)$ possesses a point below the $x$-axis and derive a contradiction. By assumption and the properties of DRPOs, there exists an integer $l'(i+1 < l' < k_{i,j,m})$ such that $\pi_2(\tilde{z}_{l'}) < 0$. Then there also exists an $l (i+1 < l < l')$ such that $\tilde{z}_l \in V_{n,0}$ for some $n$. This implies that $T^{l'+1} L_j$ has a partial arc that connects a point in $V_{n,0}$ and a point in $TV_{n,0}$. This arc $\gamma$ is to the left of the unstable manifold which constitutes one of the boundaries of $U_n$. Then $\gamma$ intersects $y = 2m\pi$, since $U_n$ has a point at some value of $y$ greater than $y = 2m\pi$, which means that there are points of $(4(j+1) - 2)$-DRPOs in $\gamma$. This, in turn, implies the existence of points of $(4k_{i,j,m} + 1 - 2)$-DRPOs in $\gamma$. This is a contradiction.

Next, we consider the case in which $\tilde{z}_{k_{i,j,m}}$ and/or $\tilde{z}_{k_{i,j,m}}$ is not unique. In this case, we let $\tilde{z}_{k_{i,j,m}}$ be the closest point to $T^{k_{i,j,m} + 1} L_j$ and/or $\tilde{z}_{k_{i,j,m}}$ be the closest point to $T^{k_{i,j,m} - 1} L_j$. Assuming $O(\tilde{z}_0)$ has a point below the $x$-axis and repeating the arguments of the previous paragraph, we again arrive at a contradiction.

The point $\tilde{z}_0$ of the $(4k_{i,j,m} + 1 - 2)$-DRPO remains below $y = 2m\pi$, and the point $\tilde{z}_{2k_{i,j,m}}$ remains above $y = 2m\pi$. Using these facts and double reversibility, we find that the Aubry graph of the DRPO intersects that of the $m/1$-Birkhoff orbit at two points during one period. Then Proposition 1 shows that the DRPO is an NBO. DRPOs satisfying Eq. (40) are symmetric orbits and do not have points satisfying $y < 0$. Thus, these DRPOs are SNBOs without turning points. Q.E.D.

Hereafter, we consider SAMOs and DRPOs with $i = j$. For DRPOs, the following two propositions are obtained.

**Proposition 7.** The following relations hold for the DRPO appearing in Proposition 5 and restricted by Proposition 6 whose starting point is $z_0 = (0, B_m)$.

\[
|\pi_1(z_{-4q_{i,i}})| = \pi_1(z_{4q_{i,i}}) < 24\pi q_{i,i} \text{ for } m = 2, \quad (41)
\]
\[
|\pi_1(z_{-4q_{i,i}})| = \pi_1(z_{4q_{i,i}}) < 30\pi q_{i,i} \text{ for } m = 3, \quad (42)
\]
\[
|\pi_1(z_{-4q_{i,i}})| = \pi_1(z_{4q_{i,i}}) < 8m\pi q_{i,i} \text{ for } m \geq 4. \quad (43)
\]
Proof. (1) $m = 2$. $z_{2q_i,i+1}$ is located on $y = 4\pi$. This implies that $\pi_2(z_k) \leq 4\pi$ for $0 \leq k \leq 2q_i,i + 1$ and $4\pi < \pi_2(z_k) < 8\pi$ for $2q_i,i + 2 \leq k \leq 4q_i,i + 2$. To derive the latter inequality, we use the symmetry of $G$. Thus we have

$$\pi_1(z_{4q_i,i}) = \sum_{k=1}^{4q_i,i} \pi_2(z_k) < 4\pi \times 2q_i + 8\pi \times 2q_i,i = 24\pi q_i,i. \quad (44)$$

The first equality is then derived by using the symmetry of $G$.

(2) $m = 3$. $z_{3q_i,i+1}$ is located on $y = 6\pi$. This implies that $\pi_2(z_k) \leq 6\pi$ for $0 \leq k \leq 3q_i,i + 1$ and $6\pi < \pi_2(z_k) < 12\pi$ for $3q_i,i + 2 \leq k \leq 6q_i,i + 2$. Thus we have $\pi_1(z_{4q_i,i}) < 6\pi \times 3q_i,i + 12\pi \times q_i,i = 30\pi q_i,i$.

(3) $m \geq 4$. $z_{mq_i,i+1}$ is located on $y = 2m\pi$. This implies that $\pi_2(z_k) \leq 2m\pi$ for $0 \leq k \leq mq_i,i + 1$. Thus we have $\pi_1(z_{4q_i,i}) < 2m\pi \times 4q_i,i = 8m\pi q_i,i$. Q.E.D.

