Hybrid Superstring on $AdS_3$
and
Space-Time Superconformal Symmetry

Hiroshi KUNITOMO
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

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Using a hybrid formalism, superstrings on $AdS_3 \times S^1$ are studied in a manifestly supersymmetric manner. The world-sheet fields in this description including superspace coordinates are obtained through a field redefinition from the corresponding ones in the RNS formalism. The physical states are defined with the BRST cohomology as an $N = 4$ topological string theory. The physical spectra for some lower mass levels are investigated. We identify two series of the space-time chiral primaries which have already been obtained from the analysis in the RNS formalism. We find that they are described by the chiral and the vector supermultiplets, but the on-shell physical structure of the latter depends on whether it is massive or massless. The space-time supersymmetry on this background is extended to the boundary $N = 2$ superconformal symmetry. We explicitly construct its generators and study how they act on specific supermultiplets.

§1. Introduction

The world-sheet symmetries play an important role during the first evolutions in the superstring theory. These symmetries are useful for studying the string as a two-dimensional field theory, which has led to important developments in the understanding of its perturbative behavior. From this viewpoint, the Ramond-Neveu-Schwarz (RNS) formalism is the most convenient formulation to obtain space-time covariant results. The space-time supersymmetry, on the other hand, is not manifest in this formalism but, rather, is obtained by imposing the GSO projection on representations of the (world-sheet) $N = 2$ superconformal symmetry.

In the next stage, however, the space-time symmetries become more important for studying non-perturbative aspects of superstrings. The space-time supersymmetry, in particular, becomes more important as a way to classify several BPS states belonging to some short representations of the supersymmetry. The spectra of these states are kept from changing due to the supersymmetry and can be treated independently of the string coupling constant. The Green-Schwarz (GS) formalism has manifest space-time supersymmetry, and therefore it is the most convenient way to investigate these states. Nevertheless, it is not known how to quantize the GS superstrings while keeping all the space-time supersymmetries manifest.

The formalism proposed and developed in Refs. 1)–3) is a hybrid of these two, the RNS and the GS formalisms, and can be quantized in such a manner that preserves the manifest $D$-dimensional supersymmetry for $D < 10$. The $D$-dimensional part is described by space-time superspace coordinates (with conjugates of fermionic coordinates) and some additional bosons. The remaining $(10 - D)$-dimensional part,
interpreted as representing some compactified space, is generally described by appropriate representations of the $N = 2$ superconformal field theory. The physical states are defined with the BRST cohomology as constituting an $N = 4$ topological string.\(^2\) This formalism has already been applied to the study of superstrings on a variety of backgrounds, and its validity has been demonstrated.\(^3\)–\(^5\)

In this paper, we apply the hybrid formalism to superstrings on $AdS_3 \times S^1 \times \mathcal{N}/U(1)$ with the manifest space-time (anti-de Sitter) supersymmetry on $AdS_3 \times S^1$. These backgrounds have received much attention\(^6\)–\(^11\) as the simplest solvable model for studying AdS/CFT-duality\(^12\) beyond the supergravity approximation. Strings propagating on $AdS_3$, with an NS $B$-field background, are described by the $SL(2, R)$ WZW model with level $k\(^7\),\(^8\) and are exactly solvable as a conformal field theory. Their spectrum has been studied in detail, and it has been found that it must include spectrally flowed representations.\(^10\) These representations can be naturally incorporated by using a free-field realization as the discrete light-cone Liouville theory,\(^11\) in which the discrete light-cone momentum is identified with the spectral flow parameter. Applying this framework to the RNS superstrings, two series of the space-time chiral primaries were found.\(^11\)

From these world-sheet fields in the RNS formalism, we can obtain those in the hybrid formalism through a field redefinition. The four-dimensional sector describing $AdS_3 \times S^1$ is represented by the superspace coordinates $(X^0, X^1, \phi_L, \bar{Y}; \Theta^\pm, \bar{\Theta}^\pm)$, the conjugates of the fermionic coordinates $(P_\pm, \bar{P}_\pm)$, and an additional boson $\rho$. The compactified space $\mathcal{N}/U(1)$ sector is characterized by the $N = 2$ superconformal symmetry, generated by $(T_{\mathcal{N}/U(1)}, G_{\mathcal{N}/U(1)}^\pm, I_{\mathcal{N}/U(1)})$, with central charge $c = 9 - 6/k$.

The physical states in this formalism are defined with the BRST cohomology as an $N = 4$ topological string theory.\(^2\) We explicitly identify physical spectra corresponding to the two series of the space-time chiral primaries which have been obtained in the RNS formalism.\(^11\) The supermultiplet structures of these series are also clarified. The first series is described by the chiral supermultiplet including a scalar, a two-component spinor and an auxiliary field. In the second series, the two cases in which the $\mathcal{N}/U(1)$ sector is excited and not excited must be distinguished. We find that the former (latter) is given by the massive (massless) vector supermultiplet. The space-time supersymmetry on this background $AdS_3 \times S^1$ can be extended to the boundary $N = 2$ superconformal symmetry with the central charge $c = 6kp$. This is closed off shell in the hybrid formalism. We explicitly construct its generators and study how they act on the specific fields in the supermultiplet. Because the spectral flow operation does not commute with the world-sheet Hamiltonian, the generators act as spectrum generating operators on generic states with non-vanishing light-cone momentum $p \neq 0$. Only two of the supersymmetries are closed on a supermultiplet, while the others generate new physical states with different masses. On the special states with vanishing light-cone momentum $p = 0$, by contrast, all of the infinite supersymmetries are realized on a supermultiplet.

This paper is organized as follows. We begin with a brief review of the RNS superstring on $AdS_3 \times S^1$ in §2. The (super) WZW model on this background is described in terms of a free-field realization as the discrete light-cone Liouville the-
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ory.\textsuperscript{11) We reformulate it as an }N = 4\text{ topological string theory for later convenience. In }§3,\text{ hybrid world-sheet fields are introduced through a field redefinition. The model can be completely rewritten in terms of these hybrid fields. This makes all the space-time supersymmetries manifest. The generators of the boundary }N = 2\text{ superconformal algebra are constructed explicitly. In }§4\text{ we construct the Hilbert space of the hybrid superstring by extending the spectral flow operation to the currents consisting of the space-time spinor fields. The physical spectrum for some lower mass levels is investigated in }§5.\text{ They are identified with the two series of the space-time chiral primaries and represented by the chiral and the vector supermultiplets, respectively. We study, in }§6,\text{ how the off-shell supersymmetries are realized on these supermultiplets. Section 7 is devoted to summary and discussion. Three appendices are added for some useful details. In Appendix A, we rewrite the space-time }N = 2\text{ superconformal generators using the currents of the global }sl(1|2)\text{ symmetry.}\textsuperscript{13) The hybrid fields introduced in this paper are analogs of the chiral coordinates }\gamma^m, \theta^a, \bar{\theta}^a\text{ in flat four-dimensional space-time, which is convenient for studying the chiral supermultiplet.}\textsuperscript{14) We give a similarity transformation to obtain the real coordinates with proper hermiticity in Appendix B. The space-time and the world-sheet superconformal generators are transformed. We show, in Appendix C, how the fields are transformed under the bosonic symmetry of the }N = 2\text{ superconformal symmetry. These transformation laws are needed to confirm that the symmetries actually satisfy the }N = 2\text{ superconformal algebra.}

