Nordic Hydrology, 18, 1987, 111-120

Determination of Steady State Drawdowns in a Horizontal Double Aquifer System

a New Approach

V. S. Pandey, S. K. Pandey and R. N. Pandey
Banaras Hindu University, Varanasi 221005, India

Using Ditkin's operator the exact analytical expressions for drawdowns for coupled two finite aquifer system in which pumpage from the upper and lower aquifer is balanced by a reduction in evapotranspiration from the upper aquifer are obtained. Also, the expressions for drawdowns for the limiting cases (viz. when the horizontal dimension of the aquifers is considered to be infinite) are deduced from one of the general results. It is observed that these results are the same as obtained by Motz. The application of Ditkin's operator in obtaining the solution for such problems is new and to our knowledge it has not been reported elsewhere in the literature.

Introduction

Several analytical and numerical solution for coupled two aquifer systems are reported in the literature. Among some prominent researchers on this topic we mention the works of Polubarinova – Kochina (1962); Spiegel (1962); Hantush (1967); Neuman and Witherspoon (1969); Saleem and Jacob (1971, 1973, 1974); Bredehoeft and Pinder (1970); Prickett and Lonnquist (1971); Trescott (1975) and others (Hennart *et al.* 1981; Herrera and Rodarte 1972; Herrera *et al.* 1980; Javandel and Witherspoon 1980, 1983; Pikul *et al.* 1974; Rodarte 1976).

In the present paper, the authors have obtained the exact solutions in case of a coupled finite two aquifer system in which the pumpage from the upper and lower aquifers is balanced by a reduction in evapotranspiration from the upper aquifer

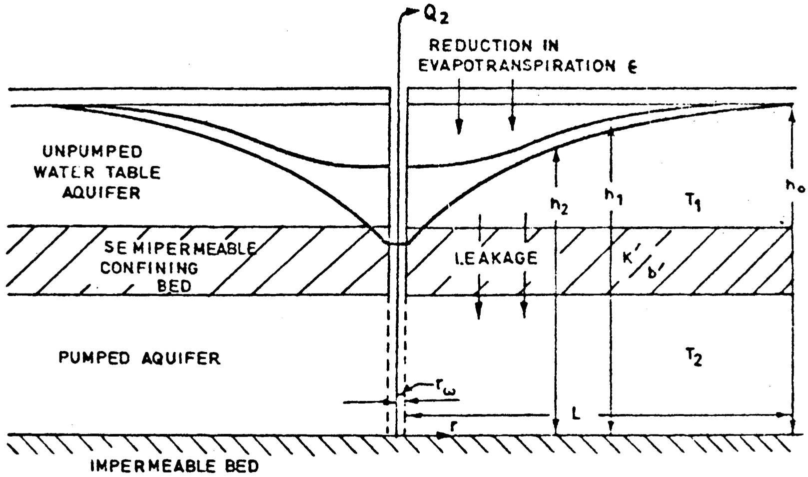

Fig. 1. Definition sketch for coupled water table and artesian aquifers.

(Fig. 1). Further, as a special case, when the dimension of the aquifer becomes sufficiently large (theoretically infinite), it has been shown that the limiting form of the general solutions are identical to Motz (1978) solution. It may be remarked that analytical solution procedure developed by Motz (1978) is quite different from our approach.

Formulation of the Problem

Pumping from an artesian aquifer can cause drawdowns in an overlying water-table aquifer when pumping is balanced by a reduction in evapotranspiration. Water pumped from the underlying artesian aquifer is obtained through an overlying aquitard; the leakage is derived from reduction in evapotranspiration caused by lowering the water table. The boundary value problem for steady radial flow in such a system is similar to Motz (1978) as

$$\begin{aligned} &\frac{d^2 s_1}{dr^2} + \frac{1}{r}\frac{ds_1}{dr} - \left(\frac{1}{B_0^2} + \frac{1}{B_1^2}\right) s_1 + \frac{1}{B_1^2} s_2 = 0 \\ &\frac{d^2 s_2}{dr^2} + \frac{1}{r}\frac{ds_2}{dr} - \frac{s_2}{B_2^2} + \frac{s_1}{B_2^2} = 0 \end{aligned} \tag{1}$$