Proposition 8. Consider the DRPOs appearing in Proposition 5 and restricted by Proposition 6. Let $z_0 = (0, B_m)$ and $\dot{z}_0 = (0, B_{m+1})$, where $B_m$ and $B_{m+1}$ are in Eq. (40). Then, for the DRPOs starting from $z_0$ and $\dot{z}_0$, the following relation holds for $m \geq 1$:

$$\pi_1(z_{4mq_i,i+2}) + 2\pi < \pi_1(\dot{z}_{4mq_i,i+2}). \quad (45)$$

Proof. For $m = 1$, we know that $\pi_1(z_{4q_i,i+2}) = 2\pi(4q_i,i + 2)$ and $\pi_1(\dot{z}_{8q_i,i+2}) = 4\pi(8q_i,i + 2)$. Then, using Eq. (41), we have

$$\pi_1(\dot{z}_{8q_i,i+2}) - 24\pi q_i,i < \pi_1(\dot{z}_{4q_i,i+2}). \quad (46)$$

It is easy to verify the relation

$$\pi_1(z_{4q_i,i+2}) + 2\pi < \pi_1(\dot{z}_{8q_i,i+2}) - 24\pi q_i,i. \quad (47)$$

Combining these relations, we have Eq. (45). Proofs for $m \geq 2$ are similar and thus omitted. Q.E.D.

§4. Existence of oscillatory orbits

In this section, we prove the existence of oscillatory orbits. Let us consider only SAMOs corresponding to the diagonal positions in the table of Theorem 1. These are referred to $q_{ii}(q_{ii}+1)/2$-SAMOs with $q_{ii} = 4i - 3$ for $i = 1, 2, \cdots$.

Theorem 2. If, for any given $i \geq 1$, there exists a point of a $q_{ii}(q_{ii}+1)/2$-SAMO in $M_i$, then there exist in $L_i$ initial points of oscillatory orbits satisfying Eq. (12).

In the limit $i \to \infty$, the critical values $a_c(i,i)$ accumulate at $a_c$ [see Eq. (36)]. Thus, oscillatory orbits appear as soon as the last KAM curve disintegrates. We thus have the following.

Corollary 1. Oscillatory orbits exist for any $a > a_c$.

Proof of Theorem 2. An interval $I_i$ exists by the hypothesis of Theorem 2 and Proposition 4. Also an interval $L_i$ obviously exists. Then we obtain the order
Oscillatory Orbits in the Standard Mapping

Fig. 2. Schematic illustration of $T^m \Gamma$ (thick black curve) in the vicinity of $z_{q_i}^m$ labeled $p$. The label “Sm” indicates $W_s(P_{mq_{m+1},0})$, “s-m” indicates $W_s(P_{mq_{m},0})$, and “Um” indicates $W_u(P_{mq_{m-1},0})$. The circle represents one of the initial points of the $q_{i,i+1/2}$-SAMO.

relation Eq. (40) of Proposition 5 with $i = j$. We define $k_i^m = mq_{ii} + 1$. In this case, $q_i^m$-DRPOs with $q_i^m = 4k_i^m - 2$ start from $(0, A_m)$ and $(0, B_m)$.

Let $\Gamma_m^B$ be the arc $[B_m, B_{m+1}]_{L_i}$ and $\Gamma_m^A$ be the arc $[A_{m+1}, A_m]_{L_i}$. In the following, we only consider $\Gamma_m^B$ for arbitrary $m \geq 1$. (The case of $\Gamma_m^A$ can be treated in a similar manner.) Therefore from this point, we abbreviate $\Gamma_m^B$ as $\Gamma_m$. We consider the behavior of $T^m \Gamma_m$. One of the endpoints of $T^m \Gamma_m$ is $z_{q_i}^m = z_0 \pmod{2\pi}$ of a $q_i^m$-DRPO and the other is $z_{q_i}^m$ of a $q_i^{m+1}$-DRPO.