§2. RNS superstrings on }AdS_3 \times S^1 \times \mathcal{N}/U(1)\text{ }

Let us start by briefly reviewing how superstrings propagating on }AdS_3 \times S^1 \times \mathcal{N}/U(1)\text{ are described in the RNS formalism. It is given by the tensor product of the }N = 2\text{ superconformal field theories representing each component space. The }AdS_3\text{ sector is described by a world-sheet supersymmetric extension of the }sl(2)\text{ current algebra,}\textsuperscript{*)}

\begin{align}
    j^3(z)j^3(w) &\sim -\frac{(k + 2)/2}{(z - w)^2}, \\
    j^3(z)j^{\pm\pm}(w) &\sim \frac{\pm j^{\pm\pm}(w)}{z - w}, \\
    j^{++}(z)j^{--}(w) &\sim \frac{k + 2}{(z - w)^2} - \frac{2j^3(w)}{z - w},
\end{align}

(2.1)

by introducing three free fermions,

\begin{align}
    \psi^+(z)\psi^-(w) &\sim \frac{2}{z - w}, \\
    \psi^3(z)\psi^3(w) &\sim -\frac{1}{z - w}.
\end{align}

(2.2)

The level of this algebra is taken to be such that the algebra of the total currents,\n
\begin{align}
    J^{\pm\pm} = j^{\pm\pm} \pm \psi^\pm \psi^3,
\end{align}

\textsuperscript{*) We consider only the holomorphic sector in this paper. It is easy to combine it with the anti-holomorphic sector if necessary.\textsuperscript{6)–11)}}
is of level $k$. For the bosonic currents (2.1), we use the free-field realization\(^{11,15}\)

$$j^{\pm} = e^{\mp \beta i(X^0 + X^1)} \left( -\frac{1}{\beta} i\partial X^1 \pm \frac{1}{Q} \partial \phi_L \right),$$

$$j^3 = \frac{1}{\beta} i\partial X^0,$$  \(2.4\)

where $\beta = \sqrt{\frac{2}{k+2}}$ and $Q = \sqrt{\frac{2}{k}}$.

The $S^1$ sector is simply represented by a pair of free fields $(Y, \psi^4)$ satisfying

$$Y(z)Y(w) \sim -\log(z-w),$$

$$\psi^4(z)\psi^4(w) \sim \frac{1}{z-w}.$$  \(2.5\)

We define a $U(1)$ current as

$$J^Y = \frac{2}{Q} i\partial Y$$  \(2.6\)

for later use.

This description of the RNS superstring on $AdS_3 \times S^1$ possesses the $N = 2$ world-sheet superconformal symmetry generated by

$$T^{(4)} = -\frac{1}{2} \partial X^\mu \partial X_\mu - \frac{1}{2} \partial \phi_L \partial \phi_L - \frac{Q}{2} \partial^2 \phi_L - \frac{1}{2} \partial Y \partial Y$$

$$- \frac{1}{4} \psi^+ \partial \psi^- - \frac{1}{4} \psi^- \partial \psi^+ + \frac{1}{2} \psi^{3} \partial \psi^{3} - \frac{1}{2} \psi^{4} \partial \psi^{4},$$

$$G^{\pm(4)} = \frac{Q}{2} \psi^\pm j^{\mp} + \frac{1}{2} (\psi^4 \mp \psi^3) (i\partial Y \pm Qj^3) \pm \frac{Q}{4} (\psi^4 \mp \psi^3) \psi^+ \psi^-,$$

$$I^{(4)} = Q^2 j^3 + \frac{1}{2} (1 + Q^2) \psi^+ \psi^- + \psi^4 \psi^3,$$  \(2.7\)

with the central charge $c = 6 + 6/k$.

For the compactified space $N/U(1)$ sector, we denote the $N = 2$ superconformal generators as $(T_{N/U(1)}, G_{N/U(1)}^{\pm}, I_{N/U(1)})$. Their central charge has to be $c = 9 - 6/k$ for criticality. The string state is given by an arbitrary unitary (rational) representation of this $N = 2$ superconformal algebra characterized by two quantum numbers, $(\Delta_N, Q_N)$, the conformal weight and the $U(1)$ charge.

In addition to these matter fields, the superconformal ghosts $(b, c)$ and $(\beta, \gamma)$ satisfying

$$c(z)b(z) \sim \gamma(z)\beta(w) \sim \frac{1}{z-w}$$  \(2.8\)

must be introduced to quantize the RNS superstring covariantly. The ghost sector also possesses the $N = 2$ superconformal symmetry generated by

$$T_{gh} = -2\beta \partial c - \partial bc - \frac{3}{2} \beta \partial \gamma - \frac{1}{2} \partial \beta \gamma,$$
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\[ G_{gh}^+ = \frac{3}{2} \beta \partial c + \partial \beta c, \quad G_{gh}^- = -2b \gamma, \]

\[ I_{gh} = 2bc + 3\beta \gamma, \quad (2.9) \]

although only its \( N = 1 \) subset, \( (T_{gh}, G_{gh} = G_{gh}^+ + G_{gh}^-), \) is familiar. The physical Hilbert space is defined by the cohomology

\[ \mathcal{H}_{\text{phys}} = \ker Q_{\text{BRST}} / \text{Im} Q_{\text{BRST}} \quad (2.10) \]

of the BRST charge

\[ Q_{\text{BRST}} = \oint \frac{dz}{2\pi i} \left( c \left( T_m + \frac{1}{2} T_{gh} \right) + \gamma \left( G_m + \frac{1}{2} G_{gh} \right) \right), \quad (2.11) \]

where

\[ T_m = T^{(4)} + T_{N/U(1)}, \quad G_m = G^{(4)} + G^{-(4)} + G^+_{N/U(1)} + G^-_{N/U(1)}. \quad (2.12) \]

In order to construct space-time supercharges, we need to bosonize the world-sheet fermions and the \( U(1) \) current \( I_{N/U(1)} \) as

\[ \psi^+ \psi^- = 2i \partial H_0, \quad \psi^3 \psi^4 = i \partial H_1, \quad (2.13a) \]

\[ I_{N/U(1)} = -\alpha i \partial H_2, \quad (2.13b) \]

where \( \alpha^2 = 3 - Q^2. \) The bosons \( H_I(z) \) \((I = 0, 1, 2)\) satisfy the standard OPEs:

\[ H_I(z) H_J(w) \sim -\delta_{IJ} \log(z - w). \quad (2.14) \]

