Perturbative Uniqueness of Black Holes near the Static Limit in All Dimensions

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(Received April 7, 2004)

The behaviour of stationary gravitational perturbations is studied for generalised static black holes in spacetimes of greater than three dimensions, using the formulation developed by the present author and Ishibashi. For the case in which the horizon has a spatial section with constant curvature, it is proved that irrespective of the value of the cosmological constant $\Lambda$, there exists no stationary perturbation that is regular at the horizon(s) and falls off at infinity in the case $\Lambda \leq 0$, except for those corresponding to the stationary rotation of black holes and the variation of the background parameters. This result indicates that regular neutral black hole solutions that are either asymptotically flat, de Sitter or anti-de Sitter can be parametrised by mass, (multiple component) angular momentum and $\Lambda$ near the spherically symmetric and static limit. A similar conclusion is obtained for topological black holes. It is also pointed out that this perturbative uniqueness near the static limit may not hold in the case in which the horizon geometry is described by a generic Einstein space with non-constant sectional curvature. Further, non-uniqueness in the asymptotically anti-de Sitter case under a weaker boundary condition at infinity related to the AdS/CFT argument is discussed.

\section{Introduction}

In recent years, inspired by proposals of various higher-dimensional universe models,\textsuperscript{1}–\textsuperscript{9)} the investigation of black holes in higher dimensions has become quite active. One of the main issues in these investigations is whether the celebrated black hole uniqueness theorem in four dimensions\textsuperscript{10}–\textsuperscript{12)} also holds in higher dimensions.

Results obtained to this time concerning this issue are quite intriguing.\textsuperscript{13)} On one hand, it has been shown that Israel’s theorem on the rigidity and uniqueness of static black holes also holds in higher dimensions, that is, the asymptotically flat and static regular black hole solutions are always spherically symmetric for the vacuum system.\textsuperscript{14, 15)} This rigidity theorem has been extended to the electro-vacuum and Einstein-Maxwell-Dilaton systems with non-degenerate horizons.\textsuperscript{16, 17)}

On the other hand, concerning rotating black holes, we now have two families of asymptotically flat regular vacuum solutions with different horizon topologies in five dimensions, the Myers-Perry solution\textsuperscript{18)} and the Emparan-Reall black ring solution.\textsuperscript{19)} The former is a higher-dimensional analogue of the Kerr solution, and its black hole surface is diffeomorphic to a sphere, while the latter has no analogue in four dimensions and its horizon has a spatial section diffeomorphic to $S^2 \times S^1$. Thus, the uniqueness theorem is violated for rotating black holes. Recently, it has also been shown that a new family of black ring solutions violating the uniqueness...
appears when the system is coupled with form fields,\(^{20}\) although the uniqueness holds for supersymmetric black holes in the five-dimensional minimal \(N = 1\) supergravity theory.\(^{21}\) Further, Emparan and Myers have pointed out that the Myers-Perry solutions in greater than five dimensions may suffer from a Gregory-Laflamme type instability for high angular momenta and argued that there may exist a new family of rotating solutions that bifurcates from the Myers-Perry family maintaining the symmetry.\(^{22}\) Such a symmetry preserving branch does not exist in five dimensions, as shown by Morisawa and Ida.\(^{23}\)

Along with these results in the asymptotically flat case, non-uniqueness has also been observed in Kaluza-Klein-type spacetimes. For example, Kudoh and Wiseman numerically constructed a family of non-uniform black string solutions in six dimensions.\(^{24}-^{26}\) This family bifurcates from the uniform black string solution at the Gregory-Laflamme critical point. As discussed by Horowitz and Maeda,\(^{27}\) this family cannot be smoothly connected to a family of localised black hole solutions with spherical horizon. In fact, Wiseman numerically found that these two families bifurcate at a singular solution with a conic horizon,\(^{28}\) confirming Kol’s conjecture.\(^{29}\)

One common feature of these new families of solutions giving rise to non-uniqueness is that they do not have a static and spherically symmetric limit. This feature, together with the uniqueness of asymptotically flat static black holes, strongly indicates that the violation of uniqueness in higher dimensions occurs only when the black horizon is significantly deformed due to high angular momentums or has a non-trivial topology. The main purpose of the present paper is to confirm this observation by searching for regular stationary perturbations of static and maximally symmetric black hole solutions. To be precise, we prove the following theorem.

**Theorem 1.** For any spherically symmetric vacuum solution that represents a regular black hole spacetime of dimension \(d(\geq 4)\), a scalar-type perturbation corresponding to a variation of the black hole mass and vector-type perturbations representing rotation of the black hole are the only stationary bounded perturbations that are regular at and outside the horizon (and fall off at infinity in the case \(\Lambda \leq 0\)). These exceptional vector perturbations can be parametrised by \([(d - 1)/2]\) parameters after identification with background isometries, irrespective of the value of the cosmological constant \(\Lambda\), where \([(d - 1)/2]\) is the rank of \(\text{SO}(d - 1)\). A similar result holds for a regular static topological black hole spacetime for which a spatial section of the horizon is an \(n\)-dimensional compact space with non-positive constant curvature \(K\). The only difference between the two cases is the number of degrees of freedom of the exceptional vector perturbation, which is \(d - 2\) for \(K = 0\) and zero for \(K < 0\).

For the asymptotically constant-curvature vacuum case, this theorem implies that the Myers-Perry solution for \(\Lambda = 0\) and the Gibbons-Lü-Page-Pope solution\(^{30}\) for \(\Lambda \neq 0\) are the only regular black hole solutions near the static and spherically symmetric limit. Further, it shows that a similar perturbative uniqueness holds for topological black holes as well. Because no uniqueness theorem has been proved for asymptotically de Sitter black holes in four dimensions, this theorem also has a non-trivial implication for the four-dimensional case.

The present paper is organised as follows. In the next section, we give a brief
summary of a gauge-invariant formulation for stationary perturbations of generalised static vacuum black holes. The basic equations given there are basically specialisations to the case of stationary perturbations of the equations for generic perturbations derived in Refs. 31)–34), but some are new. In order for solutions to these basic equations to represent regular perturbations of black hole spacetimes, an appropriate boundary condition at the horizon and an asymptotic condition at infinity should be satisfied. These boundary conditions are specified in §3, and it is shown that the exceptional vector perturbations indeed do satisfy them. Then, in the subsequent three sections, it is proved that, except for the exceptional vector perturbations and the trivial scalar perturbations corresponding to variations of the background parameters, there exists no stationary solution satisfying the boundary conditions to the basic equations for the asymptotically flat, de Sitter and anti-de Sitter cases, respectively. Section 7 is devoted to concluding remarks.

§2. Basic equations for stationary perturbations

Our analysis of black hole perturbations fully utilises the gauge-invariant formulation developed in Refs. 31), 32), 35) and 34). In this section, we briefly summarise the basic concepts of that formulation and give master equations for stationary perturbations.

2.1. Background spacetime

In the present paper, we consider the background spacetime whose metric has the form

\[ ds^2 = g_{ab}(y)dy^a dy^b + r^2(y)d\sigma_n^2, \quad (2.1) \]

where \( g_{ab}(y)dy^a dy^b \) is the static metric of a two-dimensional spacetime \( N^2 \), and \( d\sigma_n^2 = \gamma_{ij}(z)dz^idx^j \) is the metric of an \( n \)-dimensional complete, compact Einstein space \( K^n \) whose Ricci curvature \( \hat{R}_{ij} \) satisfies the condition

\[ \hat{R}_{ij} = (n - 1)K\gamma_{ij} \quad (2.2) \]

with \( K = 0, \pm 1 \). Although perturbative uniqueness can be proved only in the case in which \( K^n \) has a constant curvature, most analysis in the present paper is done without assuming this condition, in order to see what happens when \( K^n \) does not have a constant curvature.

We assume that this metric satisfies the vacuum Einstein equations for \( (n + 2) \)-dimensional spacetimes and represents a black hole. Then, the two-dimensional metric \( g_{ab} \) is given by

\[ g_{ab}(y)dy^a dy^b = -f(r)dt^2 + \frac{dr^2}{f(r)}, \quad (2.3) \]

where

\[ f(r) = K - \left( \frac{r_0}{r} \right)^{n-1} - \lambda r^2; \quad \lambda = \frac{2\Lambda}{n(n + 1)}. \quad (2.4) \]
This metric describes a regular black hole spacetime only when $\lambda$ satisfies the condition \(^34\), \(^\ast\)

$$\lambda < \begin{cases} \lambda_c^2 := \frac{n-1}{n+1} \frac{n+1}{n-1} r_0^{-2} & \text{for } K = 1, \\ 0 & \text{for } K = 0, -1. \end{cases} \quad (2.5)$$

Under this condition, in general, the region outside the black hole corresponds to the range $r_h < r < \infty$, where $r_h$ is the horizon radius, which is given by the (smallest positive) solution to

$$f(r_h) = 0 \iff \lambda = \frac{1}{r_h^2 \left[ K - \left( \frac{r_0}{r_h} \right)^{n-1} \right]}.$$

(2.6)

When we study the behaviour of perturbations, we consider this entire range for $\lambda \leq 0$. For $\lambda > 0$, however, we consider only the region $r_h < r < r_c$, where $r_c$ is the radius of the cosmological horizon given by the larger positive root of $f(r) = 0$, because the perturbative uniqueness of asymptotically de Sitter black holes can be proved only by looking at the behaviour of perturbations in this region.