The point $z_{q_i}^m$ is on the same vertical line as $P_{mq_{i}}^m, 0$. We refer to the intersection points of $W_s^l(P_{mq_{i+1},0})$ and $W_u^r(P_{mq_{i},0})$ in the vicinity of $P_{mq_{i},0}$, as shown in Fig. 2. Let $R$ be the open region bounded by $[C, D] \subset W_s^l(P_{mq_{i+1},0})$ and $[C, D] \subset W_u^r(P_{mq_{i},0})$, and let $S$ be that bounded by $[C, F] \subset W_s^l(P_{mq_{i+1},0})$ and $[C, F] \subset W_u^r(P_{mq_{i},0})$. The point $z_{q_i}^m$ is in $R \cap S$, and the starting point of the SAMO indicated by the open circle remains in $TS$. 
We claim that $T_{q_{i,m}}^m \Gamma_m$ intersects $[C, D] \subset W_s^i(P_{mq_{i,m}+1,0})$, as shown in Fig. 2. We now show this. First, $T_{q_{i,m}}^m \Gamma_m$ does not intersect $[C, D] \subset W_u^r(P_{mq_{i,m}+1,0})$. In fact, if it does so at a point $t$, then $T^{-q_{i,m}}t$ is in $W_u^r(P_{mq_{i,m}+1,0})$ near $I_{mq_{i,m}}$ and its $x$-coordinate is greater than $2\pi(mq_{i,m} - 1)$, whereas $T^{-q_{i,m}}t$ should be on $\{x = 0\}$. This is impossible. Second, Proposition 8 gives

$$\pi_1(z_{q_{i,m}}) + 2\pi < \pi_1(z_{q_{i,m}}').$$
(48)

If the point $z_{q_{i,m}}'$ is not contained in $R$, our claim is proved. In fact, the region $R'$ surrounded by arcs CH, HG, GE and EC (see Fig. 2) does not include $z_{q_{i,m}}'$, because for every point in this region we have $x \leq 2\pi(mq_{i,m} - 1)$. The region $R \setminus R'$ is the forward image of $V_{q_{i,m}}^{u,m+1,0}$. From the proof of Proposition 6, it follows that $z_{q_{i,m}}'$ is not located in $R \setminus R'$. Therefore $R$ does not include $z_{q_{i,m}}'$.

Now, let $z$ be an intersection point of $T_{q_{i,m}}^m \Gamma_m$ and $[C, D] \subset W_s^i(P_{mq_{i,m}+1,0})$. We extend the arc $[z_{q_{i,m}}, z]$ to $T_{q_{i,m}}^m \Gamma_m$ from $z_{q_{i,m}}$ in the opposite direction until it intersects $[E, F] \subset W_s^i(P_{mq_{i,m}+1,0})$. We refer to this point of intersection as $w$. We choose this arc to be $[w, z_{q_{i,m}}] \cup [z_{q_{i,m}}, z]$ so that it is simple and contained in $R \cap S$. It is to be noted that the partial arc $[w, z_{q_{i,m}}]$ is not necessarily the arc of $T_{q_{i,m}}^m \Gamma_m$. The image of the open region bounded by $[w, z_{q_{i,m}}] \cup [z_{q_{i,m}}, z]$ and $[w, z] \in W_s^i(P_{mq_{i,m}+1,0})$ contains a point of the SAMO indicated by the circle in Fig. 2. This fact implies that $T^{mq_{i,i}+1}[w, z_{q_{i,m}}] \cup [z_{q_{i,m}}, z]$ intersects $M_{i,mq_{i,m}+m(mq_{i,i}+1)/2,m}$ at no fewer than two points. Equation (37) is used to determine the value of the suffix. One point is $T^{mq_{i,i}+1}z_{q_{i,m}}$, and the other point $r$ is the intersection point of $T^{mq_{i,i}+1}(z_{q_{i,m}}, z)$ and $M_{i,mq_{i,m}+m(mq_{i,i}+1)/2,m}$. Let $\xi = [z_{q_{i,m}+mq_{i,i}+1}, r] \subset T^{mq_{i,i}+1}[z_{q_{i,m}}, w] \cup [z_{q_{i,m}}, z]$. Then $T^{mq_{i,i}+1}z_{q_{i,m}+mq_{i,i}+1}$ is located below $y = 2(m + 1)\pi$ and a point of the SAMO exists in $M_{i,mq_{i,m}+m(m+1)(m+1)/2,m+1}$. These points are those of $(4(q_{i,m} + (m + 1)q_{i,i} + 2)\text{-DRPO})$.

These initial points $(0, A_{(m)}^n)$ and $(0, B_{(m)}^n)$ are contained in $[B_{m}, B_{m+1}] \in L_i$, and $A_{(m)}^n$ and $B_{(m)}^n$ satisfy the order relations.

$$B_m < B_{2}^{(m)} < B_{3}^{(m)} < \cdots < A_{3}^{(m)} < A_{2}^{(m)} < B_{m+1}.$$  
(49)