The superconformal ghosts must be also bosonized as\(^{16}\)

\[ c = e^\sigma, \quad b = e^{-\sigma}, \]
\[ \gamma = \eta e^\phi = e^\phi - \chi, \]
\[ \beta = e^{-\phi} \partial \xi = \partial \chi e^{-\phi + \chi}, \quad (2.15) \]

with

\[ \phi(z) \phi(w) \sim -\log(z - w), \]
\[ \sigma(z) \sigma(w) \sim \chi(z) \chi(w) \sim +\log(z - w). \quad (2.16) \]

Here, it is important to note that the Hilbert space of the original bosonic ghosts, \((\beta, \gamma),\) is different from that of the bosonized fields \((\phi, \xi, \eta),\) or equivalently \((\phi, \chi),\) since the zero-mode, \( \xi_0, \) is not included in the bosonization formulas (2.15). The former (latter) is called the small (large) Hilbert space \( \mathcal{H}_{\text{small}} \) \( (\mathcal{H}_{\text{large}}) \). The BRST cohomology (2.10) is defined in \( \mathcal{H}_{\text{small}}. \) This extension of the Hilbert space is essential to realize supersymmetry in the RNS formalism.

Now, four supercharges can be constructed in the familiar \(-\frac{1}{2}\)-picture as\(^{8),11}\)

\[ G_{\frac{1}{2}} = k \frac{1}{4} \oint \frac{dz}{2\pi i} e^{-\frac{1}{2} \phi + \frac{1}{2} (H_0 + H_1 + \sqrt{3} H'_2)} \]
The generators in (2.17), however, states. This can be improved by changing the picture for half of the supercharges picture further. These

\[ G_+^{(\frac{1}{2})} = k \frac{i}{2} \int \frac{dz}{2\pi i} e^{-\frac{1}{2} \phi + \frac{i}{2} (H_0 + H_1 + \sqrt{3}H'_2)}, \]

\[ G_-^{(-\frac{1}{2})} = k \frac{i}{2} \int \frac{dz}{2\pi i} e^{-\frac{1}{2} \phi + \frac{i}{2} (H_0 + H_1 - \sqrt{3}H'_2)}, \]

\[ G_-^{(-\frac{1}{2})} = k \frac{i}{2} \int \frac{dz}{2\pi i} e^{-\frac{1}{2} \phi + \frac{i}{2} (H_0 - H_1 - \sqrt{3}H'_2)}, \]

where \( H'_2 \) is the linearly transformed boson defined by

\[ \sqrt{3}H'_2 = \alpha H_2 + QY; \]

\[ \sqrt{3}Y' = -QH_2 + \alpha Y. \]

The generators in (2.17), however, satisfy the peculiar algebra\(^9,11\)

\[ \left\{ G_+^{(\frac{1}{2})}, G_-^{(-\frac{1}{2})} \right\} = \frac{1}{Q} \int \frac{dz}{2\pi i} e^{-\phi} \psi^\pm, \quad \left\{ G_+^{(\frac{1}{2})}, G_-^{(-\frac{1}{2})} \right\} = \frac{1}{Q} \int \frac{dz}{2\pi i} e^{-\phi} (\psi^3 \mp \psi^4), \]

which is equivalent to the familiar supersymmetry only for the on-shell physical states. This can be improved by changing the picture for half of the supercharges \( G_+^{(\frac{1}{2})} \) to \( \frac{1}{2} \), so that the right-hand sides of the algebra (2.19) become ghost independent, i.e. expressed in the 0-picture:

\[ G_+^{(\frac{1}{2})} = k \frac{i}{2} \int \frac{dz}{2\pi i} \left( e^{\frac{i}{2} \phi + \frac{i}{2} (H_0 + H_1 - \sqrt{3}H'_2) - \sigma - \chi} \right. \]

\[ + \frac{Q}{\sqrt{2}} e^{\frac{i}{2} \phi - \frac{i}{2} (H_0 - H_1 + \sqrt{3}H'_2)} e^{-\beta i(X^0 + X^1)} \left( \frac{1}{\beta} i \partial X^1 - \frac{1}{Q} \partial \phi_L \right) \]

\[ - \frac{Q}{\sqrt{2}} e^{\frac{i}{2} \phi - \frac{i}{2} (H_0 + H_1 + \sqrt{3}H'_2)} \left( \frac{1}{\beta} i \partial X^0 + \frac{1}{Q} i \partial Y + i \partial H_0 \right) \]

\[ - \frac{Q}{\sqrt{2}} e^{\frac{i}{2} \phi + \frac{i}{2} (H_0 + H_1 - \sqrt{3}H'_2) - \frac{i}{2} \phi + \frac{i}{2} (H_0 + H_1 - \sqrt{3}H'_2) G_{N/U(1)}}, \]

\[ G_-^{(-\frac{1}{2})} = k \frac{i}{2} \int \frac{dz}{2\pi i} \left( e^{\frac{i}{2} \phi + \frac{i}{2} (H_0 - H_1 - \sqrt{3}H'_2) - \sigma - \chi} \right. \]

\[ + \frac{Q}{\sqrt{2}} e^{\frac{i}{2} \phi + \frac{i}{2} (H_0 - H_1 + \sqrt{3}H'_2)} e^{\beta i(X^0 + X^1)} \left( \frac{1}{\beta} i \partial X^1 + \frac{1}{Q} \partial \phi_L \right) \]

\[ - \frac{Q}{\sqrt{2}} e^{\frac{i}{2} \phi + \frac{i}{2} (H_0 + H_1 + \sqrt{3}H'_2)} \left( \frac{1}{\beta} i \partial X^0 - \frac{1}{Q} i \partial Y + i \partial H_0 \right) \]

\[ - \frac{Q}{\sqrt{2}} e^{\frac{i}{2} \phi + \frac{i}{2} (H_0 - H_1 - \sqrt{3}H'_2) + \frac{i}{2} \phi + \frac{i}{2} (H_0 - H_1 - \sqrt{3}H'_2) G_{N/U(1)}}, \]

We simply write \( (G_+^{(\frac{1}{2}), G_-^{(-\frac{1}{2})}}) = (G_+^{(\frac{1}{2}), G_-^{(-\frac{1}{2})}}) \) hereafter and do not change the picture further. These picture-changed supercharges satisfy the familiar space-time
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\((AdS)\) supersymmetry algebra \(sl(1|2)\):

\[
\begin{align*}
\left\{ g^+_{\pm \frac{1}{2}}, g^-_{\pm \frac{1}{2}} \right\} &= \mathcal{L}_{\pm 1}, \quad \left\{ g^+_\mp \frac{1}{2}, g^-_{\mp \frac{1}{2}} \right\} = \mathcal{L}_0 \pm \frac{1}{2} \mathcal{I}_0,
\end{align*}
\]

where \(\mathcal{L}_{\pm 1} = -\oint \frac{dz}{2\pi i} J^{\pm}, \quad \mathcal{L}_0 = -\oint \frac{dz}{2\pi i} J^3, \quad \mathcal{I}_0 = \oint \frac{dz}{2\pi i} J^Y\)

are the generators of the bosonic subgroup \(SL(2) \times U(1)\). This global \(sl(1|2)\) symmetry can be extended to the infinite-dimensional \(N=2\) superconformal symmetry, and we explicitly construct its generators in \(\S 3\).