In this black hole background, the linearised Einstein equations can be reduced to simple master equations if we decompose the metric perturbation $\delta g_{\mu\nu} = h_{\mu\nu}$ into tensor-type, vector-type and scalar-type components, according to their transformation behaviour as tensors on $K^n$. \(^32\) Now, we present these equations for the case of stationary perturbations.

### 2.2. Tensor perturbations

It is convenient to expand tensor perturbations in terms of the eigentensors satisfying

$$\hat{\triangle} L T_{ij} = (k_T^2 + 2nK) T_{ij}; \quad T^i_i = 0, \quad \hat{D}^j T_{ij} = 0,$$  \quad (2.7)

where $\hat{\triangle} L$ is the Lichnerowicz operator

$$\hat{\triangle} L h_{ij} = -\hat{D} \cdot \hat{D} h_{ij} - 2 \hat{R}_{ikjl} h^{kl} + 2(n-1) K h_{ij}.$$  \quad (2.8)

Here, $\hat{D}_i$ and $\hat{R}_{ijkl}$ represent the covariant derivative and the curvature tensor with respect to the metric $\gamma_{ij}$ on $K^n$, respectively. When $K^n$ is a constant curvature space, $\hat{\triangle} L$ can be written simply as

$$\hat{\triangle} L = -\hat{D} \cdot \hat{D} + 2nK.$$  \quad (2.9)

Hence, $k_T^2$ corresponds to the eigenvalue of the operator $-\hat{D} \cdot \hat{D}$ on second-rank symmetric tensors. In this case, $k_T^2$ takes discrete and positive values, except in the case $K = 0$. In particular, for $K^n = S^n$, its spectrum is given by

$$k_T^2 = l(l + n - 1) - 2, \quad l = 2, 3, \ldots.$$  \quad (2.10)

In contrast, for $K = 0$, i.e., when $K^n$ is a torus $T^n$, the smallest eigenvalue is zero, and the corresponding eigentensors are given by trace-free constant symmetric matrices.

\(^\ast\) For the interpretation of this solution in the case $K \leq 0$ as a topological black hole, see Refs. 36) and 37).
There exist \(n(n + 1)/2 - 1\) such matrices, and these matrices represent variations of the moduli parameters of \(T^n\).

In general, the moduli degrees of freedom of a constant curvature space correspond to the tensor harmonics satisfying \(k_T^2 = -2K\). Hence, these constant matrices exhaust all moduli degrees of freedom for \(K = 0\), and there exists no moduli degree of freedom for \(K = 1\). Further, from the integral identity

\[
2 \int_{K^n} d\Omega^n \hat{D}_i T^i j k \hat{D}^j [i T] k = (k_T^2 + nK) \int_{K^n} d\Omega^n T_{jk} T^{jk} \geq 0, \tag{2.11}
\]

we obtain

\[
k_T^2 \geq -nK = n \tag{2.12}
\]

for \(K = -1\). Hence, for \(K = -1\), there exist moduli degrees of freedom only when \(n = 2\), in accordance with Mostow’s rigidity theorem.\(^{38)–42}\) Note that these tensor harmonics satisfying \(k_T^2 = -2K\) are the only non-trivial tensor harmonics for \(n = 2\).

Through the expansion in terms of these tensor harmonics, each mode of a stationary tensor perturbation can be expressed in terms of a gauge-invariant variable \(H_T(r)\) as

\[
h_{ab} = 0, \ h_{ai} = 0, \ h_{ij} = 2r^2 H_T(r) T_{ij}. \tag{2.13}
\]

Note that stationary tensor perturbations are always static.

The linearised Einstein equations are reduced to\(^{32}\)

\[
\left[ f(r^{n/2} H_T)’ \right]' = r^{n/2 - 2} U_T H_T, \tag{2.14}
\]

where the prime denotes differentiation with respect to \(r\), and

\[
U_T = \frac{n(n + 2)}{4} f(r) + \frac{n(n + 1)}{2} \left( \frac{r_0}{r} \right)^{n-1} + k_T^2 - (n - 2)K. \tag{2.15}
\]

Note that for the modes satisfying \(k_T^2 = -2K\), \(H_T = \text{const}\) is always a solution of this equation and represents a perturbation corresponding to the variation of the background metric with respect to its moduli parameters.

### 2.3. Vector perturbations

Stationary vector perturbations can be expanded in terms of the vector harmonics satisfying

\[
(\hat{D} \cdot \hat{D} + k_V^2) V_i = 0; \quad \hat{D}_j V^j = 0 \tag{2.16}
\]

as

\[
h_{ab} = 0, \quad h_{ai} = r f_a(r) V_i, \quad h_{ij} = 2r^2 H_T(r) V_{ij}, \tag{2.17}
\]

where

\[
V_{ij} := -\frac{1}{2k_V} (\hat{D}_i V_j + \hat{D}_j V_i). \tag{2.18}
\]

The eigenvalue \(k_V^2\) takes discrete and positive values, except when \(K = 0\), and \(k_V^2 \geq n - 1\) for \(K = 1\).\(^{34}\) In particular, for \(K^n = S^m\), the spectrum of \(k_V^2\) is given by

\[
k_V^2 = l(l + n - 1) - 1, \quad l = 1, 2, \ldots. \tag{2.19}
\]
In contrast, when $K = 0$ and $K^n$ is expressed as $K^n = T^p \times K'$, where $K'$ is a Ricci flat space with no isometry, there exist $p$ independent covariantly constant vector fields providing harmonic vectors with $k^2_T = 0$. These modes and those with $k^2_T = n - 1$ for $K = 1$ comprise the exceptional modes discussed below.

First, for generic modes satisfying $k^2_V > (n - 1)K$, the linearised Einstein equations can be written in terms of the gauge-invariant variables

$$F_a := f_a + \frac{r}{v} D_a H_T$$

as

\begin{align}
D_a (r^{n+1} F^{(1)}) - m_V r^{n-1} \epsilon_{ab} F^b &= 0, \\
D_a (r^{n-1} F^a) &= 0,
\end{align}

where

\begin{align}
F^{(1)} &= \epsilon^{ab} r D_a \left( \frac{f_b}{r} \right) = \epsilon^{ab} r D_a \left( \frac{F_b}{r} \right), \\
m_V &= k^2_V - (n - 1)K.
\end{align}

From the $a = t$ component of (2.21a), we find that $F^r = 0$ for stationary perturbations, and (2.21b) becomes trivial. Hence, the Einstein equations are equivalent to

\begin{align}
\left( r^{n+1} F^{(1)} \right)' + m_V r^{n-1} F^t &= 0, \\
F^{(1)} &= r \left( \frac{F_t}{r} \right)'.
\end{align}

Note that stationary vector perturbations are not static.

Next, for exceptional modes with $k^2_V = (n - 1)K$, $V_{ij}$ vanishes.\(^{34}\) (The factor $1/k_V$ in the definition of $V_{ij}$ is introduced just for convenience and is not essential.) For these modes, $H_T$ does not appear, and perturbations are described by $f_a$ alone. Because $f_a$ transforms under the gauge transformation $\delta z^i = L V^i$ as $\delta f_a = -r D_a L$, $f_r$ can be set to zero through such a gauge transformation. In this gauge, the Einstein equations are given by (2.24) with $m_V = 0$ and $F^t = f^t$. This equation can be easily solved, and the general solution is given by

$$F^{(1)} = r \left( \frac{f_t}{r} \right)' = -\frac{J}{r^{n+1}}.$$  

From this, we have

$$f_t = \frac{J}{n+1} \frac{1}{r^n} + Cr,$$

where $C$ is an integration constant.

The gauge condition $f_r = 0$ does not remove the gauge degree of freedom completely and leaves a residual gauge freedom such that $L = L(t)$. If we require that $f_t$
be independent of time, \( L(t) \) is restricted to the form \( L(t) = C't \), where \( C' \) is a constant. Under this gauge transformation, the above integration constant \( C \) changes to \( C - C' \). Hence, we can set this constant to zero through a gauge transformation. However, as discussed in the next section, this residual gauge degree of freedom plays an important role in proving the regularity of the above solution. In any case, this argument shows that the physical degrees of freedom of these exceptional modes can be parametrised by the single constant \( J \) for each mode. As shown in Appendix A, we can regard each of these modes as representing a rotational perturbation of a static black hole, and the parameter \( J \) represents the total angular momentum of the perturbation.