with

$$r \left.\frac{ds_1}{dr}\right|_{r=r_w} = -\frac{Q_1}{2\pi T_1}$$

$$r \left. \frac{ds_2}{dr} \right|_{r=r_w} = - \frac{Q_2}{2\pi T_2} \qquad (2)$$

and

$$s_1(L) = s_2(L) = 0$$

where s_1, s_2 represent the drawdowns in upper and lower aquifers respectively defined as $s_1 = h_0 - h_1$, $s_2 = h_0 - h_2$; h_0 is the initial hydraulic head; h_1, h_2 are the hydraulic head in the upper and lower aquifers affected by pumping; r is the radial distance; K'/b' is the leakage of confining bed; ε is the rate at which evapotranspiration is reduced per unit of water-table drawdown in the upper water-table aquifer; T_1, T_2 are the transmissivity of aquifers; r_w is the radius of the pumping well; Q_1, Q_2 (constants) are the pumping rates of aquifers; L is the horizontal dimension of the aquifer system which is considered to be sufficiently large and

$$\frac{1}{B_0^2} = \frac{\varepsilon}{T_1}, \quad \frac{1}{B_1^2} = \frac{K'}{b' T_1} \quad \text{and} \quad \frac{1}{B_2^2} = \frac{K'}{b' T_2}$$

Method of Solution

Using the dimensionless entities

$$V_1 = \frac{s_1}{h_0}, \quad V_2 = \frac{s_2}{h_0}, \quad X = \frac{r}{L}, \quad P_i = \frac{Q_i}{2\pi T_i h_0}, \quad (i=1,2);$$

$$R_1 = L^2 \left(\frac{1}{B_0^2} + \frac{1}{B_1^2}\right), \quad R_2 = \frac{L^2}{B_1^2} \quad \text{and} \quad R_3 = \frac{L^2}{B_2^2},$$

the differential Eq. (1) and boundary conditions Eq. (2) may be put in the form

$$\begin{aligned} \frac{d^2 V_1}{dX^2} + \frac{1}{X} \frac{dV_1}{dX} - R_1 V_1 + R_2 V_2 &= 0 \\ \frac{d^2 V_2}{dX^2} + \frac{1}{X} \frac{dV_2}{dX} - R_3 V_2 + R_3 V_1 &= 0 \end{aligned} \qquad (3)$$

and

$$X \left. \frac{dV_1}{dX} \right|_{X=X_w} = - P_1, \qquad (4)$$

$$X \left. \frac{dV_2}{dX} \right|_{X=X_w} = - P_2, \qquad (5)$$

$$V_1(X) = V_2(X) = 0 \quad \text{at} \quad X = 1 \tag{6}$$

The solution of the differential Eq. (3) may be obtained by using Ditkin's operator

$$D \equiv \frac{1}{X} \frac{d}{dX} \left(X \frac{d}{dX} \right)$$

The differential Eq. (3) in terms of Ditkin's operator D can be rewritten as

$$(D-R_1)V_1 + R_2 V_2 = 0 \tag{7}$$

$$R_3 V_1 + (D-R_3) V_2 = 0 \tag{8}$$

Since the operator D is linear we can eliminate V_2 by pre-operating Eq. (7) by (D-R_3) and multiplying Eq. (8) by R_2 respectively and subtracting the latter from the former, we get

$$[D^2 - (R_1 + R_3) D + (R_1 - R_2) R_3] V_1 = 0 \quad ,$$

or,

$$[(D-n_1)(D-n_2)] V_1 = 0 \tag{9}$$

where

$$n_1 \equiv \frac{1}{2} [(R_1 + R_3) + \sqrt{(R_1+R_3)^2 - 4R_3(R_1-R_2)}]$$

$$n_2 \equiv \frac{1}{2} [(R_1 + R_3) - \sqrt{(R_1+R_3)^2 - 4R_3(R_1-R_2)}]$$

Keeping in view the definition of operator D, the solution of Eq. (9) can directly be written as

$$V_1 = A_1 I_0(\sqrt{n_1}X) + A_2 K_0(\sqrt{n_1}X) + A_3 I_0(\sqrt{n_2}X) + A_4 K_0(\sqrt{n_2}X) \tag{10}$$

Using Eq. (10) in Eq. (7), we get

$$V_2 \equiv - \frac{1}{R_2}[R_4 (A_1 I_0(\sqrt{n_1}X) + A_2 K_0(\sqrt{n_1}X)) + R_5 (A_3 I_0(\sqrt{n_2}X) + A_4 K_0(\sqrt{n_2}X))] \tag{11}$$

where

$$R_4 = n_1 = R_1 \quad , \qquad R_5 = n_2 - R_1$$

and A_i (i = 1, 2, 3, 4) are constants of integration.