Before closing this section, we reconsider the physical state conditions to develop the hybrid formalism in the next section. As mentioned above, the physical states are originally defined with the BRST cohomology in the small Hilbert space \(\mathcal{H}_{\text{small}}\). However, because \(\mathcal{H}_{\text{small}}\) is not sufficient to realize the space-time supersymmetry, we should modify them so as to obtain conditions in \(\mathcal{H}_{\text{large}}\) as

\[
Q_{\text{BRST}} |\psi\rangle = 0, \quad \eta_0 |\psi\rangle = \eta_0 |A\rangle = 0,
\]

where \(|\psi\rangle, |A\rangle \in \mathcal{H}_{\text{large}}\). Additionally, we require that the physical states have ghost number 1, i.e.,

\[
Q_{gh} |\psi\rangle = |\psi\rangle,
\]

counted by the charge\(^*\)

\[
Q_{gh} = \oint \frac{dz}{2\pi i} (cb - \xi \eta).
\]

The conditions given in (2.23) have a natural interpretation as an \(N=4\) topological string theory, as we now describe.

We note that there is the hidden twisted \(N=4\) superconformal symmetry in \(\mathcal{H}_{\text{large}}\) generated by\(^2\)

\[
T = T_m + T_{gh}, \quad G^+ = J_{\text{BRST}}, \quad G^- = b, \quad \tilde{G}^+ = \eta, \quad \tilde{G}^- = \xi T - b \{Q_{\text{BRST}}, \xi\} + \partial^2 \xi, \quad I^{++} = \eta c, \quad I^{--} = b \xi, \quad I = cb - \xi \eta.
\]

The conditions (2.23) can be written in terms of these \(N=4\) generators as

\[
G^+_0 |\psi\rangle = 0, \quad \delta |\psi\rangle = G^+_0 |A\rangle, \quad I_0 |\psi\rangle = |\psi\rangle, \quad \tilde{G}^+_0 |\psi\rangle = \tilde{G}^+_0 |A\rangle = 0,
\]

\(^*\) This definition of the ghost number is related to the familiar one, \(N_c = \oint \frac{dz}{2\pi i} (cb - \gamma \beta)\), through the relation \(Q_{gh} = N_c - R\), where \(R = \oint \frac{dz}{2\pi i} (\xi \eta - \partial \phi)\) is the picture counting operator. The difference between the two definitions is a constant in a given picture.

\(^2\)
which are the definitions of the physical states in the $N = 4$ topological string theory. Because the $\eta_0$-cohomology is trivial, we can always solve Eq. (2.26c) as $|\psi\rangle = \tilde{G}_0^+ |V\rangle$ and $|\Lambda\rangle = \tilde{G}_0^+ |\tilde{\Lambda}\rangle$, and this allows us to rewrite (2.26) in the more symmetric forms

$$G_0^+ \tilde{G}_0^+ |V\rangle = 0,$$  \hspace{1cm} (2.27a)

$$\delta |V\rangle = G_0^+ |\Lambda\rangle + \tilde{G}_0^+ |\tilde{\Lambda}\rangle,$$  \hspace{1cm} (2.27b)

$$I_0 |V\rangle = 0.$$  \hspace{1cm} (2.27c)

In this paper, we regard the first condition, (2.27a), as the equation of motion and the second, (2.27b), as the gauge transformation, according to the standard terminology of string field theory. These conditions will be solved in order to investigate some physical states in §5.

§3. Hybrid superstrings on $AdS_3 \times S^1 \times N/\mathbb{U}(1)$

We develop the hybrid formalism on $AdS_3 \times S^1$ in this section. We introduce the hybrid fields through a field redefinition from the RNS fields, which allows the entire space-time supersymmetry to be manifest. Using these new fields, the space-time $N = 2$ superconformal generators are constructed explicitly.

The basic fields of the RNS superstrings on $AdS_3 \times S^1$ are the matter fields $(X^0, X^1, \phi_L, Y, \psi^0, \psi^1, \psi^3, \psi^4)$ and the superconformal ghosts $(b, c, \beta, \gamma)$, as explained in the previous section. Here we use the bosonized forms $(H_0, H_1, H'_2, \phi, \chi, \sigma)$ for the world-sheet fermions (2.13a), the $\mathbb{U}(1)$ current (2.13b) and the superconformal ghosts (2.15). In order to introduce the hybrid fields, we first carry out a linear transformation on these six bosons as

$$\phi_- = - \frac{i}{2} H_0 - \frac{i}{2} H_1 + \frac{i}{2} \sqrt{3} H'_2 - \frac{3}{2} \phi + \chi + \sigma,$$

$$\phi_+ = \frac{i}{2} H_0 - \frac{i}{2} H_1 - \frac{i}{2} \sqrt{3} H'_2 + \frac{1}{2} \phi,$$

$$\phi_+ = \frac{i}{2} H_0 + \frac{i}{2} H_1 + \frac{i}{2} \sqrt{3} H'_2 - \frac{3}{2} \phi + \chi + \sigma,$$

$$\phi_- = - \frac{i}{2} H_0 + \frac{i}{2} H_1 - \frac{i}{2} \sqrt{3} H'_2 + \frac{1}{2} \phi,$$

$$\sqrt{3} i \tilde{H}'_2 = -3 \phi + 2 \chi + \sigma.$$

These relations yield

$$i \tilde{H}_2 = i H_2 - \alpha (\phi - \chi),$$

$$i \tilde{Y} = i Y - Q (\phi - \chi).$$  \hspace{1cm} (3.2)

Then, we define the space-time spinor fields and their conjugates as

$$\Theta^\alpha = k^{-\frac{1}{4}} e^{\phi_+}, \hspace{1cm} \bar{\Theta}^\alpha = k^{\frac{1}{4}} e^{\phi_-},$$

$$\mathcal{P}_\alpha = k^{\frac{1}{4}} e^{-\phi_+}, \hspace{1cm} \bar{\mathcal{P}}_\alpha = k^{-\frac{1}{4}} e^{-\phi_-}, \hspace{1cm} (\alpha = \pm).$$  \hspace{1cm} (3.3)
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satisfying

$$\Theta^\alpha(z)P_\beta(w) \sim \frac{\delta^\alpha_\beta}{z-w}, \quad \Theta^\alpha(z)\bar{P}_\beta(w) \sim \frac{\delta^\alpha_\beta}{\bar{z}-\bar{w}}. \quad (3.4)$$

The basic fields in the hybrid formalism are finally obtained as the superspace coordinates (and conjugates of fermionic coordinates) with an additional boson: \((X^0, X^1, \phi_L, \bar{Y}; \Theta^\alpha, \bar{\Theta}^\alpha, P_\alpha, \bar{P}_\alpha, \rho)\). The \(U(1)\) boson in the \(N/U(1)\) sector is modified to \(\hat{H}_2\), which requires modifications of the \(N=2\) superconformal generators to \((\hat{T}_{N/U(1)}, \hat{G}_{\pm}^{N/U(1)}; \hat{I}_{N/U(1)})\), which are uniquely determined through the relation

$$\hat{I}_{N/U(1)} = -\alpha i\partial \hat{H}_2. \quad (3.5)$$

We note here that these new generators of the \(N/U(1)\) sector completely (anti-)commute with the hybrid fields in the \(AdS_3 \times S^1\) sector.