Because the vector harmonics satisfying \( k_1^2 = (n - 1)K \) are in one-to-one correspondence with the Killing vector fields of \( K^n \), these solutions form a linear space isomorphic to the linear space of Killing vector fields of \( K^n \). However, they are not all physically distinct, because two solutions related by an isometry of the background spacetime must be considered physically equivalent. In the case in which \( K^n = S^n \) and the orientation-preserving spatial isometry group is given by \( SO(n + 1) \), the Killing vectors are in one-to-one correspondence with the antisymmetric matrices of rank \( n + 1 \), and the transformation of a Killing vector by an isometry is mapped to a conjugate transformation of the corresponding antisymmetric matrix by an element of \( SO(n + 1) \). Because the conjugate classes of these anti-symmetric matrices are classified according to their \( [(n + 1)/2] \) eigenvalues, where \( [(n + 1)/2] \) is the rank of \( SO(n + 1) \), physically distinct exceptional modes are classified with respect to \( [(n + 1)/2] \) constants. Thus, they have exactly the same number of degrees of freedom as that of the angular momentum parameters of the Myers-Perry solution\(^{18}\) and the Gibbons-Lü-Page-Pope solution\(^{30}\). In fact, we can directly check that these exceptional modes can be obtained by expanding these solutions with respect to the angular momentum parameters. Note that when \( K^n \) is distinct from \( S^n \), the parameter \( J \) does not represent the angular momentum in the standard sense, although it is still related to a conserved quantity of the system. The number of physical degrees of freedom of the exceptional modes can differ from \( [(n + 1)/2] \) in that case.

2.4. Scalar perturbations

Scalar perturbations can be expanded in terms of harmonic functions satisfying

\[
(\hat{D} \cdot \hat{D} + k^2)S = 0
\]  (2.27)

and the vector and tensor harmonics derived from them,

\[
S_i = -\frac{1}{k} \hat{D}_i S, \quad S_{ij} = \frac{1}{k^2} \hat{D}_i \hat{D}_j S + \frac{1}{n} \gamma_{ij} S.
\]  (2.28)

Because we assume that \( K^n \) is compact, \( k^2 \) takes discrete values starting from zero, and, in particular for \( K^n = S^n \), its spectrum is given by

\[
k^2 = l(l + n - 1), \quad l = 0, 1, 2, \cdots.
\]  (2.29)

The above definitions of \( S_i \) and \( S_{ij} \) become meaningless for \( k^2 = 0 \). Because the harmonic function for \( k^2 = 0 \) is constant, the corresponding perturbation merely
represents a change of the background metric with respect to a variation of the
mass parameter. Thus, this is a trivial perturbation with respect to the uniqueness
problem. Therefore, we only consider modes with $k^2 > 0$ from this point, unless
otherwise stated.

In addition to $k^2 = 0$, harmonics with $k^2 = n$ for $K = 1$ are also exceptional,
because $S_{ij}$ vanishes for such harmonic functions. Modes corresponding to such
harmonics are gauge modes as shown in Ref. 32), and we do not consider such
modes. This implies that $k^2 > n$ can be assumed when $K = 1$, because for $K = 1$,
the second smallest eigenvalue of $k^2$ is greater than or equal to $n$.43)

Ignoring these exceptional modes, stationary scalar perturbations of the metric
can be expanded in terms of the harmonics as

$$h_{ab} = f_{ab}(r)S, \ h_{ai} = r f_a(r)S_i, \ h_{ij} = 2r^2 (H_L(r)\gamma_{ij}S + H_T(r)S_{ij}). \quad (2.30)$$

We adopt the following combinations as a basis for gauge-invariant quantities con-
structed from these expansion coefficients:32)

$$F := H_L + \frac{1}{n}H_T + \frac{f}{r}X_r, \quad (2.31a)$$

$$F_t^t := f_t^t + f'X_r, \quad (2.31b)$$

$$F_r^r := f_r^r + 2fX_t + f'X_r, \quad (2.31c)$$

$$F_r^t := f_t^r - f'X_t, \quad (2.31d)$$

where

$$X_r := \frac{r}{k}f_r + \frac{r^2}{k^2}H_T', \quad X_t := \frac{r}{k}f_t. \quad (2.32)$$

As shown in Ref. 32), the linearised Einstein equations for stationary scalar
perturbations can be written in terms of these gauge-invariant variables as

$$F_r^r + F_t^t = -2(n - 2)F, \quad (2.33a)$$

$$F_t^t = 0, \quad (2.33b)$$

$$f'X' = \left(\frac{2(n - 1)}{r^2}(f - K) + \frac{4(n + 1)\lambda}{n} + \frac{2f'}{r} - \frac{(f')^2}{2f} + \frac{2f''}{n}\right)X$$

$$- \left(\frac{2(f - K)}{r^2} + \frac{4(n^2 - 1)}{n}\lambda + \frac{2(n - 1)}{r}f' - \frac{(f')^2}{2f}
+ \frac{2(n - 1)f'' - 2(2K - nK)}{r^2}\right)Y, \quad (2.33c)$$

$$Y' = \frac{f'}{2f}(X - Y), \quad (2.33d)$$

where

$$X := r^{n-2}(F_t^t - 2F), \quad Y := r^{n-2}(F_r^r - 2F). \quad (2.34)$$

As discussed in Ref. 32), we can reduce this set of equations to a second-order
ODE in various ways. For example, the master variable $\Phi$ introduced in Ref. 32)
for generic perturbations can be written as a complicated linear combination of $X$
and $Y$ and satisfies a second-order ODE for stationary perturbations. However, this equation is not useful in the analysis of stationary perturbations, because its effective potential is not positive definite in the case of a non-vanishing cosmological constant for generic values of $n$. Further, it is not easy to determine the asymptotic behaviour of $F$ and $F^a_b$, because their expressions in terms of $Φ$ are rather complicated. Therefore, in the present paper, we utilise second-order ODEs for $X$ and $Y$, which turn out to have structures convenient for the investigation of the uniqueness issue.

The second-order ODE for $Y$ can be easily obtained by eliminating $X$ from (2.33c) with the help of (2.33d). The result can be expressed as

$$
\left( \frac{f^2}{r^{n-2}(f')^2} Y' \right)' = \frac{f U_Y}{r^n f'^2} Y,
$$

where

$$
U_Y := k^2 - 2(n-1)K + (n-2)f.
$$

Similarly, by eliminating $Y$, we obtain the following second-order ODE for $X$:

$$
\left( \frac{f^2}{r^{n-4} P} X' \right)' = \frac{f U_X}{r^{n-2} P^2} X,
$$

where

$$
P := 4 \left[ (n+1)x + m - K \right] f + \left[ (n+1)x - 2K \right]^2,
$$

$$
U_X := 4(n-2) \left[ (n+1)(n+2)x + 3(m - K) \right] f^2
+ \left[ 5(n-2) \left\{ (n+1)x - 2K \right\}^2 + 4n(n+1) \left\{ m - (n-2)K \right\} x
+ 4m \left\{ m - (n-1)K \right\} + 4(n-2)K^2 \right] f
+ 3 \left\{ m - (n-2)K \right\} \left[ (n+1)x - 2K \right]^2.
$$

§3. Boundary conditions

In the arguments regarding black hole uniqueness, the boundary conditions at the horizon and at infinity play a crucial role. Obviously, if we impose boundary conditions that are too weak, the uniqueness will always be violated. On the other hand, the uniqueness theorem obtained under conditions that are too strong will not be sufficiently powerful in general.

3.1. Regularity condition at the horizon

Because we are considering only regular black holes, the spacetime metric describing a black hole must be regular at the horizon. Hence, it is natural to require that a metric perturbation be regular at the horizon with respect to a coordinate system in which the background metric is regular. For the background metric (2.1) with (2.3), such a coordinate system is given by the Szekeres-type coordinates $U, V$ and $z^i$ that satisfy

$$
UV = -\frac{1}{\kappa^2} e^{2Kr_*}, \quad |V/U| = e^{2\kappa t},
$$

(3.1)
where \( \kappa = f'(r_h)/2 \) is the surface gravity of the black hole, and \( r_* \) is the coordinate defined by
\[
dr_* = \frac{dr}{f(r)}. \tag{3.2}
\]
Because the black hole horizon is non-degenerate, we have \( \kappa \neq 0 \), and \( UV \) can be expressed near the horizon in terms of a regular positive function \( g(r) \) as
\[
UV = -f(r)g(r). \tag{3.3}
\]
In the present paper, we require that the components of perturbations of the metric and the Weyl tensors in this coordinate system be bounded at the horizon.

