Path Integral Formulation of Noncommutative Superspace
in the IKKT Matrix Model

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We propose a physical interpretation of our novel fermionic solution for the IKKT matrix model, which we obtained in a previous paper. We extend the configuration space of the bosonic field to supernumber space and obtain a noncommutative parameter that is not bi-Grassmann, but an ordinary number. This establishes the connection between Seiberg’s noncommutative superspace and our solution of the IKKT matrix model.

§1. Introduction

For several years, the noncommutative space in string theory has been studied actively. First, only the noncommutative structure of bosonic coordinates was studied. Recently, these concepts have been extended to the superspace. Many attempts to study this new noncommutativity have been made. For example, solutions that represent noncommutative superspace were introduced in the context of the super-Lie matrix model. Many attempts have also been made in the context of field theory, quantum mechanics and string theory. In this paper, we would like to shed light on this noncommutativity from the viewpoint of the IKKT matrix model.

While the IKKT matrix model can be investigated using an expansion around the commutative (diagonal) background, it is also interesting to start with the solution

$$[X^a, X^b] = -iC^{ab}N, \theta^\alpha = 0,$$

where $C^{ab}$ is an antisymmetric constant. The result is noncommutative Yang-Mills theory. The quantities $X_a$, which satisfy (1-1), serve as noncommutative coordinates. In the string picture, this corresponds to the appearance of noncommutative geometry in a constant B field background.

By contrast, for backgrounds of graviphoton fields, the endpoints of open strings possess the structure of Seiberg’s noncommutative superspace. If the IKKT matrix model contains the entire string theory, it must have a solution that corresponds to this noncommutativity. We suggested a possibility for this vacuum in a previous paper. However, in this solution the noncommutative parameter is not an ordinary number, but bi-Grassmann. In string theory, this parameter is an ordinary number which depends on $\alpha'$ and the graviphoton field strength. Therefore, we need to clarify the physical interpretation of this bi-Grassmann structure in order to claim that this solution of the IKKT matrix model corresponds to string theory with graviphoton
field backgrounds.

In this paper, we interpret this noncommutative parameter as the vacuum expectation value of this solution. In the process, we deform the path of the integral to a special path in the supernumber configuration space. Our computation involves new techniques, which we explain in detail below.

This paper is organized as follows. In §2, we review the concept of the supernumber, which is necessary to define path integrals over our new configuration space. In §3, we introduce a novel method to compute path integrals, and we compute the commutation relations of the coordinates of the superspace, $x^a, \theta^a$ and $\bar{\theta}^\dot{a}$. In §4, we discuss our method and its meaning.

§2. Supernumbers

In this section, we introduce the concepts of supernumbers and integrals over supernumbers.\(^{40}\)

Let $\zeta^i, i = 1 \cdots M$, be a set of generators for an algebra, which satisfy the following anticommutation relation:

$$\zeta^i \zeta^j = -\zeta^j \zeta^i \quad \text{for all } i, j. \tag{2.1}$$

This algebra is called a ‘Grassmann algebra’, and it is denoted by $\Lambda_M$. The elements $1, \zeta^i, \zeta^i \zeta^j, \cdots$ form the basis of $\Lambda_M$.

We call the elements of $\Lambda_M$ ‘supernumbers’. Every supernumber can be expressed in the form

$$z = z_B + z_S = z_B + \sum_{m=1}^{M} \frac{1}{m!} c_{i_1 \cdots i_m} \zeta^{i_m} \cdots \zeta^{i_1}. \tag{2.2}$$

The quantity $z_B$ is an ordinary number and is called the ‘body’, while $z_S$ is called the ‘soul’. We can divide supernumbers into two sectors, one whose basis consists of an even number of $\zeta^i$, ‘$z_c$’, and one whose basis consists of an odd number of $\zeta^i$, ‘$z_a$’:

$$z = z_c + z_a, \tag{2.3}$$

$$z_c = z_B + \sum_{m=1}^{\lfloor M/2 \rfloor} \frac{1}{(2m)!} c_{i_1 \cdots i_{2m}} \zeta^{i_{2m}} \cdots \zeta^{i_1}, \tag{2.4}$$

$$z_a = \sum_{m=1}^{\lfloor M/2 \rfloor} \frac{1}{(2m-1)!} c_{i_1 \cdots i_{2m-1}} \zeta^{i_{2m-1}} \cdots \zeta^{i_1}. \tag{2.5}$$

The supernumber $z$, which is $z_c = 0$, is called an ‘a-number’ and the supernumber $z$, which is $z_a = 0$, is called a ‘c-number’. The set of c-numbers is denoted by $\mathbb{R}_c$, and the set of a-numbers is denoted by $\mathbb{R}_a$.