In terms of these hybrid fields, the space-time supercharges are written

$$G^+_1 = \oint \frac{dz}{2\pi i} \bar{P}_+, \quad G^-_1 = \oint \frac{dz}{2\pi i} \left( P_+ + \Theta^+ e^{\gamma_1} (n^0 + X^1) \left( \frac{1}{\beta} i\partial X^1 + \frac{1}{Q} \partial \phi_L \right) \right) - \Theta^+ \left( \frac{1}{\beta} i\partial X^0 \pm \frac{1}{Q} i\partial \bar{Y} \mp i\partial \rho \pm \Theta^\pm P_+ \mp \Theta^\mp \bar{P}_+ \right), \quad (3.6)$$

where we have carried out the similarity transformation generated by

$$R_0 = -\oint \frac{dz}{2\pi i} \left( \frac{\sqrt{2}}{Q} e^{-i\rho} \Theta^+ \Theta^- \hat{G}^-_{N/U(1)} \right), \quad (3.7)$$

so that the supersymmetry may be closed in the \(AdS_3 \times S^1\) sector. In our case, this space-time supersymmetry is enlarged to the \(N=2\) superconformal symmetry with \(c = 6kp\). We can explicitly construct their generators as

$$L_n = \oint \frac{dz}{2\pi i} \left( \gamma^n \left( -\frac{1}{\beta} i\partial X^0 - n \frac{1}{Q} \partial \phi_L \right) \right. \left. + \frac{1}{2} (n^2 - 1) \gamma^n (\Theta^+ P_+ - \Theta^- P_- + \Theta^+ \bar{P}_+ - \Theta^- \bar{P}_- ) \right. \left. - \frac{1}{2} n(n+1) \gamma^{n-1} (\Theta^+ P_+ + \Theta^+ \bar{P}_- ) + \frac{1}{2} n(n-1) \gamma^{n+1} (\Theta^- P_+ + \Theta^- \bar{P}_+) \right), \quad (3.8a)$$

$$G^+_r = \oint \frac{dz}{2\pi i} \left( \frac{1}{2} - r \right) \gamma^{r+\frac{1}{2}} \bar{P}_+ + \left( \frac{1}{2} + r \right) \gamma^{r-\frac{1}{2}} \bar{P}_- \right), \quad (3.8b)$$

$$G^-_r = \oint \frac{dz}{2\pi i} \left( \frac{1}{2} - r \right) \gamma^{r+\frac{1}{2}} (P_+ + \Theta^+ e^{\gamma_1} (n^0 + X^1) \left( \frac{1}{\beta} i\partial X^1 + \frac{1}{Q} \partial \phi_L \right) \right) \right. \left. - \Theta^+ \left( \frac{1}{\beta} i\partial X^0 \pm \frac{1}{Q} i\partial \bar{Y} \mp i\partial \rho \pm \Theta^\pm P_+ \mp \Theta^\mp \bar{P}_+ \right) \right) \right), \quad (3.8c)$$
where $\gamma = e^{-\beta i(X^0 + X^1)}$. We use these explicit forms to obtain the transformation laws of some lower-level physical fields in §6.

In order to obtain the physical state conditions, we must also rewrite the $N = 4$ topological superconformal generators (2.25) in terms of the hybrid fields. Here we present only the topological superconformal generators (2.25) in terms of the hybrid fields. Here we present only the $N = 2$ subset, in the forms they take after the similarity transformation $R_0$, which are necessary and sufficient to obtain all the $N = 4$ generators:

\begin{align}
G^+ &= \frac{Q}{\sqrt{2}} e^{-ip} (1 + \Theta^+ \bar{\Theta}^- - \Theta^- \bar{\Theta}^+) d_+ \bar{d}_- + \tilde{G}_N^{+/(U(1)},
G^- &= \frac{Q}{\sqrt{2}} e^{ip} d_+ \bar{d}_- + \tilde{G}_N^{-/(U(1)},
I &= i \partial \bar{\rho} - Qi \partial \bar{Y} + \tilde{I}_N^{U/(1)},
\end{align}

where

\begin{align}
d_+ &= \mathcal{P}_+,
\bar{d}_- &= \bar{\mathcal{P}}_- - \Theta^+ e^{\pm \beta i(X^0 + X^1)} \left( \frac{1}{\beta} i \partial X^1 \pm \frac{1}{Q} \partial \phi_L \right)
+ \Theta^+ \left( \frac{1}{\beta} i \partial X^0 \pm \frac{1}{Q} i \partial \bar{Y} \mp i \partial \rho \pm \Theta^\pm \mathcal{P}_\pm \mp \bar{\Theta}^+ \bar{\mathcal{P}}_+ \right)
\end{align}
\[\pm \Theta^\pm \Theta^\mp \mp + 2 \Theta^\pm \Theta^\mp \Theta^\mp \left( \frac{1}{Q} i \partial \tilde{Y} - i \partial \rho \right) + \frac{2}{Q^2} \Theta^\pm \Theta^\mp \partial \Theta^\mp + 2 \partial (\Theta^\pm \Theta^\mp) \Theta^\mp, \tag{3.10}\]

and \(\times\) denotes normal ordering with respect to the coefficient fields and the currents \(d_\alpha\) and \(\bar{d}_\alpha\). The zero modes \(G^+_0\), \(\tilde{G}^+_0\) and \(I_0\) are (anti-)commutative with the supercharges (3.8). This guarantees a supersymmetric physical spectrum. Although these forms are useful for explicit calculation, and will be used in §5, the hybrid fields possess some non-trivial hermiticity properties. We must carry out a further similarity transformation to obtain fields with conventional hermiticity. We present this similarity transformation in Appendix B.