### 3.1.1. Tensor perturbations

For a tensor perturbation, the metric components in the \((U, V, z^i)\) coordinate system are bounded only when \( H_T \) is bounded at the horizon. Under this condition, from (3.3) and (B.4), we see that a perturbation of the Weyl tensor, \( \delta C_{\ast \ast \ast \ast} \), in this coordinate system is bounded if and only if \( H_T' \) and \( H_T'' \) are bounded at \( r = r_h \). Thus, the regularity condition at horizon is given by
\[
H_T, H_T', H_T'' = O(1). \tag{3.4}
\]

### 3.1.2. Vector perturbations

For a stationary vector perturbation, \( F_t = f_t \) is the only non-vanishing gauge-invariant variable. Because we have
\[
f_U = \frac{f}{2\kappa U}(f_t + f_r), \quad f_V = \frac{f}{2\kappa V}(-f_t + f_r), \tag{3.5}
\]
the metric components in the \((U, V, z^i)\) coordinate system are bounded only when \( F_t = f_t \) is bounded at the horizon. Under this condition, from (B.11), it follows that the Weyl tensor in this coordinate system is bounded at the horizon if and only if \( F^{(1)} \) is bounded there. To summarise, the regularity condition is given by
\[
F_t, F^{(1)} = O(1). \tag{3.6}
\]

### 3.1.3. Scalar perturbations

For a scalar perturbation, the determination of the boundary condition at the horizon is not so simple, because all metric components can be non-vanishing. First, from the relations
\[
f_{UU} = \frac{f^2}{4\kappa^2 U^2}(f_{tt} + f_{rr} - 2 f_t^r), \tag{3.7a}
\]
\[
f_{VV} = \frac{f^2}{4\kappa^2 V^2}(f_{tt} + f_{rr} + 2 f_t^r), \tag{3.7b}
\]
\[
f_{UV} = \frac{f}{4\kappa^2 U V} f_a^a, \tag{3.7c}
\]
we obtain the conditions
\[
f_t^t, f_r^r, f_r^t = O(1), \quad f_t^t - f_r^r = O(f) \tag{3.8}
\]
at the horizon. From this, we have $X^t = O(1)$. Next, from (3.5) and the regularity of $h_{ij}$, we obtain the conditions

$$f^t, f_r, H_T, H_L = O(1)$$

at the horizon. Under these conditions, the coefficients of $S_{ij}$ in $\mathcal{L}_\eta C_{abij}$ given in (B.17) are bounded at the horizon. Therefore, the corresponding coefficients in $[\delta C_{abij}]$ should be bounded in the $(U, V, z^i)$ coordinates. Taking account of (B.19) and the equation $F^a = -2(n-2)F$, this leads to the conditions

$$F, F^a = O(1), \quad F^t - F^r = O(f)$$

at the horizon. From this and the definitions of $F^a$, it follows that

$$X_r, X'_r, H'_T = O(1)$$

at the horizon. Using (B.17), we can easily check that all components of $\mathcal{L}_\eta C_{abij}$ in the $(U, V, z^i)$ coordinates are bounded at the horizon under these boundary conditions. This implies that the gauge-invariant combinations $[\delta C_{abij}]$ in the same coordinates should be bounded at the horizon. This requirement yields the additional conditions

$$F', (F^t)'', (F^r)'', f(F^t)'', F'' = O(1)$$

at the horizon.

In terms of the variables $X$ and $Y$, the boundary conditions at the horizon obtained by the above argument are expressed simply as

$$X, Y, X', Y' = O(1).$$

Note that the conditions on the second derivatives follow from these conditions, with the help of the Einstein equations (2.33). Note also that in the case $\Lambda > 0$, the same conditions should be imposed at the cosmological horizon.

### 3.2. Asymptotic condition at infinity

In the cases $\Lambda \leq 0$, the region outside the horizon extends to $r = \infty$. Therefore, we must impose some asymptotic condition. In the present paper, we require that all components of the metric perturbation $h_{\mu\nu}$ in the natural orthonormal basis of the background metric fall off at infinity, i.e.,

$$f^t, f_r, H_T, H_L \to 0, \quad f^{-1/2} f_t, f^{1/2} f_r \to 0, \quad H_T, H_L \to 0.$$  

First, for a tensor perturbation, this requirement gives the single condition

$$H_T \to 0.$$  

Next, for a stationary vector perturbation, these conditions can be expressed in terms of the gauge-invariant variables $F_t = f_t$ and $F^{(1)} = -r(f_t/r)'$ as

$$f^{-1/2} F_t, rf^{-1/2} F^{(1)} \to 0.$$
Note that we do not need an asymptotic condition on $F_r$, because $F_r$ vanishes for a stationary perturbation.

Finally, for a stationary scalar perturbation, the Einstein equation $F_a^a = -2(n - 2)F$ yields

$$\frac{2}{r^{n-2}}(r^{n-2}f X_r)' + f_a^a + 2(n - 2)\left(H_L + \frac{1}{n}H_T\right) = 0. \quad (3.17)$$

From this, we obtain the asymptotic condition

$$\frac{f}{r} X_r \to 0. \quad (3.18)$$

Hence, the above asymptotic conditions on the metric perturbations can be expressed in terms of the gauge-invariant variables $X$ and $Y$ as

$$\frac{X}{r^{n-2}}, \frac{Y}{r^{n-2}} \to 0. \quad (3.19)$$

### 3.3. Exceptional vector perturbations

As shown in §2.3, we have an exact general solution for the exceptional vector perturbation with $k^2_V = (n - 1)K$ for $K \geq 0$. There is a subtlety concerning the regularity of this solution. In the case $\Lambda \leq 0$, the above asymptotic condition is satisfied by (2.26) only for the choice $C = 0$. The solution corresponding to this choice is obtained when we treat, for example, the Kerr metric with small angular momentum as a perturbed form of the Schwarzschild metric in the standard $(t, r, \theta, \phi)$ coordinates.

However, under this gauge condition, the regularity condition at the horizon, (3.6), is not satisfied for $J \neq 0$. Nevertheless, this solution is regular at the horizon, because we can set $C = -J/(n + 1)/r^{n+1}_h$ through a gauge transformation, for which the solution satisfies (3.6). The apparent singular behaviour of the perturbation for the former gauge arises because the Killing vector $\partial_t$ vanishes at the bifurcation sphere of the horizons for the Schwarzschild metric, while it becomes space-like and does not vanish there for the Kerr metric. That is, the mapping used to compare the two metrics is singular at the bifurcation sphere. In contrast, for the latter gauge, which corresponds to the redefinition of the angular coordinate from $\phi$ to $\hat{\phi} = \phi - \Omega_h t$, with $\Omega_h$ being the angular velocity of the horizon, $\partial_t$ becomes parallel to null generators of the horizon and vanishes at the bifurcation sphere. Hence, the mapping between two spacetimes is regular at the horizon. The situation in the $\Lambda > 0$ case is similar.

In this case, to prove the regularity of the solution, we have to employ different choices of $C$ at the black hole horizon and at the cosmological horizon.

In connection to the above consideration, we now give a comment on another type of exceptional mode. As is well known, the Kerr solution can be extended to a larger regular family by introducing the NUT parameter $\nu$. This family reduces to the Schwarzschild solution in the simultaneous limit $J \to 0$ and $\nu \to 0$. In particular, the Taub-NUT solution with $J = 0$ expressed as

$$ds^2 = -f(dt + 2\nu \cos \theta d\phi)^2 + (r^2 + \nu^2)(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dr^2}{f}, \quad (3.20)$$
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with

\[ f = \frac{r^2 - 2Mr - \nu^2}{r^2 + \nu^2}, \tag{3.21} \]

can be treated as a perturbed form of the Schwarzschild solution when \(|\nu|\) is small. This perturbation is of the vector type and given by \(h_{ti} = -2\nu fV_i\), with

\[ V_\theta = 0, \quad V_\phi = \cos \theta. \tag{3.22} \]

It is easy to see that this \(V_i\) satisfies the divergence-free condition and that \(f_t = f/r\) is a solution to the Einstein equations satisfying the regularity condition at the horizon. (Of course, the NUT solution is not asymptotically flat and does not satisfy the above asymptotic condition at infinity.) Further, \(V_i\) is also harmonic. However, the corresponding eigenvalue is given by \(k_V^2 = -1\), i.e., \(l = 0\). This apparently contradicts our previous claim that \(k_V^2 > 0\) for \(K = 1\). This contradiction arises simply because \(V_i\) is singular and not square integrable on \(S^2\).

This peculiarity of the solution arises from two pathological features of the Taub-NUT solution. Firstly, each \(r = \text{const}\) submanifold is diffeomorphic to \(S^3\) for \(\nu \neq 0\). In particular, the causality condition is violated, and there exist closed time-like curves in the region satisfying \(f(r) > 0\), corresponding to the outside of a black hole. Second, each \(t = \text{const}\) section of this \(S^3\) is actually a cylinder with two boundaries for \(f(r) > 0\), although they shrink to points at the bifurcation surface of the horizons. Further, this section becomes time-like near the boundary. Due to these features, taking the limit \(\nu \to 0\) is a singular procedure.

In the present paper, we only consider \(L^2\)-normalizable perturbations. Therefore, solutions with peculiarities like this Taub-NUT solution are excluded from the argument of perturbative uniqueness.

§4. **Asymptotically flat case**

The analysis of the perturbative uniqueness of asymptotically flat black hole solutions for \(K = 1\) and \(\Lambda = 0\) is rather straightforward, because the perturbation equations are exactly soluble. We mainly consider the case of a spherically symmetric black hole, i.e., \(K^n = S^n\), and only briefly mention generic Einstein cases.