We can define a function over $\mathbb{A}_M$ as

$$f(z) \equiv \sum_{m=0}^{M} \frac{1}{m!} f^{(m)}(z_B)(z_S)^m, \tag{2.6}$$
where the function \( f^{(m)}(z_B) \) is the \( m \)-th derivative with respect to \( x \) of the usual function \( f(x) \). Also, we can define the integrals of \( f(z) \) over \( \mathbb{R}_a \) and \( \mathbb{R}_c \).

Integrals over \( \mathbb{R}_a \) are called ‘Berezin’s integrals’. The functions of \( z_a \) can be written \( f(z_a) = a + bz_a \), and therefore we define the integrals over \( \mathbb{R}_a \) as

\[
\int dz_a = 0, \\
\int dz_a z_a = 1, \\
\int dz_a \, af(z_a) + bg(z_a) = a \int dz_a \, f(z_a) + b \int dz_a \, g(z_a),
\]

where \( a \) and \( b \) are constant supernumbers that do not depend on \( z_a \).

Integrals over a path \( C \) in \( \mathbb{R}_c \) are defined as follows. Consider a path \( C \) as \( z_S = z_S(z_B) \) that starts at \( a \) and ends at \( b \). We define the integral for \( C \) as

\[
\int_a^b dzf(z) = \sum_{m=0}^{M} \frac{1}{m!} \int_{a_B}^{b_B} dz_B (1 + z'_S(z_B)) f^{(m)}(z_B) z_S^m(z_B).
\]

From this definition, we have

\[
\int_a^b dzf(z) = \sum_{m=0}^{M} \frac{1}{m!} \int_{a_B}^{b_B} dz_B (1 + z'_S(z_B)) f^{(m)}(z_B) z_S^m(z_B) \\
+ \sum_{m=0}^{M} \frac{1}{(m+1)!} \int_{a_B}^{b_B} dz_B f^{(m)}(z_B) (z_S^{m+1}(z_B))' \\
= \sum_{m=0}^{M} \frac{1}{m!} \int_{a_B}^{b_B} dz_B f^{(m)}(z_B) z_S^m(z_B) \\
+ \sum_{m=0}^{M} \left[ \frac{1}{(m+1)!} f^{(m)}(z_B) (z_S^{m+1}(z_B)) \right]_{a_B}^{b_B} \\
- \sum_{m=0}^{M} \frac{1}{(m+1)!} \int_{a_B}^{b_B} dz_B f^{(m+1)}(z_B) z_S^{m+1}(z_B) \\
= \int_{a_B}^{b_B} dz_B f^{(0)}(z_B) + \sum_{m=0}^{M} \left[ \frac{1}{(m+1)!} f^{(m)}(z_B) (z_S^{m+1}(z_B)) \right]_{a_B}^{b_B}.
\]

Thus, excluding the pure bosonic part ‘body’, this integral depends only on the starting point \( a \) and the ending point \( b \). Thus, if we put the endpoints on the real number axis i.e., \( z_a(a) = z_a(b) = 0 \), the integral is the same as that for a path that is entirely on the real number axis.
When we study the IKKT matrix model, we consider $\mathcal{R}$ to be a configuration space of bosonic fields that are the components $X^a$ of the Lie algebra and Grassmann number to be a configuration space of fermionic fields which are the components $\theta, \bar{\theta}$ of the Lie algebra.

In the following, we propose to extend the configuration space of $X^a$ from $\mathcal{R}$ to $\mathcal{R}_c$, and we perform path integrals over $\mathcal{R}_c$. As in the case above, the integrals do not vary when the path is deformed, and hence we can expect that the theory itself is independent of this path.

§3. Noncommutative superspace

In a previous paper,$^1$ we obtain the following new solution of the IKKT matrix model:

$$\sum_{AB} \theta^A \theta^B f_{AB0} = \tilde{C}^{\alpha \beta},$$

(3.1)

$$\bar{\theta}^0 = \bar{\theta}^\dot{\alpha},$$

(3.2)

$$X^{aA} = -i \theta^A (\sigma^a)_{a\beta} \bar{\theta}^\beta 0,$$

(3.3)

$$\text{Others} = 0.$$  

(3.4)

The various quantities appearing here are defined below. (Also, see Ref. 41 for spinor notation.)