§4. Spectral flow and the Hilbert space of the hybrid superstring

Now, let us study the Hilbert space of the hybrid superstring, including spectrally flowed representations.\(^{10}\) We consider the structure of the \(sl(2)\) currents (A.2), because the spectral flow is defined by their automorphism. They can be decomposed into three independent parts as

\[J^a = j^a + \sum_{l=1}^{2} J^a_{(l)}, \tag{4.1}\]

where \(j^a\) represents the bosonic currents given in (2.4), and

\[J^\pm_{(1)} = \pm \Theta^\pm \mathcal{P}_\mp, \quad J^3_{(1)} = \frac{1}{2}(\Theta^+ \mathcal{P}_+ - \Theta^- \mathcal{P}_-), \]
\[J^\pm_{(2)} = \pm \Theta^\pm \mathcal{P}_\mp, \quad J^3_{(2)} = \frac{1}{2}(\Theta^+ \mathcal{P}_+ - \Theta^- \mathcal{P}_-). \tag{4.2}\]

are the two independent fermionic currents of level \(-1\).

The flowed representation of the bosonic part is defined by\(^{10}\)

\[j^3_0 |j, m, p\rangle_0 = \left( m - \left( \frac{k + 2}{2} \right) p \right) |j, m, p\rangle_0, \]
\[j^\pm_{\mp} |j, m, p\rangle_0 = (m \pm j) |j, m \mp 1, p\rangle_0, \]
\[j^3_n |j, m, p\rangle_0 = 0, \quad j^\pm_{n \mp} |j, m, p\rangle_0 = 0, \quad (n > 0) \tag{4.3}\]

which is realized as the oscillator vacuum of the free-field realization of the discrete light-cone Liouville theory as\(^{11}\)

\[|j, m, p\rangle_0 = e^{-i(m\beta - \frac{k}{2})X^0 - im\beta X^1 - Qj\phi_L (0)|0\rangle_B }, \tag{4.4}\]

where \(|0\rangle_B\) is the bosonic \(SL(2, R)\) invariant vacuum. For string theory on \(AdS_3\), we must include all the spectrally flowed representations of the continuous \(\hat{C}^{(p)}_{\frac{1}{2} + is}\) and the discrete representation \(\hat{D}^{(p)}_j\) for consistency.\(^{10,11}\)
Now, the extension to the fermionic part is obtained by simply replacing $j^a$ with the fermionic current $J^a_{(1)}$ or $J^a_{(2)}$ and assuming that the fermionic $SL(2,R)$ invariant vacuum is its singlet. These conditions are realized by setting $j = m = 0$ and replacing $k + 2 \rightarrow -1$ in (4.3). The flowed representation is then given by the state $|p\rangle_F$, defined by

$$
\Theta^+_{n-p}|p\rangle_F \rightarrow 0, \quad (P^+)_n|p\rangle_F = 0, \quad (n > 0)
$$

$$
\Theta^-_n|p\rangle_F \rightarrow 0, \quad (P^-)_n|p\rangle_F = 0, \quad (n > 0)
$$

$$
\bar{\Theta}^+_{n-p}|p\rangle_F \rightarrow 0, \quad (\bar{P}^+)_n|p\rangle_F = 0, \quad (n > 0)
$$

$$
\bar{\Theta}^-_{n-p}|p\rangle_F \rightarrow 0, \quad (\bar{P}^-)_{n-1-p}|p\rangle_F = 0, \quad (n > 0)
$$

which is explicitly constructed (up to a sign) on the fermionic $SL(2,R)$ invariant vacuum $|p = 0\rangle_F$ as

$$
|p\rangle_F = \left\{ \prod_{r=0}^{p-1} \Theta^+_r \prod_{s=1}^{p} (\bar{P}^-)_s |0\rangle \right\} (4.5)
$$

Moreover, we use the representation

$$
\Theta^+_\rho|\theta,p\rangle_F = |\theta,p\rangle_F \theta^+, \quad (P^+)_\rho|\theta,p\rangle_F = |\theta,p\rangle_F \frac{\partial}{\partial \theta^+},
$$

$$
\Theta^-_\rho|\theta,p\rangle_F = |\theta,p\rangle_F \theta^-, \quad (P^-)_\rho|\theta,p\rangle_F = |\theta,p\rangle_F \frac{\partial}{\partial \theta^-},
$$

$$
\bar{\Theta}^+_\rho|\theta,p\rangle_F = |\theta,p\rangle_F \bar{\theta}^+, \quad (\bar{P}^+)_\rho|\theta,p\rangle_F = |\theta,p\rangle_F \frac{\partial}{\partial \bar{\theta}^+},
$$

$$
\bar{\Theta}^-_\rho|\theta,p\rangle_F = |\theta,p\rangle_F \bar{\theta}^-, \quad (\bar{P}^-)_\rho|\theta,p\rangle_F = |\theta,p\rangle_F \frac{\partial}{\partial \bar{\theta}^-}
$$

for the fermionic “zero-modes”, where $\theta = (\theta^+, \bar{\theta}^\pm)$ are the fermionic coordinates. An explicit expression of this state $|\theta,p\rangle_F$ is obtained from the oscillator state (4.6) as

$$
|\theta,p\rangle_F = (\Theta^+_{-\rho} - \theta^+)(\Theta^-_{\rho} - \theta^-)(\bar{\Theta}^+_\rho - \bar{\theta}^+)(\bar{\Theta}^-_{\rho} - \bar{\theta}^-)|p\rangle_F.
$$

Eventually, the spectrally flowed representation of the total algebra (4.1) is obtained as the tensor product $|j,m,p,\theta\rangle = |j,m,p\rangle_0 \otimes |\theta,p\rangle_F$.

For the $S^1$ direction $\vec{Y}$ and the additional boson $\rho$, we take the oscillator ground states as

$$
|q,l\rangle = e^{i\rho q \vec{Y} - i(l+p)\rho} |0\rangle
$$

where the eigenvalue $l + p$ of the $\rho$-momentum is determined such that the physical state conditions (2.27) are consistent with the spectral flow. The unflowed part $l$ must be restricted to $l = 0, \pm 1$ to avoid the infinite degeneracy due to the picture ambiguity in the RNS formalism.

Together with the representations $|\Delta_N, Q_N\rangle$ of the $N = 2$ superconformal algebra in the $N/U(1)$ sector characterized by

$$
I^{N/U(1)}_0 |\Delta_N, Q_N\rangle \rightarrow \Delta_N |\Delta_N, Q_N\rangle,
$$

$$
J^{N/U(1)}_0 |\Delta_N, Q_N\rangle \rightarrow Q_N |\Delta_N, Q_N\rangle,
$$

(4.10)
we can define the total Hilbert space of the hybrid superstrings on $AdS_3 \times S^1 \times \mathcal{N}/U(1)$ as the tensor product of all these ground states,

$$|j, \theta, l, \Delta_N, Q_N\rangle = |j, m, p\rangle_0 \otimes |\theta, p\rangle_F \otimes |q, l\rangle \otimes |\Delta_N, Q_N\rangle,$$

where the zero-modes $(j, m, p, q)$ are denoted simply by $j$. It is useful to note that the world-sheet energy of this state is given by

$$L_0|j, \theta, l, \Delta_N, Q_N\rangle = \left(-\frac{j(j-1)}{k} + (m-l)p - \frac{k}{4}p^2 + \frac{1}{k}q^2 + \Delta_N - \frac{l^2}{2}\right)|j, \theta, l, \Delta_N, Q_N\rangle,$$

where we use the $U(1)$ condition (2.27c), which now leads to $l + p - Q^2q + Q_N = 0$.