It can easily be verified (see appendix) that the solutions Eqs. (10) and (11) satisfy the differential Eq. (3).

Now we analyze the solutions Eqs. (10) and (11) for two different cases as follow:

Case 1 - Determination of Drawdown Distributions by Pumping only the Upper Aquifer

If the upper aquifer is pumped we conclude that $Q_2 = 0$ (with the result $P_2 = 0$) and the boundary condition Eq. (5) reduces to

$$X \frac{dV_2}{dX}\bigg|_{X=X_w} \equiv 0 \tag{12}$$

leaving the other two boundary conditions Eqs. (4) and (6) unchanged.

Using the boundary conditions Eqs. (4), (12) and (6) in Eqs. (10) and (11) we get four simultaneous algebraic equations which on simplification provide the constants of integration A_i ($i = 1, 2, 3, 4$). Substituting these values in Eqs. (10) and (11) we get the dimensionless drawdown distributions as

$$V_1 = \frac{P_1}{X_w(n_1-n_2)}\Big[\frac{R_5}{\sqrt{n_1}}\Big\{\frac{I_0(\sqrt{n_1}X)K_0(\sqrt{n_1})-I_0(\sqrt{n_1})K_0(\sqrt{n_1}X)}{I_0(\sqrt{n_1})K_1(\sqrt{n_1}X_w)+I_1(\sqrt{n_1}X_w)K_0(\sqrt{n_1})}\Big\}$$
$$- \frac{R_4}{\sqrt{n_2}}\Big\{\frac{I_0(\sqrt{n_2}X)K_0(\sqrt{n_2})-I_0(\sqrt{n_2})K_0(\sqrt{n_2}X)}{I_1(\sqrt{n_2}X_w)K_0(\sqrt{n_2})+I_0(\sqrt{n_2})K_1(\sqrt{n_2}X_w)}\Big\}\Big] \tag{13}$$

and

$$V_2 = \frac{P_1R_3}{X_w(n_1-n_2)}\Big[\frac{1}{\sqrt{n_1}}\Big\{\frac{I_0(\sqrt{n_1}X)K_0(\sqrt{n_1})-I_0(\sqrt{n_1})K_0(\sqrt{n_1}X)}{I_0(\sqrt{n_1})K_1(\sqrt{n_1}X_w)+I_1(\sqrt{n_1}X_w)K_0(\sqrt{n_1})}\Big\}$$
$$- \frac{1}{\sqrt{n_2}}\Big\{\frac{I_0(\sqrt{n_2}X)K_0(\sqrt{n_2})-I_0(\sqrt{n_2})K_0(\sqrt{n_2}X)}{I_1(\sqrt{n_2}X_w)K_0(\sqrt{n_2})+I_0(\sqrt{n_2})K_1(\sqrt{n_2}X_w)}\Big\}\Big] \tag{14}$$

where I_1 and K_1 are the modified Bessel functions of the first and second kind and of order one.

Case 2 • Determination of Drawdown Distributions by Pumping the Lower Aquifer

Now we assume that the lower aquifer is being pumped which obviously imply that $Q_1 = 0$ (i.e. $P_1 = 0$). Again, it is understood that the two boundary conditions Eqs. (5), (6) remain unchanged while Eq. (4) takes the form

$$X \frac{dV_1}{dX}\bigg|_{X=X_w} \equiv 0 \tag{15}$$

For the sake of clarity in the form of dimensionless drawdown distributions in contrast to case 1 we denote V_1^* and V_2^* for drawdowns in upper and lower aquifers respectively. It may further be noticed that the forms of the differential equations and boundary conditions alongwith dimensionless parameters remain the same, as in case 1.