Let us denote the $U(N)$ generators as $T^A$ and choose an integer $n$ which is sufficiently large, but much smaller than $N$, so that $N/n \gg 1$. Then we have

$$\hat{A} = 0, 1, 2, \cdots, N^2 - 1,$$

$$A = 1, 2, \cdots, 2n,$$

$$T^A = Q_1, P_1, Q_2, P_2, \cdots, Q_n, P_n,$$

(3.5)

where the $Q_k$ and $P_k$ are $N/n \times N/n$ matrices satisfying

$$[Q_j, P_k] = i \delta_{jk}.$$  

(3.6)

Also, we represent the identity by $T^0$, that is,

$$T^0 = 1_N.$$  

(3.7)

It follows that the only nontrivial structure constants $f_{\hat{A} \hat{B} \hat{C}}$ among the generators $T^A$ are of the form

$$f_{AB0} = \begin{pmatrix}
0 & i & 0 & 0 & 0 & 0 \\
-i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i & 0 & 0 \\
0 & 0 & -i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

(3.8)
Equations (3.1)–(3.3) represent solutions of the 4-dimensional IKKT matrix model,

\[ S = \text{Tr} \left( \frac{1}{4} [X^a, X^b]^2 - \frac{1}{2} \theta \sigma^a [\bar{\theta}, X_a] \right), \]  

which yields the following equations of motion:

\[
\begin{align*}
[X^b, [X^a, X_b]] - \frac{1}{2} \{\theta, \sigma^a \bar{\theta}\} &= 0, \\
[X_a, (\theta \sigma^a)_\alpha] &= 0, \\
[(\sigma^a \bar{\theta})_\alpha, X_a] &= 0.
\end{align*}
\]  

The special feature of this solution is that it satisfies the following realization of Seiberg’s noncommutative algebra:6)

\[
\begin{align*}
\{\theta^\alpha, \bar{\theta}^\beta\} &= \tilde{C}^{\alpha\beta}, \\
[X^a, \theta^\alpha] &= i\tilde{C}^{\alpha\beta} \sigma^a_\beta \bar{\theta}^\alpha, \\
[X^a, X^b] &= (\bar{\theta})^2 \tilde{C}^{ab}, \\
\{\bar{\theta}^\dot{\alpha}, \bar{\theta}^\dot{\beta}\} &= \{\bar{\theta}^\dot{\alpha}, \theta^\beta\} = [\bar{\theta}^\dot{\alpha}, X^a] = 0,
\end{align*}
\]

where

\[
\begin{align*}
a, b &= 1 \cdots 4, \quad \alpha, \dot{\alpha} = 1, 2, \\
\tilde{C}^{ab} &\equiv \tilde{C}^{\alpha\beta} (-\sigma^{ab} \epsilon)_{\alpha\beta}, \\
\\n\tilde{C}^{\alpha\beta} &\equiv (\epsilon \sigma^{a\beta})_{\alpha\beta} \tilde{C}_{ab}. 
\end{align*}
\]

This algebra is the same as Seiberg’s noncommutative algebra,6)

\[
\begin{align*}
\{\theta^\alpha, \bar{\theta}^\beta\} &= C^{\alpha\beta}, \\
[X^a, \theta^\alpha] &= iC^{\alpha\beta} \sigma^a_\beta \bar{\theta}^\alpha, \\
[X^a, X^b] &= (\bar{\theta})^2 C^{ab}, \\
\{\bar{\theta}^\dot{\alpha}, \bar{\theta}^\dot{\beta}\} &= \{\bar{\theta}^\dot{\alpha}, \theta^\beta\} = [\bar{\theta}^\dot{\alpha}, X^a] = 0,
\end{align*}
\]

where

\[
\begin{align*}
a, b &= 1 \cdots 4, \quad \alpha, \dot{\alpha} = 1, 2, \\
C^{ab} &\equiv C^{\alpha\beta} (-\sigma^{ab} \epsilon)_{\alpha\beta}, \\
C^{\alpha\beta} &\equiv (\epsilon \sigma^{a\beta})_{\alpha\beta} C_{ab}.
\end{align*}
\]