§5. Physical spectrum

In this section, we explicitly investigate the physical spectrum for the oscillator ground states constructed in the previous section. We concentrate on the states whose $\mathcal{N}/U(1)$ sector is the world-sheet chiral primary states characterized by $\Delta_N = \frac{Q_N^2}{2}$. In this case, we find physical states for $l = -1$ ($l = 0$) identified with the first (second) series of the space-time chiral primaries, which have been obtained in the RNS formalism.\(^{11})^*$

Let us first consider the $l = -1$ case in detail. The oscillator ground state is generally given by

$$|V\rangle = |j, \theta, -1\rangle \Phi(j, \theta),$$

where $\Phi$ is the superfield, which is a function of zero-modes $(j, \theta) = (j, m, p, q, \theta^\pm, \bar{\theta}^\pm)$. The eigenvalues of the $\mathcal{N}/U(1)$ sector are omitted, because they are not independent, due to the $U(1)$ condition $Q_N = 1 - p + Q^2q$ and the chirality $\Delta_N = \frac{Q_N^2}{2}$. In terms of this superfield $\Phi$, the equation of motion (2.27a) and the gauge transformation (2.27b) can be written

$$D_- D_+ T_0 \bar{D}_+ \bar{D}_- \Phi = 0,$$

$$\delta \Phi = \bar{D}_+ \bar{A}^+ + \bar{D}_- \bar{A}^-,$$

where

$$T_0 = 1 + \theta^+ \bar{\theta}^- - \theta^- \bar{\theta}^+,$$

$$D_+ = \frac{\partial}{\partial \theta^+}, \quad D_- = \frac{\partial}{\partial \theta^-},$$

$$\bar{D}_+ = \frac{\partial}{\partial \bar{\theta}^+} + \theta^+ \nabla^- + \theta^- \left(\nabla^3 + q - l\right) + \theta^+ \bar{\theta}^- \frac{\partial}{\partial \theta^+} - \theta^- \bar{\theta}^+ \frac{\partial}{\partial \theta^-} \frac{1}{2} + 2\theta^+ \bar{\theta}^- \left(q - l - \frac{k}{2}p\right),$$

* We conjecture that there is no physical state for $l = 1$ in this sector. However, this is not explicitly shown in this paper.
\[ \tilde{D}_- = \frac{\partial}{\partial \theta^-} + \theta^- \nabla^{++} + \theta^+ \left( \nabla^3 - q + l \right) - \theta^+ \theta^- \frac{\partial}{\partial \theta^-} + \theta^+ \bar{\theta}^+ \frac{\partial}{\partial \theta^+} + \theta^- \bar{\theta}^+ \frac{\partial}{\partial \theta^-} - 2 \theta^- \bar{\theta}^+ \theta^+(q - l). \]  

(5.3)

The operators \((\nabla^{\pm \pm}, \nabla^3)\) act on \(\Phi(j, \theta)\) as

\[ \nabla^{\pm \pm} \Phi(j, m, p, q, \theta) = (m \mp 1 \pm j) \Phi(j, m \mp 1, p, q, \theta), \]

\[ \nabla^3 \Phi(j, \theta) = \left( m - \frac{k}{2} \right) \Phi(j, \theta). \]  

(5.4)

We can obtain the gauge transformation (5.2b) by taking the gauge parameter state as, e.g.,

\[ |\Lambda^-\rangle = \tilde{\Theta}^{-1}_{-1+p} |j, \theta, -2\rangle \tilde{\Lambda}^+ |j, \theta, -2\rangle + \tilde{\Theta}^{-1}_1 |j, \theta, -2\rangle \tilde{\Lambda}^- |j, \theta, -2\rangle, \]  

(5.5)

where the quantities \(\tilde{\Lambda}^\pm(j, \theta)\) are the gauge parameter superfields.

In order to solve the equation of motion (5.2a) and find an explicit form of the physical spectrum, we expand the superfields \(\Phi\) and \(\Lambda^\pm\) as

\[ \Phi^\pm = \phi^+ + \theta^+ \bar{\psi}_+ + \theta^- \bar{\psi}_- + \bar{\theta}^+ \psi_+ + \bar{\theta}^- \psi_- + \theta^+ \bar{\theta}^+ v_{++} + \theta^- \bar{\theta}^- v_{--} + \theta^+ \theta^- \psi^3 + \theta^+ \theta^+ \omega + \bar{\theta}^+ \bar{\theta}^- \varphi + \theta^+ \bar{\theta}^+ \chi^+ + \theta^- \bar{\theta}^- \chi^- + \theta^+ \bar{\theta}^+ \bar{\theta}^- \chi^+ + \theta^+ \theta^- \theta^+ \chi^- + \theta^+ \theta^- \bar{\theta}^+ \bar{\theta}^- d, \]  

(5.6a)

\[ \Lambda^+ = \varepsilon^+ + \theta^+ \alpha^+ + \theta^+ \lambda^+ + \bar{\theta}^- \lambda^- + \theta^+ \bar{\theta}^+ \eta^+ + \bar{\theta}^- \eta^- + \theta^+ \bar{\theta}^+ \zeta^+ + \bar{\theta}^- \eta^- - \theta^+ \bar{\theta}^+ f^{++} + \theta^+ \bar{\theta}^- f^{+-} + \theta^+ \theta^- m^{++} + \theta^+ \bar{\theta}^+ m^{+-} + \theta^+ \bar{\theta}^+ \bar{\theta}^- \xi^+ + \theta^- \bar{\theta}^- \xi^-, \]  

(5.6b)

\[ \Lambda^- = \varepsilon^- + \theta^+ \alpha^- + \theta^- \alpha^- + \bar{\theta}^+ \lambda^- + \bar{\theta}^- \lambda^+ + \theta^+ \bar{\theta}^- \eta^+ + \bar{\theta}^+ \eta^- + \theta^+ \bar{\theta}^- \zeta^+ + \bar{\theta}^+ \eta^- + \theta^+ \bar{\theta}^- f^{--} + \theta^+ \bar{\theta}^+ f^{--} + \theta^+ \theta^- m^{--} + \theta^+ \bar{\theta}^- m^{+-} + \theta^+ \theta^- \bar{\theta}^+ \bar{\theta}^- \xi^-, \]  