Proceeding exactly in the same way as in case 1 subject to boundary conditions

Eqs. (5), (6) and (15), we get the dimensionless drawdown distributions as

$$V_1^* = \frac{P_2 R_2}{X_w (n_1 - n_2)} \left[\frac{1}{\sqrt{n_1}} \left\{ \frac{I_0(\sqrt{n_1}X)K_0(\sqrt{n_1}) - I_0(\sqrt{n_1})K_0(\sqrt{n_1}X)}{I_0(\sqrt{n_1})K_1(\sqrt{n_1}X_w) + I_1(\sqrt{n_1}X_w)K_0(\sqrt{n_1})} \right\} - \frac{1}{\sqrt{n_2}} \left\{ \frac{I_0(\sqrt{n_2}X)K_0(\sqrt{n_2}) - I_0(\sqrt{n_2})K_0(\sqrt{n_2}X)}{I_1(\sqrt{n_2}X_w)K_0(\sqrt{n_2}) + I_0(\sqrt{n_2})K_1(\sqrt{n_2}X_w)} \right\} \right] \quad (16)$$

and

$$V_2^* = -\frac{P_2}{X_w (n_1 - n_2)} \left[\frac{R_4}{\sqrt{n_1}} \left\{ \frac{I_0(\sqrt{n_1}X)K_0(\sqrt{n_1}) - I_0(\sqrt{n_1})K_0(\sqrt{n_1}X)}{I_0(\sqrt{n_1})K_1(\sqrt{n_1}X_w) + I_1(\sqrt{n_1}X_w)K_0(\sqrt{n_1})} \right\} - \frac{R_5}{\sqrt{n_2}} \left\{ \frac{I_0(\sqrt{n_2}X)K_0(\sqrt{n_2}) - I_0(\sqrt{n_2})K_0(\sqrt{n_2}X)}{I_1(\sqrt{n_2}X_w)K_0(\sqrt{n_2}) + I_0(\sqrt{n_2})K_1(\sqrt{n_2}X_w)} \right\} \right] \quad (17)$$

Discussion and Conclusion

We find that for $Q_1 = Q_2 = Q$, the drawdown distributions in the upper aquifer when lower aquifer is pumped is equal to the drawdown distribution in the lower aquifer when upper aquifer is pumped, i.e., $V_2 = V_1^*$. Thus we conclude that the reciprocity principle holds good. This is clear from the expressions for V_2 and V_1^* keeping in mind the relation $P_2R_2 = P_1R_3$ which is derived as follows

$$P_2R_2 \equiv \frac{Q}{2\pi T_2 h_0} \frac{L^2}{B_1^2} \equiv \frac{Q}{2\pi T_2 h_0} \frac{L^2 K'}{b' T_1} = \frac{Q}{2\pi T_1 h_0} \frac{L^2 K'}{b' T_2} \equiv P_1 R_3$$

When the horizontal dimension of the aquifer system becomes the characteristic length, the quantity $X = r/L$ tends to infinity as $r \rightarrow \infty$. Also, if the radius of the pumping well r_w tends to zero it is obvious that $X_w \rightarrow 0$. Under these conditions, the limiting forms of the solutions for both the cases may be obtained from Eqs. (10) and (11). Here

$$I_0(\sqrt{n_i}X) \rightarrow \infty \text{ as } X \rightarrow \infty \text{ and } K_1(\sqrt{n_i}X_w) \simeq \frac{1}{\sqrt{n_i}X_w} \text{ as } X_w \rightarrow 0 \ ; \ (i = 1, 2)$$

Taking into account these considerations, we have the solution for both the cases as

$$V_1 \simeq \frac{-P_1}{(n_1 - n_2)} (R_5 K_0(\sqrt{n_1}X) - R_4 K_0(\sqrt{n_2}X)) \quad (18)$$

$$V_2 \simeq -\frac{P_1 R_3}{(n_1 - n_2)} (K_0(\sqrt{n_1}X) - K_0(\sqrt{n_2}X)) \quad (19)$$

$$V_1^* \simeq -\frac{P_2 R_2}{(n_1-n_2)} (K_0(\sqrt{n_1}X) - K_0(\sqrt{n_2}X)) \quad (20)$$

and

$$V_2^* \simeq \frac{P_2}{(n_1-n_2)} (R_4 K_0(\sqrt{n_1}X) - R_5 K_0(\sqrt{n_2}X)) \quad (21)$$

Again, it is observed that the reciprocity principle holds good, i.e., $V_2 \equiv V_1^*$.