We could identify this solution with Seiberg’s noncommutative superspace, recognizing that \(\tilde{C}^{\alpha\beta}\) corresponds to \(C^{\alpha\beta}\). However, the noncommutative parameter \(\tilde{C}^{\alpha\beta}\) is bi-Grassmann (3.1), while \(C^{\alpha\beta}\) is an ordinary number that depends on \(\alpha\)' and the graviphoton fieldstrength.39) Thus, it is difficult to obtain a physical interpretation of this solution.
In the following, we present a physical interpretation of this solution by showing that Seiberg’s noncommutative algebra given in (3.19)–(3.22) can be realized as the vacuum expectation values of the algebra for this solution:

\[
\langle \text{Tr} \{ \theta^\alpha, \theta^\beta \} \rangle = C^\alpha\beta, \\
\langle \text{Tr} [X^a, \theta^\alpha] \rangle = iC^\alpha\beta \sigma^a_{\beta\dot{\alpha}} \tilde{\theta}^{\dot{\alpha}}, \\
\langle \text{Tr} ([X^a, X^b]) \rangle = (\tilde{\theta})^2 C^{ab}, \\
\text{Others} = 0.
\]

Here, \( C_{\alpha\beta} \) and \( C_{ab} \) are ordinary numbers which will be obtained below. Thus these parameters are identical to Seiberg’s noncommutative parameters.

Let us now proceed to the actual computation in the path integral formulation. What we need to compute are various vacuum expectation values,

\[
\langle O \rangle = \int DXD\theta D\bar{\theta} \, O e^{-S}.
\]

The measure of this integration is given by

\[
DX^a D\theta D\bar{\theta} = \prod_{A=1}^{N^2-1} dX_A^a \, d^2 \theta_A \, d^2 \bar{\theta}_A.
\]

We decompose the components of matrices \( X^a, \theta \) and \( \bar{\theta} \) as

\[
X^a = \sum_{\hat{A}=1}^{\hat{A}=N^2-1} T^{\hat{A}} X^{a}_{\hat{A}}, \quad \theta = \sum_{\hat{A}=1}^{\hat{A}=N^2-1} T^{\hat{A}} \theta_{\hat{A}}, \quad \bar{\theta} = \sum_{\hat{A}=1}^{\hat{A}=N^2-1} T^{\hat{A}} \bar{\theta}_{\hat{A}}.
\]

As above, the gauge group is \( SU(N) \).

Because the “vacuum” given in (3.1)–(3.3) is not the ordinary vacuum but, rather, Seiberg’s noncommutative superspace vacuum, we need to carry out the “tree” approximation described below.

We deform the path of integration for \( X^a \) from \( C \) to \( C' \) (Fig. 1). Because the integrals do not depend on the path, we can perform the integration over the path \( C' \). In other words, we change the variables from \( X^a \) to \( \hat{X}^a = i\theta \sigma^a \Delta^a \). Because \( \hat{X}^a \in \mathbb{R} \) is an ordinary number, we can adopt the value at \( \hat{X}^a = 0 \) as a first-order approximation to the integrals over \( \hat{X}^a \). Thus we obtain

\[
\langle O \rangle = \int DXD\theta D\bar{\theta} \, O e^{-S}
\]
≈ \int \mathcal{D}\theta \mathcal{D}\bar{\theta} \ e^{-S}|_{X^a = -i\theta \sigma^a \bar{\theta}}.

(3.34) "tree"

We call this the "tree" approximation. However, we do not assert that this value is the first-order approximation of the entire IKKT matrix model, as it is effective only in the vicinity of Seiberg’s noncommutative superspace vacuum \( \tilde{X}^a = 0 \). Even on this path \( C'' \), there are other vacua, for example, that for which \( \tilde{X}^a \) is the usual noncommutative plane.

The action \( S[X^a = -i\theta \sigma^a \bar{\theta}] (\equiv \tilde{S}) \) is fourth order in \( \theta \) and \( \bar{\theta} \). Strictly speaking, we are not treating the gauge group \( U(N) \) but, rather, \( SU(N) \). Hence, \( \bar{\theta} = \bar{\theta}_0 1_N \) is not contained in the model. However, because we would like to treat this vacuum, which is asymmetric with respect to \( \theta \) and \( \bar{\theta} \), we employ the following technique.