4.1. **Tensor perturbations**

In terms of the variable

\[ x := \left( \frac{r_0}{r} \right)^{n-1}, \tag{4.1} \]

the master equation (2.14) can be written

\[ \frac{d^2H_T}{dx^2} + \frac{1}{x-1} \frac{dH_T}{dx} + \frac{k_V^2 + 2}{(n-1)^2 x^2 (x-1)} H_T = 0. \tag{4.2} \]

This equation is of the Fuchs type, and its general solution can be expressed in terms of hypergeometric functions as

\[ H_T = \frac{A}{r^{\ell+n-1}} F(\alpha, \alpha, 2\alpha; x) + Br^\ell F(\alpha', \alpha', 2\alpha'; x), \tag{4.3} \]
where \( l \) is a solution of the equation
\[
k_T^2 = l(l + n - 1) - 2,
\]
and
\[
\alpha := \frac{l + n - 1}{n - 1}, \quad \alpha' := -\frac{l}{n - 1}.
\]

For \( K^n = S^n \), \( l \) takes the discrete values \( l = 2, 3, \ldots \). Hence, if we require that \( H_T \) be bounded at infinity, which is a weaker condition than (3.15), we have \( B = 0 \). However, from the general formula
\[
F(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)},
\]
we find that
\[
\lim_{x \to 1} F(\alpha, \alpha, 2\alpha; x) = +\infty.
\]
Therefore, there exists no stationary tensor perturbation of the Schwarzschild-Tangherlini solution that is bounded and regular outside the horizon.

Contrastingly, for the case in which \( K^n \) is a generic Einstein space, the above general solution always falls off at infinity if the Lichnerowicz operator has an eigen-tensor for which \( k_T^2 < -2 \). In this case, because the above equation for \( H_T \) always has a solution that is regular at \( x = 1 \), there exists a regular and bounded stationary tensor perturbation, and uniqueness does not hold.

4.2. Vector perturbations

Because there exists no harmonic vector for which \( k_V^2 < n - 1 \), \( l \) defined by
\[
k_V^2 = l(l + n - 1) - 1
\]
can be assumed to satisfy \( l \geq 1 \). Because the solutions for \( l = 1 \), i.e. the exceptional modes, have been discussed in §§2.3 and 3.3, we need only consider modes with \( l > 1 \).

For a generic mode, the Einstein equations (2.24) are equivalent to
\[
\frac{dF^t}{dx^2} + \frac{2}{x - 1} \frac{dF^t}{dx} - \frac{n(x - 1) - (l - 1)(l + n)}{(n - 1)^2x^2(x - 1)} F^t = 0,
\]
where \( x = (r_0/r)^{n-1} \), as previously. The general solution to this equation can be expressed in terms of hypergeometric functions as
\[
F^t = \frac{A}{r^{d+n-1}} F(a+2, b+1, a+b+2; x) + Br^d F(-a, 1-b, -a-b; x),
\]
where
\[
a = \frac{l+1}{n-1}, \quad b = \frac{l-1}{n-1}.
\]
Note that when \( a + b \) is a non-negative integer, \( F(-a, 1-b, -a-b; x) \) should be replaced by a function \( \tilde{F}(x) \) that is regular at \( x = 0 \) and \( \tilde{F}(0) = 1 \).

If we require that \( F^t \) be bounded at \( r = \infty \), which is weaker than the asymptotic condition (3.16), \( B \) should vanish. Under this condition, from (4.6), we find that \( F^t \) diverges at the horizon. Therefore, there exists no regular and bounded stationary vector perturbation other than the exceptional modes.
4.3. Scalar perturbations

As explained in §2.4, we can assume that \( m = k^2 - n > 0 \) for \( K = 1 \). Therefore, we can express \( k^2 \) in terms of \( l > 1 \) as

\[
k^2 = l(l + n - 1). \tag{4.12}
\]

Then, (2.35) can be written in terms of \( x = (r_0/r)^{n-1} \) as

\[
\frac{d^2 Y}{dx^2} + p \frac{dY}{dx} + qY = 0, \tag{4.13}
\]

where

\[
p = \frac{2(n-2)x + 2}{(n-1)x(x-1)}, \quad q = \frac{-(l-1)(l+n) + (n-2)x}{(n-1)^2 x^2 (1 - x)}. \tag{4.14}
\]

This equation can be solved explicitly again, and its general solution is given by

\[
Y = \frac{Ar_0^{n-1}}{r^{n+l}} F_1(x) + Br_0^{n-1} r^{l-1} F_2(x), \tag{4.15}
\]

with

\[
F_1(x) = F(\nu + 1, \nu + 2, 2\nu + 2; x), \tag{4.16a}
\]

\[
F_2(x) = F(-\nu, -\nu + 1, -2\nu; x), \tag{4.16b}
\]

where \( x = l/(n-1) \), and \( A \) and \( B \) are arbitrary constants. Inserting this solution for \( Y \) into (2.33d), we obtain

\[
X = -\frac{A}{n-1} \frac{1}{r^{n+l+1}} \left[ \{2 + (n-3)x\} F_1(x) + 2(l+n-1) F_3(x) \right]
- \frac{B}{n-1} r^{n+l-2} \left[ \{2 + (n-3)x\} F_2(x) - 2l F_4(x) \right]. \tag{4.17}
\]

with

\[
F_3(x) = F(\nu, \nu + 2, 2\nu + 2; x), \tag{4.18}
\]

\[
F_4(x) = F(-\nu - 1, -\nu + 1, -2\nu; x). \tag{4.19}
\]

From these expressions, we find that the asymptotic condition (3.19) is satisfied only when \( B = 0 \) if \( l > 1 \). However, when \( B = 0 \), \( Y \) diverges at \( x = 1 \), and the regularity condition (3.13) is not satisfied. Actually, the same conclusion is obtained from the weaker condition that \( F \) and \( F_a \) are bounded at infinity. Hence, there exists no regular and bounded stationary scalar perturbation.

The following proposition summarises the main result obtained in this section.

**Proposition 1.** For the Schwarzschild-Tangherlini black hole, there exists no stationary perturbation that is regular and bounded outside the horizon, except for the exceptional vector perturbations.

Taking account of the uniqueness theorem for asymptotically flat static vacuum regular solutions in higher dimensions, this result implies that the Myers-Perry solution is the only regular stationary solution for an asymptotically flat vacuum system near the static limit in a spacetime with any number of dimensions. However, for a generic system for which \( K^n \) is a non-spherically symmetric Einstein space with \( K = 1 \), there may exist a regular static tensor perturbation that falls off at infinity.
§5. Asymptotically de Sitter case

In the case with a non-vanishing cosmological constant $\Lambda$, we cannot express general solutions to the perturbation equations in terms of known functions explicitly. Nevertheless, we can show the non-existence of stationary perturbations satisfying the boundary conditions with the help of integral identities derived from the master equations.

We first consider the case $\Lambda > 0$, for which $K$ must be unity. In this case, we can demonstrate uniqueness only by studying the behaviour of perturbations in the region bounded by the black hole horizon $r = r_h$ and the cosmological horizon $r = r_c$, for which only the regularity condition at the horizon is relevant.

5.1. Tensor perturbations

For a tensor perturbation, by multiplying (2.14) by $r^{n/2}H_T$, integrating it over $r$, and using the regularity condition (3.4), we obtain the identity

\[ 0 = \left[ f r^{n/2}H_T \left( r^{n/2}H_T \right) \right]_{r_h}^{r_c} - \int_{r_h}^{r_c} dr \left[ f \{(r^{n/2}H_T)\}^2 + r^{n-2}U_T H_T^2 \right]. \] (5.1)

When $K^n = S^n$, $U_T$ is always positive in the range $r_h < r < r_c$, because $k^2_T \geq n - 2$. Hence, from this integral identity, it follows that there exists no regular stationary tensor perturbation.

By contrast, in the case in which $K^n$ is not a constant curvature space, $k^2_T$ may become negative, and no general lower bound on it is known. Therefore, there may exist a regular stationary perturbation.

5.2. Vector perturbations

The argument concerning vector perturbations is almost the same as that for tensor perturbations. We multiply (2.24a) by $F_t/r$ and integrate it over $r$. Then, using the regularity condition (3.6), we obtain

\[ 0 = - \left[ r^n f F^t F^{(1)} \right]_{r_h}^{r_c} + \int_{r_h}^{r_c} dr \left[ r^n \left( F^{(1)} \right)^2 + m_V r^{n-2} f \left( F_t \right)^2 \right]. \] (5.2)

As in the case $\Lambda = 0$, we can restrict our consideration to the case in which $m_V = k^2_T - n + 1 > 0$. Then, only the trivial solution $F_t = 0$ satisfies this integral identity. Unlike the case of tensor perturbations, this result holds even if $K^n$ is a generic Einstein space with $K = 1$.

5.3. Scalar perturbations

The argument for a scalar perturbation is also similar to that given above. Now, we utilise the master equation for $X$, (2.37), which leads to the identity

\[ \left[ \frac{f^2}{r^{n-4}P} XX' \right]_{r_h}^{r_c} = \int_{r_h}^{r_c} dr \left[ \frac{f^2}{r^{n-4}P} (X')^2 + \frac{f U_X}{r^{n-2}P^2 X^2} \right]. \] (5.3)

For $f = 1 - x - \lambda r^2 > 0$, $P$ vanishes only when $m + 1 = 0$, which is not realised, because $m = k^2 - n > 0$. Further, if $x = 2/(n + 1)$ at $r = r_h$ or $r = r_c$, $P'$ does
not vanish there. Hence, \( f^2/P \) is positive for \( r_h < r < r_c \) and vanishes at \( r = r_h \) and \( r_c \). From this and the regularity condition (3.13), we find that the left-hand side of the above integral identity vanishes. Therefore, if \( U_X \) is non-negative, we can conclude that there exists no stationary regular scalar perturbation. From (2.38b), we see that this condition is satisfied if \( m \geq n - 1 \). Although this inequality may not hold for a generic Einstein space \( \mathcal{K}^n \) with \( K = 1 \), in the most important case, in which \( \mathcal{K}^n = S^n \), we have the necessary inequality, because \( m \) takes the discrete values \( m = (l - 1)(l + n) \) with \( l \geq 2 \):

\[
m - (n - 1) = (l - 2)(l + n + 1) + 3 > 0.
\]

(5.4)

To summarise, we have proved the following proposition.