Next, we repeat the procedure described above for an interval $\Gamma = [B_{l}^{(m)}, B_{l+1}^{(m)}] \subset L_i$ for $l \geq m + 1$. Note that the period of a DRPO starting from $B_{l}^{(m)}$ is $q_{i,m,l} = 4(q_{l}^{m} + lq_{i,i} + 1) - 2$ and that of a DRPO starting from $0, B_{l}^{(m)}$ is $q_{i,m,l} = 4q_{i,i}$. We consider the structure of $T_{q_{i,m,l}}^m \Gamma$. One of the endpoints is $z_{q_{i,m,l}}$ of a $q_{i,m,l}\text{-DRPO}$, and the other is $z_{q_{i,m,l}}'$ of a $(q_{i,m,l} + 4q_{i,i})\text{-DRPO}$. The relation

$$\pi_1(z_{q_{i,m,l}}) + 2\pi < \pi_1(z_{q_{i,m,l}}').$$
(50)

holds. The proof of Eq. (50) is similar to that of Eq. (48) and thus we omit it. Then
repeating the procedure that led to Eq. (49), we obtain
\[ B_1^{(m)} < B_2^{(m,l)} < B_3^{(m,l)} < \cdots < A_3^{(m,l)} < A_2^{(m,l)} < B_{l+1}^{(m)}. \] (51)

The suffices \((m, l)\) of Eq. (51) specify the interval in which the initial point of DRPO is located. Applying the same procedure to an arbitrary interval \(B_{l'}^{(m,l)}, B_{l'+1}^{(m,l)}\) \((l' \geq l + 1)\), we obtain the relations
\[ B_{l'}^{(m,l)} < B_2^{(m,l,l')} < B_3^{(m,l,l')} < \cdots < A_3^{(m,l,l')} < A_2^{(m,l,l')} < B_{l'+1}^{(m,l)}. \] (52)

Repeating the above procedure, for example, we obtain a sequence of inequalities
\[ B_{n_1} < B_{n_2}^{(n_1)} < B_{n_3}^{(n_1,n_2)} < B_{n_4}^{(n_1,n_2,n_3)} < \cdots, \] (53)
\[ \lim_{i \to \infty} n_i = \infty. \] (54)

From the nature of the construction and contraction of intervals, we conclude that this sequence has an accumulation point \(B^*\). We thus find that the orbit starting from \((0, B^*)\) satisfies the definition of an oscillatory orbit. Q.E.D.

As a direct consequence of Theorem 2, we obtain the following.

**Corollary 2.** There exist diffusive oscillatory orbits. The set of initial positions of oscillatory orbits is uncountable.

**Proof.** In the proof of Theorem 2, we refer to the process of obtaining \(B_{n_{j+1}}^{(n_1,n_2,\ldots,n_j)}\) from \(B_{n_j}^{(n_1,n_2,\ldots,n_j-1)}\) as a step. In each step, only partial arcs and enumerable points are removed from each interval in \(L_i\). Therefore, after an infinite number of steps, a Cantor set remains. Thus the number of oscillatory orbits is uncountable.

In each step, we can take the amplitude of the excursion in the \(y\)-coordinate to be random. The orbit so obtained is called a diffusive oscillatory orbit in the three-body problem.\(^6\) Q.E.D.

**§5. Discussion**

It is very difficult to locate the actual position of an oscillatory orbit and to plot its orbital form. However, from the construction of oscillatory orbits, we know that there are initial conditions of DRPOs close to those of oscillatory orbits. Here, we display a DRPO that is contained in \(\{0 < y < 4\pi\}\) at the beginning of its period, in \(\{0 < y < 8\pi\}\) in the middle of its period, and it moves again in \(\{0 < y < 4\pi\}\) in the end of its period. This is displayed in Fig. 3. This DRPO can be regarded as the initial part of the oscillatory orbit satisfying Eq. (53). The oscillatory orbit obtained in Theorem 2 has a point in the turnstiles \(V_{u_0,m}, V_{d_0,m}, U_{u_0,m}\) and \(U_{d_0,m}\) for \(m \geq 1\) but does not have a point in the turnstiles for \(m \leq 0\).

The system containing oscillatory orbits seems to have the following characteristic property. There is a point \(z_0\) such that the orbit starting at this point goes to infinity as \(t\) goes to infinity; that is, an escape orbit exists. Further there exists an
initial point $z'_0$ in a neighborhood of $z_0$ such that the orbit starting from $z'_0$ goes far from $z'_0$ but eventually comes back to $z'_0$. The initial point of an oscillatory orbit is located in the vicinity of $z_0$ and $z'_0$. In the present paper, we used the existence of periodic orbits possessing various oscillation amplitudes. However, we did not consider exact conditions other than the existence of escape orbits and periodic orbits. It is important to make clear the necessary and sufficient conditions for the existence of oscillatory orbits. This is a future problem.

References

3) V. M. Alekseev, Math. USSR-Sb 5 (1968), 73; Math. USSR-Sb 6 (1968), 505; Math. USSR-Sb 7 (1969), 1.