First, we extend the gauge group \( SU(N) \) to \( U(N) \) only on \( \bar{\theta} \) as

\[
\mathcal{D}\theta \mathcal{D}\bar{\theta} \rightarrow \mathcal{D}\theta \mathcal{D}\bar{\theta} d^2\bar{\theta}_0, \quad (3.35)
\]

\[
\bar{\theta} \rightarrow \bar{\theta} = \sum_{\tilde{A} = 0}^{\tilde{A} = N^2 - 1} T^{\tilde{A}} \bar{\theta}_{\tilde{A}}. \quad (3.36)
\]

This model contains the zero modes (3.1) and (3.2), but it does not contain the zero modes that are obtained by exchanging \( \theta \) and \( \bar{\theta} \). We define the expectation value of this model as \( \langle \mathcal{O} \rangle_{\text{deformed}} \).

Similarly to the case of an instanton background, when we treat a theory that has fermionic zero modes, the expectation value of an operator \( \mathcal{O} \) is zero if it does not contain all the zero modes:\(^{42})
\[
\langle 1 \rangle_\text{deformed} = \int \mathcal{D} \theta \mathcal{D} \bar{\theta} d^2 \theta_0 e^{-\bar{S}} = 0.
\] (3.37)

Because we have the fermionic zero modes (3.1) and (3.2) in this model, we must compute the expectation value of the operator which contains all the fermionic zero modes, \( \bar{\theta}_0^2 \) and \( \bar{\theta}_{sol}^2 \), in order to obtain a non-zero expectation value:

\[
\langle \bar{\theta}_0^2 \bar{\theta}_{sol}^2 \rangle_\text{deformed} = \int \mathcal{D} \theta \mathcal{D} \bar{\theta} d^2 \theta_0 \bar{\theta}_0^2 \bar{\theta}_{sol}^2 e^{-\bar{S}} \neq 0.
\] (3.38)

In particular, we can compute the expectation value

\[
\langle \bar{\theta}_0^2 \text{Tr}(\{\theta^\alpha, \theta^\beta\}) \rangle_\text{deformed} = \int \mathcal{D} \theta \mathcal{D} \bar{\theta} d^2 \theta_0 \bar{\theta}_0^2 \text{Tr}(\{\theta^\alpha, \theta^\beta\}) e^{-\bar{S}} = C^{\alpha\beta}.
\] (3.39)

The zero modes of \( \theta \) are contained in the left-hand side of the above equation and give a non-zero contribution to \( \text{Tr}(\{\theta^\alpha, \theta^\beta\}) \). Hence, the expectation value is non-zero. Here, \( C^{\alpha\beta} \) is an ordinary number that depends on \( f_{AB} \) and \( \sigma^a_{\alpha\dot{\alpha}} \) and the manner in which we take the limits \( N, n \to \infty \). Recalling the feature of Berezin integral expressed by

\[
\int d\psi \psi f = g \Rightarrow f = g,
\] (3.40)

we obtain

\[
\langle \bar{\theta}_0^2 \text{Tr}(\{\theta^\alpha, \theta^\beta\}) \rangle_\text{deformed} = \int \mathcal{D} \theta \mathcal{D} \bar{\theta} d^2 \theta_0 \bar{\theta}_0^2 \text{Tr}(\{\theta^\alpha, \theta^\beta\}) e^{-\bar{S}} = \int d^2 \theta_0 \bar{\theta}_0^2 A = C^{\alpha\beta},
\] (3.41)

\[
A = \int \mathcal{D} \theta \mathcal{D} \bar{\theta} \text{Tr}(\{\theta^\alpha, \theta^\beta\}) e^{-\bar{S}} = \langle \text{Tr}(\{\theta^\alpha, \theta^\beta\}) \rangle = C^{\alpha\beta}.
\] (3.42)

We can compute (3.42) within the \( SU(N) \) theory. Because \( C^{\alpha\beta} \) is an ordinary number, we obtain a noncommutative parameter which is not bi-Grassmann, but an ordinary number.