**Proposition 2.** For a de Sitter-Schwarzschild black hole, there exists no stationary perturbation that is regular in the region \( r_h \leq r \leq r_c \), except for the exceptional vector perturbations.

From the above arguments, it follows that near the static and spherically symmetric limit, the rotating version of the de Sitter-Schwarzschild solution with the same number of parameters as the Myers-Perry solution is the only regular stationary vacuum solution for \( \Lambda > 0 \). In four dimensions, this solution is identical to the Carter solution with a vanishing NUT parameter, and in higher dimensions, it is identical to the Gibbons-Lü-Page-Pope solution with \( \Lambda > 0 \). This uniqueness may not hold when \( \mathcal{K}^n \) is not a constant curvature space, due to the existence of a regular static perturbation of the tensor or scalar type in that case.

### §6. Asymptotically anti-de Sitter case

The method based on integral identities can also be applied to the case \( \Lambda < 0 \). However, because the region outside the horizon extends to infinity in this case, we have to check that the boundary contribution at infinity vanishes for solutions satisfying the asymptotic condition. Further, we also have to consider the cases \( K = 0 \) and \( K = -1 \).

#### 6.1. Tensor perturbations

For large \( r \), the master equation (2.14) can be approximated by

\[
H_T'' + \frac{n + 2}{r}H_T' \approx 0,
\]

(6.1)

whose general solution can be written \( A/r^{n+1} + B \), with constants \( A \) and \( B \). Hence, if we require the asymptotic condition (3.15), \( H_T \) should behave as \( \sim 1/r^{n+1} \) at infinity. From this and the regularity condition at the horizon, (3.4), it follows that the left-hand side of the identity

\[
\left[ f r^{n/2}H_T \left( r^{n/2}H_T \right)' \right]_{r_h}^\infty = \int_{r_h}^\infty dr \left[ f \left( (r^{n/2}H_T)' \right)^2 + r^{-2}U_T H_T^2 \right]
\]

(6.2)
vanishes. Hence, in the case in which \( K^n \) is a constant curvature space, from the positivity of \( UT \), we can conclude that there exists no regular stationary tensor perturbation that satisfies the fall off condition (3-15).

6.2. Vector perturbations

For large \( r \), the master equation (2.24) can be approximated by

\[
\left[ r^{n+2} (rF^t) \right]' \approx 0, \tag{6.3}
\]

whose general solution can be written \( A/r^{n+2} + B/r \). Because \( f^{-1/2}F_t \) behaves as \( \sim rF^t \) at infinity in an asymptotically anti-de Sitter spacetime, the asymptotic condition (3-16) requires \( B \) to vanish. Then, in the identity obtained from (2.24),

\[
-\left[ r^n f F^t F^{(1)} \right]_{r_h}^\infty = \int_{r_h}^{\infty} dr \left[ r^n \left( F^{(1)} \right)^2 + m_V r^{n-2} f (F^t)^2 \right], \tag{6.4}
\]

the boundary terms on the left-hand side vanish if the regularity condition (3.6) at the horizon is also satisfied. From this, it follows that \( F^t \) should vanish identically if \( m_V > 0 \). Therefore, the exceptional modes discussed in §3.3 are the only regular stationary vector perturbations that satisfy the fall-off condition (3-16), irrespective of the value of \( K \).

6.3. Scalar perturbations

In order to determine the asymptotic behaviour of scalar perturbations for \( \Lambda < 0 \), we utilise the equation for \( Y \), (2.35). For large \( r \), this equation can be approximated as

\[
Y'' - \frac{n-4}{r} Y' - \frac{n-2}{r^2} Y \approx 0. \tag{6.5}
\]

The general solution to this equation can be expressed as

\[
Y \approx \frac{A}{r} + B r^{n-2}, \tag{6.6}
\]

and the asymptotic condition (3-19) is satisfied only if \( B = 0 \). For this solution, the boundary terms in the identity obtained from (2.35),

\[
\left[ \frac{f^2}{r^{n-2} (f')^2} YY' \right]_{r_h}^\infty = \int_{r_h}^{\infty} dr \frac{1}{r^n (f')^2} \left[ r^2 f^2 (Y')^2 + fU_Y Y^2 \right], \tag{6.7}
\]

vanish if the solution also satisfies the regularity condition (3.13) at the horizon. Thus, uniqueness holds if \( U_Y \) is non-negative. For \( K \leq 0 \), this condition is trivially satisfied. In contrast, for \( K = 1 \), it leads to the condition on the spectrum \( k^2 - 2(n-1) \geq 0 \). As in the case \( \Lambda \geq 0 \), this condition is satisfied for \( K^n = S^n \), but it may not be satisfied generically.

The main result obtained in this section can be summarised as the following proposition.
Proposition 3. For anti-de Sitter-Schwarzschild black holes and topological black holes, there exists no stationary perturbation that is regular and falls off at infinity, except for the exceptional vector perturbations.

This result indicates that the Gibbons-Lü-Page-Pope solution with $\Lambda < 0$, which is a rotational extension of the anti-de Sitter-Schwarzschild solution characterised by the same number of parameters as for the Myers-Perry solution, is the only regular stationary asymptotically anti-de Sitter solution near the static and spherically symmetric limit. Concerning topological black holes with $K = -1$, no such rotational extension exists if $C^n$ is compact, because there is no exceptional mode in this case (cf. Refs. 44 and 45). This uniqueness may be violated for the case in which $C^n$ is a generic Einstein space with a non-constant sectional curvature, due to the existence of a regular static perturbation of tensor type for $K = 0, \pm 1$, or of scalar type for $K = 1$.

§7. Concluding remarks

In the present paper, we have determined all stationary solutions to the perturbation equations that are regular at the horizon and fall off at infinity (in the case $\Lambda \leq 0$) for a static black hole background whose horizon has a spatial section with constant curvature. As summarised in Theorem 1 given in the introduction, we have found that these solutions are exhausted by the trivial perturbations corresponding to variations of the background parameters and the exceptional vector perturbations representing stationary rotations of black holes for any value of the cosmological constant. We have also pointed out that there may exist additional regular and bounded solutions if a spatial section of the horizon does not have constant curvature.

As mentioned in the introduction, this result strongly indicates that the Myers-Perry solution and the Gibbons-Lü-Page-Pope solution are the only asymptotically flat, de Sitter or anti-de Sitter regular stationary vacuum solutions near the static and spherically symmetric limit. However, our arguments do not give an exact proof of this perturbative uniqueness near the static and spherically symmetric limit, because there may exist a regular family of solutions that approaches the limit in a singular way, as in the case of the Taub-NUT solution discussed in §3.3. Logically, there is also the possibility that there exist infinitely many families whose bifurcation points accumulate at the static limit, although this is quite unlikely.

It is clear that we have to study perturbations of the Myers-Perry solutions in order to show that such a pathological situation is not realized and to determine whether the Myers-Perry family actually bifurcates at high angular momenta, as suggested by Emparan and Myers. At present, however, such a study would be quite difficult because there exists no tractable formulation for perturbations of the Myers-Perry solutions. The development of such a formulation is the most important problem to be solved in the present context.

Finally, we would like to point out that the fall-off condition at infinity is essential for uniqueness to hold in the asymptotically anti-de Sitter case, unlike in the asymptotically flat case, in which uniqueness holds even if we only require perturba-
tions to be bounded at infinity. In fact, for the general solution given in the previous section, $\delta g_{\mu \nu}$ with respect to the natural orthonormal frame is bounded at infinity even if $B \neq 0$, irrespective of the type of perturbations. Further, we can easily show that one solution is always regular at the horizon for each mode. Hence, in the asymptotically anti-de Sitter case, for any boundary value of $\delta g_{\mu \nu}$ at infinity, there exists a stationary solution to the perturbation equation that is regular everywhere and satisfies the given boundary condition at infinity. Furthermore, this solution is unique except for the freedom to add solutions corresponding to the exceptional vector perturbations. This result is consistent with the general results concerning asymptotically anti-de Sitter and static solutions given in Refs. 46) and 47), and has a close connection with the AdS/CFT argument.48), 49)

Acknowledgements

The author would like to thank Akihiro Ishibashi for valuable comments and Yoshiyuki Morisawa and Daisuke Ida for useful conversations. This work is supported by the JSPS grant No. 15540267.

Appendix A

Interpretation of Exceptional Modes for Vector Perturbations

In this appendix, we derive the relation between the angular momentum of the system and the parameter $J$ for each exceptional mode of the vector perturbation in §2.3.