It is interesting to note that the limiting solutions Eqs. (18) - (21) are identical to that obtained by Motz (1978). However the method of solution developed by Motz is quite different from our approach.

The use of Ditkin's operator has also been made in determining drawdowns for other coupled aquifer systems (Pandey 1985).

Acknowledgement

The authors are thankful to Dr. Nikhilesh Roy and Dr. V. B. Mishra, Professors of Aquifer Hydrology, Department of Civil Engineering, Institute of Technology, Banaras Hindu University for helpful suggestions and interpretations. Also, one of the authors (V. S. Pandey) is thankful to C.S.I.R. New Delhi for providing financial assistance.

Notations

- ε – rate of which evapotranspiration is reduced per unit of water table drawdown in the upper aquifer
- r – radial distance
- r_w – radius of the pumping well
- Q_1, Q_2 – pumping rates of aquifers
- L – horizontal dimension of the aquifer system
- X – dimensionless space variable
- X_w – dimensionless radius of the pumping well
- P_1, P_2 – dimensionless pumping rates of the aquifers
- D – Ditkin's operator
- I_0, K_0 – modified Bessel functions of the first and second kind and of index zero
- I_1, K_1 – modified Bessel functions of the first and second kind and of index one.

References

Bredehoeft, J. D., and Pinder, G. F. (1970) Digital analysis of a real flow in multiaquifer groundwater systems: a quasi-three-dimensional model, *Water Resour. Res., Vol. 6(3),* pp 883-888.

Hantush, M. S. (1967) Flow to wells in aquifers separated by a semipervious layer, *J. Geophys. Res., Vol. 72(6),* pp 1709-1720.

Hennart, J. P., Yates, R. and Herrera, I. (1981) Extension of the integrodifferential approach to inhomogeneous multiaquifer systems, *Water Resour. Res., Vol. 17(4),* pp 1044-1050.

Herrera, I., and Rodarte, L. (1972) Integrodifferential equations for systems of leaky aquifers and applications, 1, the nature of approximate theories, *Water Resour. Res., Vol. 9(4),* pp 995-1005.

Herrera, I., Hennart, J. P., and Yates, R. (1980) A critical discussion of numerical models for multiaquifer systems, *Advance Water Resour., Vol. 3,* pp 159-163.

Javandel, I., and Witherspoon, P. A. (1980) A semi-analytical solution for partial penetration in two-layer aquifers, *Water Resour. Res., Vol. 16(6),* pp 1099-1106.

Javandel, I., and Witherspoon, P. A. (1983) Analytical solution of a partially penetrating well in a two-layer aquifer, *Water Resour. Res., Vol. 19(2),* pp 567-577.

Motz, L. H. (1978) Steady-state drawdowns in coupled aquifers, *J. Hydraul. Div. Amer. Soc. Civil Eng., Vol. 104(HY7),* pp 1061.

Neuman, S. P., and Witherspoon, P. A. (1969) Transient flow of groundwater to wells in multiple aquifer systems, Geotech. Eng. Rept., 69, 1, University of California, Berkeley.

Pandey, V. S. (1985) Some Problems in Subsurface Flow, Ph.D. Thesis, Banaras Hindu University, India.

Pikul, M. F., Street, R. L., and Remson, I. (1974) A numerical model based on coupled one-dimensional Richards and Boussinesq equations, *Water Resour. Res., Vol. 10(2),* pp 295-302.

Polubarinova – Kochina, P. Ya. (1962) Theory of Ground Water Movement, Princeton, Princeton University Press, New York.

Prickett, T. A., and Lonnquist, C. G. (1971) Selected digital computer Techniques, Bulletin 55, Illinois State Water Survey, Urbana, III.

Rodarte, L. (1976) Theory of multiple leaky aquifers, 1, the integrodifferential and differential equations for small and large values of time, *Water Resour. Res., Vol. 12(2),* pp 163-170.

Saleem, Z. A., and Jacob, C. E. (1971) Optimal use of coupled leaky aquifers, *Water Resour. Res., Vol. 7(2),* pp 382-393.