In the same way, we have

\[
\langle \text{Tr}([X^a, X^b]) \rangle_\text{deformed} = \langle \text{Tr}([-i\theta \sigma^a \bar{\theta}, -i\theta \sigma^a \bar{\theta}]) \rangle_\text{deformed} = \int d^2 \theta_0 B = C^{ab},
\] (3.43)

\[
B = \langle \text{Tr}([X^a, X^b]) \rangle = \bar{\theta}_0^2 C^{ab}.
\] (3.44)

Here \( C^{ab} \equiv C^{\alpha\beta}(-\sigma^{ab})_{\alpha\beta} \) and \( C^{\alpha\beta} = (\epsilon^{ab})^{\alpha\beta} C_{ab} \), which are consistent with (3.23) and (3.24). Similarly, we have

\[
\langle \bar{\theta}_0^2 \text{Tr}([X^a, \theta^\alpha]) \rangle_\text{deformed} = \langle \bar{\theta}_0^2 \text{Tr}([-i\theta \sigma^a \bar{\theta}, \theta^\alpha]) \rangle_\text{deformed}
\]
\[
\begin{align*}
\int d^2 \bar{\theta} \bar{\theta} C &= \frac{-i}{2} C^{\alpha\beta} (\sigma^a \epsilon)_{\alpha} \bar{\epsilon}^\beta, \\
C &= \langle \text{Tr}([X^a, \theta^\alpha]) \rangle = i C^{\alpha\beta} \sigma^a_{\beta\alpha} \bar{\theta} \bar{\epsilon}^\alpha, \\
\langle \text{Tr} \{\bar{\theta}^\alpha, \bar{\theta}^\beta\} \rangle &= \langle \bar{\theta}^\alpha \text{Tr} \{\bar{\theta}^\alpha, \bar{\theta}^\beta\} \rangle_{\text{deformed}} \\
&= 0, \\
\text{Others} &= 0.
\end{align*}
\]

Thus we obtain the full algebra (3.25)–(3.28).

In string theory, the noncommutative parameter \(C^{\alpha\beta}\) is a free parameter that depends on the background of graviphoton fields. Therefore, we need to properly choose the manner in which we take limits \(N, n \to \infty\) in \(SU(N)\) theory in order to obtain the arbitrary parameter \(C^{\alpha\beta}\).

\section*{§4. Discussion}

In this paper we have proposed a physical interpretation of our novel fermionic solution. Seiberg’s noncommutative superspace algebra is realized as the vacuum expectation values of the algebra for this solution. We obtain an ordinary noncommutative parameter with the path integral formulation.

We adopted a special method in this paper to carry out the calculation. The important point of this method is its asymmetrical treatment of \(\theta\) and \(\bar{\theta}\). We believe that this asymmetrical treatment is necessary to extract information from Seiberg’s noncommutative superspace vacuum. For finite \(N\), this asymmetrical feature does not appear. In the true \(N \to \infty\) limit, this asymmetrical feature emerges from the measure. For finite \(N\), it seems that we have

\[\langle 1 \rangle \neq 0, \tag{4.1}\]

in contrast to (3.37). However, this contribution originates in other vacua. If we would like to obtain information concerning Seiberg’s noncommutative superspace vacuum, we must take the \(N \to \infty\) limit as the asymmetrical feature appears. In this limit, the contributions from other vacua disappear.

There is another subtlety that we should mention. In the true \(N \to \infty\) limit, we have

\[\text{Tr}([\theta X^a, \sigma_a \bar{\theta}]) \neq \text{Tr}(([\theta, X^a]) \sigma_a \bar{\theta}). \tag{4.2}\]

If we would like to obtain vacua that are symmetric with respect to \(\bar{\theta}\) and \(\theta\), we probably must treat the action

\[S = S\text{Tr} \left( \frac{1}{4} [X^a, X^b]^2 - \frac{1}{2} \theta \sigma^a [\bar{\theta}, X_a] \right), \tag{4.3}\]

instead. Here, \(S\text{Tr}\) represents a symmetric trace.

We focused on the IKKT model in this paper. However, the technique that we adopted here may also be applicable to other models that have fermionic zero modes,
in particular in the case that there are nontrivial relations between the bosonic fields and fermionic fields. When a bosonic field $\Phi(x)$ has a non-zero expectation value $\Phi_{cl}$, we expand the field around $\Phi_{cl}$ as $\Phi(x) = \tilde{\Phi}(x) + \Phi_{cl}$ and treat the new field $\tilde{\Phi}(x)$ as a quantum field. However, in the case that we have a fermionic zero mode $\psi_0$, expanding the fermionic field around $\psi_0$ is not valid. Extending the configuration space to supernumber space may be an effective method for wider class of problems.

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