To begin, we note that for an exceptional mode, the metric can be written

$$ds^2 = ds_0^2 + 2rf_i(r)V^i dt dz^i,$$

where $ds_0^2$ is the background metric. Because $V^i$ is a Killing vector for $k_r^2 = (n - 1)K$, this perturbed metric is invariant under translations generated by the vector field

$$\eta := V^i \partial_i.$$ (A.2)

Hence, it is “rotationally symmetric”. This implies that we can calculate the angular momentum of this spacetime using the Komar integral on $K^n$. In fact, from the Einstein equations for the background metric

$$R_{\mu \nu} = (n + 1)\lambda g_{\mu \nu},$$ (A.3)

we obtain the identity

$$d \ast d\eta = 2(n + 1)\lambda I_\eta \Omega_{n+2}$$ (A.4)

for any Killing vector $\eta$, where $\eta_* = \eta_\mu dx^\mu$, $\ast$ is the Hodge dual operator, $I_\eta$ is the inner product operator, and $\Omega_{n+2}$ is the spacetime volume form. From this, it follows that the integration of the $n$-form $\ast d\eta_*$ over a $n$-subspace $\Sigma_n$ is independent of the choice of $\Sigma_n$ if $\Sigma_n$ is tangential to the vector field $\eta$. If we calculate this integral in the present case in which

$$\eta_* = \frac{J}{n + 1} V \cdot V \frac{dt}{r^{n-1}},$$ (A.5)
we obtain
\[ \int_{\Sigma_n} *d\eta_* = C_V \frac{n-1}{n+1} J, \quad (A.6) \]
where
\[ C_V := \int_{K^n} \nabla \cdot \nabla d\Omega_n; \quad d\Omega_n = \sqrt{\gamma} d^n z. \quad (A.7) \]
Therefore, \( J \) is proportional to the angular momentum of the spacetime defined by the Komar integral.

In order to determine the proportionality constant in this relation, we utilise the perturbation equation with a source term for an exceptional mode derived in Ref. 34). In the stationary case, it reads
\[ \left( r^{n+1} F^{(1)} \right)' = 2 \kappa^2 r^{n+1} \tau^t, \quad (A.8) \]
where \( \tau^t \) is related to the energy-momentum tensor of the source as
\[ T^t_i = r \tau^t V^i. \quad (A.9) \]
In a flat background, the angular momentum \( J[V] \) of the system with respect to the “rotational” Killing vector \( V^i \) can be expressed as
\[ J[V] = \int_0^\infty dr r^n \int_{K^n} d\Omega_n T^t_i V^i. \quad (A.10) \]
Therefore, integration of (A.8) over \( r \) yields
\[ (r^{n+1} F^{(1)})(r = \infty) = -\frac{2 \kappa^2}{C_V} J[V]. \quad (A.11) \]
Comparing this expression and (2.25), we obtain
\[ J = \frac{2 \kappa^2}{C_V} J[V]. \quad (A.12) \]
This final result can be regarded as exact, because it is consistent with the expression in terms of the Komar integral, (A.6), and it gives the relation
\[ J[V] = \frac{1}{2 \kappa^2} \frac{n+1}{n-1} \int_{\Sigma_n} *d\eta_. \quad (A.13) \]

Appendix B

In this appendix, we derive explicit expressions for the background values and the perturbation of the Weyl curvature.
B.1. \( C_{****} \)

The Weyl tensor for our static background metric is given by

\[
C_{abcd} = \frac{n(n-1)}{2(n+1)} \Psi (g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{B.1a}
\]

\[
C_{aibj} = -\frac{n-1}{2(n+1)} r^2 \Psi g_{ab} \gamma_{ij}, \tag{B.1b}
\]

\[
C_{ijkl} = \frac{r^4 \Psi}{n+1} (\gamma_{ik} \gamma_{jl} - \gamma_{il} \gamma_{jk}), \tag{B.1c}
\]

where

\[
\Psi := \frac{\Box r}{r} + 2 \frac{K - (Dr)^2}{r^2} = \frac{(n+1)r^{-n-1}}{r^{n+1}}. \tag{B.2}
\]

B.2. Tensor perturbations

For a tensor perturbation, all components of the perturbation of the Weyl tensor, \( \delta C_{****} \), are gauge invariant. For a solution to (2.14), their non-vanishing components in the \((t, r, z^i)\) coordinate system are given by

\[
\frac{1}{r^2} \delta C_{titj} = \left[ \frac{f'}{2} H''_T + \frac{n-1}{n+1} \Psi H_T \right] T_{ij}, \tag{B.3a}
\]

\[
f \frac{r}{2} \delta C_{rirj} = \left[ \left( -(n-1) \lambda + \frac{(n-2)K}{r^2} - \frac{n-3}{2(n+1)} \Psi \right) r H'_T 
+ \left( \frac{k^2 + 2K}{r^2} + \frac{n-1}{n+1} \Psi \right) H_T \right] T_{ij}, \tag{B.3b}
\]

\[
\sqrt{r^3} \delta C_{rijk} = \sqrt{r} H'_T \left( \hat{D}_k T_{ij} - \hat{D}_j T_{ik} \right), \tag{B.3c}
\]

\[
\frac{1}{r^4} \delta C_{ijkl} = \left[ -\frac{f}{r} H'_T + \left( \frac{2Psi}{n+1} - \frac{K}{r^2} \right) H_T \right] T^{(0)}_{ijkl} + \frac{H_T}{r^2} T^{(1)}_{ijkl}. \tag{B.3d}
\]

From these, we find that the non-vanishing components of \( \delta C_{****} \) in the \((U, V, z^i)\) coordinate system are given by

\[
\delta C_{Uij} = \frac{f^2}{4\kappa^2 U^2} \left[ -r^2 H''_T - 2r H'_T \right] T_{ij}, \tag{B.4a}
\]

\[
\delta C_{ViVj} = \frac{f^2}{4\kappa^2 V^2} \left[ -r^2 H''_T - 2r H'_T \right] T_{ij}, \tag{B.4b}
\]

\[
\delta C_{Uij} = \frac{r^2 f}{4\kappa^2 U V} \left[ \left( f' + \frac{n-2}{r} f \right) H'_T + \frac{n-1}{n+1} \Psi H_T \right] T_{ij}, \tag{B.4c}
\]

\[
\delta C_{Uijk} = \frac{r^2 f}{2\kappa U} H'_T (\hat{D}_k T_{ij} - \hat{D}_j T_{ik}), \tag{B.4d}
\]

\[
\delta C_{Vijk} = \frac{r^2 f}{2\kappa V} H'_T (\hat{D}_k T_{ij} - \hat{D}_j T_{ik}). \tag{B.4e}
\]

The expression for \( \delta C_{ijkl} \) is the same as that above.
B.3. Vector perturbations

For a vector perturbation, $\delta C_{***}$ are not gauge invariant. Under the vector-type coordinate transformation

$$\xi^a := \bar{\delta}g^a = 0, \quad \xi^i := \bar{\delta}z^i = r^2 L \nabla^i,$$  \hspace{1cm} (B.5)

they transform as

$$\bar{\delta}(\delta C_{\mu\nu\lambda\sigma}) = -\mathcal{L}_{\xi} C_{\mu\nu\lambda\sigma}. \hspace{1cm} (B.6)$$

Hence, taking account of the fact that $H_T$ transforms under (B.5) as

$$\bar{\delta}H_T = kL, \hspace{1cm} (B.7)$$

we find that the following combination is gauge invariant:

$$[\delta C_{\mu\nu\lambda\sigma}] := \delta C_{\mu\nu\lambda\sigma} + \mathcal{L}_{\eta} C_{\mu\nu\lambda\sigma}, \hspace{1cm} (B.8)$$

where

$$\eta^a = 0, \quad \eta^i = \frac{r^2}{k} H_T \nabla^i. \hspace{1cm} (B.9)$$

Explicitly, the non-vanishing components of $\mathcal{L}_{\eta} C_{\mu\nu\lambda\sigma}$ are as follows:

$$\mathcal{L}_{\eta} C_{abci} = -\frac{n-1}{2(n+1)} \frac{r^2}{k} \psi (g_{ac} D_b H_T - g_{bc} D_a H_T) \nabla_i, \hspace{1cm} (B.10a)$$

$$\mathcal{L}_{\eta} C_{aibj} = \frac{n-1}{n+1} r^2 \psi H_T \nabla_{ij}, \hspace{1cm} (B.10b)$$

$$\mathcal{L}_{\eta} C_{aijk} = -\frac{r^4}{(n+1)k} \psi D_a H_T (\gamma_{ij} \nabla_k - \gamma_{ik} \nabla_j), \hspace{1cm} (B.10c)$$

$$\mathcal{L}_{\eta} C_{ijkl} = -\frac{2r^4}{n+1} \psi H_T (\gamma_{ik} \nabla_{jl} + \gamma_{jl} \nabla_{ik} - \gamma_{il} \nabla_{jk} - \gamma_{jk} \nabla_{il}). \hspace{1cm} (B.10d)$$