Saleem, Z. A., and Jacob, C. E. (1973) Drawdown distribution due to well fields in coupled leaky aquifers, 1, infinite aquifer system, *Water Resour. Res., Vol. 9(6),* pp 1671-1678.

Saleem, Z. A., and Jacob, C. E. (1974) Drawdown distribution due to well fields in coupled leaky aquifers, 2, finite aquifer system, *Water Resour. Res., Vol. 10(2)* pp 336-342.

Spiegel, Z. (1962) Hydraulics of certain stream – connected aquifer systems, Special Report, New Mexico State Engineer, Sante Fe, N.M.

Trescott, P. C. (1975) Documentation of finite difference model for simulation of three-dimensional groundwater flow, Open File Report, 74-438, U.S. Geological Survey, Washington, D.C.

Appendix

To show that V_1 is a solution of Eq. (9) we have

$$[(D-n_1)(D-n_2)]V_1 = (D-n_1)[(D-n_2)\{A_1 I_0(\sqrt{n_1}X) + A_2 K_0(\sqrt{n_1}X) + A_3 I_0(\sqrt{n_2}X) + A_4 K_0(\sqrt{n_2}X)\}] \quad (23)$$

In view of the relations

$$z I_1'(z) + I_1(z) \equiv z I_0(z)$$

and

$$z K_1'(z) + K_1(z) \equiv - z K_0(z)$$

we have

$$D[I_0(\sqrt{n}X)] = n I_0(\sqrt{n}X)$$

and

$$D[K_0(\sqrt{n}X)] \equiv [n K_0(\sqrt{n}X)]$$

Using these expressions in the right hand side of Eq. (23) it becomes

$$L.H.S. = (D-n_1)\ [(n_1-n_2)\{A_1 I_0(\sqrt{n_1}X) + A_2 K_0(\sqrt{n_1}X)\}]$$

since

$$(D-n_2)[A_3 I_0(\sqrt{n_2}X) + A_4 K_0(\sqrt{n_2}X)] = 0$$

Again,

$$(D-n_1)\ [(n_1-n_2)\{A_1 I_0(\sqrt{n_1}X) + A_2 K_0(\sqrt{n_1}X)\}]$$
$$= (n_1-n_2)\ [(D-n_1)\{A_1 I_0(\sqrt{n_1}X) + A_2 K_0(\sqrt{n_2}X)\}]$$
$$= 0$$

which shows that V_1 satisfies the differential Eq. (23).

Using the expressions V_1, V_2 from Eqs. (10), (11) in the differential Eq. (3) we get

$$\frac{d^2 V_1}{dX^2} + \frac{1}{X}\frac{dV_1}{dX} - R_1 V_2 + R_2 V_2$$

$$= n_1 [A_1 I_0(\sqrt{n_1}X) + A_2 K_0(\sqrt{n_1}X)] + n_2 [A_3 I_0(\sqrt{n_2}X) + A_4 K_0(\sqrt{n_2}X)]$$
$$- R_1 [A_1 I_0(\sqrt{n_1}X) + A_2 K_0(\sqrt{n_1}X) + A_3 I_0(\sqrt{n_2}X) + A_4 K_0(\sqrt{n_2}X)]$$
$$- R_4 [A_1 I_0(\sqrt{n_1}X) + A_2 K_0(\sqrt{n_1}X)] - R_5 [A_3 I_0(\sqrt{n_2}X) + A_4 K_0(\sqrt{n_2}X)]$$

$$= (n_1 - R_1 - R_4)[A_1 I_0(\sqrt{n_1}X) + A_2 K_0(\sqrt{n_1}X)]$$

$$+ (n_2 - R_1 - R_5)[A_3 I_0(\sqrt{n_2}X) + A_4 K_0(\sqrt{n_2}X)]$$

$$= 0 \qquad \text{(since } R_4 = n_1 - R_1,\ R_5 = n_2 - R_1\text{)}$$

Similarly it can be shown that V_1 and V_2 satisfy the Eq. $(3)_2$.

Received: 29 September, 1986

Address:
New F/6, Jodhpur Colony,
Banaras Hindu University,
Varanasi 221005,
India.