In the $(t,r,z^i)$ coordinate system, the non-vanishing components of $[\delta C_{***}]$ are given by

$$[\delta C_{rttr}] = \left[ - (n-1) F^{(1)} + \frac{F^t}{r} \left\{ \frac{n^2 + 3n - 2}{4n(n+1)} r^2 \psi \right. \right.$$

$$+ \frac{n^2 - 5n + 2}{2n(n+1)} K - \frac{n-1}{n} m_{\nabla} + \frac{n^2 + 7n - 2}{2n(n+1)} \frac{f}{f} \left. \right\} \right] \nabla_i, \hspace{1cm} (B.11a)$$

$$[\delta C_{rrij}] = -\frac{r}{2} F^{(1)} d\nabla_{ij}, \hspace{1cm} (B.11b)$$

$$[\delta C_{ttrj}] = -\frac{r}{4} F^{(1)} d\nabla_{ij}, \hspace{1cm} (B.11c)$$

$$[\delta C_{tijk}] = -\frac{r f}{2} F^t \hat{D}_i d\nabla_{jk} + \frac{r f}{2n} \left[ - \frac{(2n-1)r^2 \psi + 2(n^2 + n - 1)K + 2f}{n+1} \right. \left. \right.$$

$$+ r F^{(1)} (\gamma_{ij} \nabla_k - \gamma_{ik} \nabla_j), \hspace{1cm} (B.11d)$$

where

$$d\nabla_{ij} := \hat{D}_i \nabla_j - \hat{D}_j \nabla_i. \hspace{1cm} (B.12)$$
Note that from (B.10), we have \( \delta C_{\mu\nu\lambda\sigma} = \delta C_{\mu\nu\lambda\sigma} \) for these components. The components in the \((U, V, z^i)\) coordinate system are expressed in terms of these non-vanishing components in the \((t, r, z^i)\) coordinate system as

\[
\frac{1}{V}[\delta C_{UVU_i}] = \frac{1}{U}[\delta C_{UVVi}] = \frac{1}{4k^3} \left( \frac{f}{UV} \right)^2 [\delta C_{rttr}],
\] (B.13a)

\[
[\delta C_{UVij}] = 2[\delta C_{UiVj}] = \frac{f}{2k^2} \frac{1}{UV} [\delta C_{rttij}],
\] (B.13b)

\[
\frac{1}{U}[\delta C_{Vijk}] = -\frac{1}{V}[\delta C_{Uijk}] = \frac{1}{2k} \left( \frac{1}{UV} \right) [\delta C_{tijk}].
\] (B.13c)

**B.4. Scalar perturbations**

For a scalar perturbation, under the gauge transformation

\[
\xi_a = T_a S,
\]

\[
\xi_i = r^2 L S_i,
\] (B.14)

\[
X_a \text{ and } H_T \text{ transform as}
\]

\[
\bar{\delta}X_a = T_a S,
\]

\[
\bar{\delta}H_T = k L.
\] (B.15)

Hence, the combinations (B-8) with

\[
\eta_a = X_a S,
\]

\[
\eta_i = \frac{r^2}{k} H_T S_i
\] (B.16)

are gauge invariant.

Explicitly, the non-vanishing components of \( \mathcal{L}_\eta C_{\mu\nu\lambda\sigma} \) are given by

\[
\mathcal{L}_\eta C_{trtr} = \frac{n(n-1)}{2(n+1)} \frac{\Psi}{n+1} \left[ -2D \cdot X + \frac{n+1}{r} Dr \cdot X \right] S,
\] (B.17a)

\[
\mathcal{L}_\eta C_{trci} = \frac{n-1}{2(n+1)} \frac{\Psi}{k} \epsilon_{ca} (nk^2X^a + r^2 D^a H_T) S_i,
\] (B.17b)

\[
\mathcal{L}_\eta C_{aibj} = \frac{(n-1)r^2 \Psi}{2(n+1)} [(n-1)g_{ab} Dr \cdot X - r(D_a X_b + D_b X_a)] \gamma_{ij} S
\]

\[
- \frac{n-1}{n(n+1)} kr^2 \Psi H_T g_{ab} (nS_{ij} + \gamma_{ij} S),
\] (B.17c)

\[
\mathcal{L}_\eta C_{aijk} = \frac{-r^2 \Psi}{2(n+1)} \left[ k^2(n-1)X_a + 2r^2 D_a H_T \right] (\gamma_{ij} S_k - \gamma_{ik} S_j),
\] (B.17d)

\[
\mathcal{L}_\eta C_{ijkt} = \frac{n-3}{n+1} \frac{r^3 \Psi}{2(n+1)k} Dr \cdot X (\gamma_{ik} \gamma_{jl} - \gamma_{il} \gamma_{jk}) S
\]

\[
+ \frac{2r^4 k}{(n+1)} \Psi H_T \left[ 2n(\gamma_{ij} S_{ljq} - \gamma_{jl} S_{iq}) + 2(\gamma_{ik} \gamma_{jl} - \gamma_{il} \gamma_{jk}) \right] S.
\] (B.17e)

The non-vanishing components of the gauge-invariant combinations in the \((t, r, z^i)\) coordinates are

\[
[\delta C_{rttr}] = \left[ -\left( \frac{n^2 - n + 2}{4(n+1)} \Psi + \frac{f - K}{2r^2} \right) F_c^e + 2(f F_t^i)'' - (f' F_t^i)' \right] S,
\] (B.18a)
\[ [\delta C_{rtti}] = \frac{k_f}{2} \left[ (F^t_i)' - \frac{1}{r} F^t_i + \frac{f'}{2f} (F^t_i - F^r_r) \right] S_i, \] 

(B.18b)

\[ [\delta C_{titj}] = f \left\{ \left\{ \frac{n-1}{n+1} r^2 \Psi + (n-2) r f' \right\} F + \frac{1}{2} r^2 f' F' - \frac{r}{2} F^t_t \right. \\
- \left( \frac{k^2}{2n} - \frac{n-1}{2(n+1)} r^2 \Psi \right) F^t_t \right\} \gamma_{ij} S + \frac{k^2}{2} f F^t_t S_{ij}, \] 

(B.18c)

\[ [\delta C_{rirj}] = \frac{k_r}{2} \left[ -r^2 f F'' - \left( \frac{r^2}{2} f' + 2rf \right) F' - \frac{n-1}{n+1} r^2 \Psi F + \frac{r}{2} (f F^r_r)' \right. \\
+ \left( \frac{k^2}{2n} - \frac{n-1}{2(n+1)} r^2 \Psi \right) F^r_r \right\} \gamma_{ij} S - \frac{k^2}{2f} F^r_r S_{ij}, \] 

(B.18d)

\[ [\delta C_{rijk}] = \frac{k_r}{2} [2r F' - F^r_r] (\gamma_{ik} S_j - \gamma_{ij} S_k), \] 

(B.18e)

\[ [\delta C_{ijkl}] = \frac{r^2}{n^2(n+1)} \left\{ -2n^2(n+1) r f F' + 2n \left\{ (n+1)m + 2nr^2 \Psi \right\} F \right. \\
+ n^2(n+1) f F^r_r \right\} (\gamma_{ik} \gamma_{jl} - \gamma_{il} \gamma_{jk}) S \\
- \frac{k^2r^2}{2} F (\gamma_{ik} S_{jl} + \gamma_{jl} S_{ik} - \gamma_{il} S_{jk} - \gamma_{jk} S_{il}). \] 

(B.18f)

The corresponding components in the \((U, V, z^i)\) coordinates are

\[ [\delta C_{UVUV}] = \frac{1}{4k^4} \left( \frac{f}{UV} \right)^2 [\delta C_{rtrt}], \] 

(B.19a)

\[ \frac{f}{U} [\delta C_{UVVi}] = -\frac{f}{V} [\delta C_{UVU_i}] = \frac{1}{4k^3} \left( \frac{f}{UV} \right)^2 [\delta C_{rtri}], \] 

(B.19b)

\[ \frac{4k^2 U^2}{f^2} [\delta C_{UiUj}] = \frac{4k^2 V^2}{f^2} [\delta C_{ViVj}] = \frac{k^2 (F^t_t - F^r_r)}{2f} S_{ij} \\
+ \left[ -r^2 F'' - nr F' + \left( \frac{k^2}{2n} - \frac{n-1}{2(n+1)} r^2 \Psi \right) \frac{F^r_r - F^t_t}{f} \right] \gamma_{ij} S, \] 

(B.19c)

\[ [\delta C_{ViVj}] = \frac{f}{4k^2 U V} \left\{ -r^2 f F'' - (r^2 f') F' \\
- \left( \frac{2(n-1)}{n+1} r^2 \Psi - 2(n-2) \lambda r^2 + \frac{n-2}{n} k^2 \right) F \right. \\
+ \left. \frac{r}{2} [f (F^r_r - F^t_t)']' \right\} \gamma_{ij} S + (n-2) k^2 F S_{ij}, \] 

(B.19d)

\[ [\delta C_{Uiijk}] = \frac{f}{2k^2 U} [\delta C_{rijk}], \quad [\delta C_{Vijk}] = \frac{f}{2k^2 V} [\delta C_{rijk}], \] 

(B.19e)

with \([\delta C_{ijkl}]\) as above.

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