(5.6c)

where the component fields are functions of \(j = (j, m, p, q)\). The gauge transformation (5.2b) can be given in terms of these component fields as

\[ \delta \phi = \lambda_+^+ + \lambda_-^-, \]

\[ \delta \psi_+ = - \eta_+^- - \eta_+^-, \]

\[ \delta \psi_- = \eta_+^-, \]

\[ \delta \bar{\psi}_+ = \bar{\eta}_+^- + \bar{\eta}_+^- - \nabla^{-} \varepsilon^+ + (\nabla^3 - q - 1) \varepsilon^-, \]

\[ \delta \bar{\psi}_- = - \bar{\eta}_+^+ - \bar{\eta}_+^- + (\nabla^3 + q + 1) \varepsilon^+ + \nabla^{+++} \varepsilon^-, \]

\[ \delta v_{++} = - f^{--} + \nabla^{-} \lambda_+^+ + (\nabla^3 - q) \lambda_+^-, \]

\[ \delta v_3 = \frac{1}{2} f^{++} + \frac{1}{2} \nabla^{+++} \lambda_+^+ + \frac{1}{2} \nabla^{-} \lambda_-^+ + \frac{1}{2} \nabla^3 (\lambda_+^+ + \lambda_-^+) + \frac{1}{2} q (\lambda_+^+ - \lambda_-^+), \]

\[ \delta v_{--} = - f^{++} + (\nabla^3 + q) \lambda_+^+ + \nabla^{+++} \lambda_+^-, \]
where \( K \) is the spectral flow. It should be noted that when \( p \neq 0 \), this contribution depends on \( m \), as indicated explicitly. Equation (5.9c) is interpreted as the Dirac equation on \( m \) and \( \alpha \), as indicated explicitly. Equation (5.9c) is interpreted as the Dirac equation on \( m \) and \( \alpha \).

To fix these invariances, we choose the gauge conditions

\[ \phi = v_{++} = v_3 = v_{--} = v_Y = \omega = 0, \]

\[ \psi_+ = \bar{\psi}_+ = \bar{\psi}_- = \bar{\chi}_+ = \bar{\chi}_- = 0. \]

Then, the remaining fields form the chiral supermultiplet including two bosons and two fermions: \( \Phi(m) = (\varphi(m), d(m - 1); \chi^+(m), \chi^-(m - 1)). \) Here and hereafter, we omit the \((j, p, q)\) dependence, because they are uniform in a given supermultiplet, that is, inert under supersymmetry transformations, as seen in the next section. The equations of motion for these reduced fields become

\[ d(m - 1) = 0, \]

\[ \left( \mathcal{K}(m) + \Delta_N - \frac{1}{2} \right) \varphi(m) = 0, \]

\[ \left( \begin{array}{cc} m - \frac{k}{2}p - q - 1 & m - 1 + j \\ m - j & m - \frac{k}{2}p + q \end{array} \right) \left( \begin{array}{c} \chi^+(m) \\ \chi^-(m - 1) \end{array} \right) = 0, \]

where \( \mathcal{K}(m) \) is the Klein-Gordon operator on \( AdS_3 \times S^1 \), defined by

\[ \mathcal{K}(m) = -j \frac{(j - 1)}{k} + \frac{q^2}{k} + mp - \frac{k}{4}p^2, \]

where the last two terms, \( mp \) and \( -\frac{k}{4}p^2 \), are among the mass terms induced by the spectral flow. It should be noted that when \( p \neq 0 \), this contribution depends on \( m \), as indicated explicitly. Equation (5.9c) is interpreted as the Dirac equation on \( AdS_3 \times S^1 \).

Using the generators (3.8a) and (3.8d), one can easily compute the space-time conformal weight and the \( R \)-charge of the physical boson \( \varphi \) as \( \Delta_{s-t} = -m + \frac{k}{2}p \) and
\[ Q_{s-t} = 2q = k(Q_N + p - 1). \]

To compare this with the results in the RNS formalism, we take \( \mathcal{N}/U(1) = SU(2)/U(1) \times T^4 \), where the \( SU(2)/U(1) \) sector is represented by the Kazama-Suzuki model\(^{17}\) with \( c = 3 - \frac{6}{k} \). If we consider the ground state for the \( T^4 \) sector, the \( U(1) \) charge spectrum of the world-sheet chiral primaries is given by \( Q_N = 2\Delta_N = s/k \) \( (s = 0, 1, \ldots, k - 2) \). Then the space-time \( R \)-charge becomes

\[ Q_{s-t} = s + (p - 1)k, \quad (5.11) \]

which coincides with the spectrum of the first series of the space-time chiral primaries obtained in Ref. 11). Actually, the space-time conformal weight can be independently calculated as

\[ \Delta_{s-t} = -m - \frac{k}{2}p = \frac{Q_{s-t}}{2} \quad (5.12) \]

by taking \( j = m = -q + \frac{k}{2}p = \frac{1}{2}(k - s) \) to solve the on-shell condition \( (5.9b)^* \). The fermionic partner can be obtained similarly by solving the Dirac equations given in \( (5.9c) \). They form the on-shell chiral supermultiplet \((\varphi(m), \chi^+(m))\) of the space-time chiral primaries with \( j = m = \frac{1}{2}(k - s) \).

For \( l = 0 \), the \( U(1) \) condition leads to \( q = \frac{1}{2}k(p + Q_N) \). The equation of motion \( (2.27a) \) becomes

\[
\left( 2D_-D_+T_0\bar{D}_+\bar{D}_- + D_+\bar{D}_-D_-T_0\bar{D}_+ - D_-T_0\bar{D}_-D_+\bar{D}_+ \\
+ D_-\bar{D}_+D_+T_0\bar{D}_- - D_+T_0\bar{D}_+D_-\bar{D}_- + D_+\bar{D}_-D_+T_0\bar{D}_+ - D_-T_0\bar{D}_+ \right) \Phi = 0. \quad (5.13)
\]

The explicit forms of the gauge transformation \( (2.27b) \) are different for the following two cases. For \( Q_N \neq 0 \), it is given by

\[ \delta \Phi = \bar{D}_+\bar{D}_-T_0\bar{\Sigma}, \quad (5.14) \]

where the gauge parameter superfield comes from

\[ |A^-\rangle = |j, \theta, -1\rangle \bar{\Sigma}(j, \theta). \quad (5.15) \]

The case \( Q_N = 0 \) corresponds to the ground state in the \( \mathcal{N}/U(1) \) sector, which is often called “compactification independent”. The gauge symmetry is enlarged in this case to

\[ \delta \Phi = \bar{D}_+\bar{D}_-T_0\bar{\Sigma} + D_+D_-\Sigma, \quad (5.16) \]

where the additional transformation comes from the state

\[ |A^+\rangle = |j, m, p, q + 1, \theta, 2\rangle \Sigma(j, \theta), \quad (5.17) \]

with \( U(1) \) charge \(-3 + Q^2\) in the \( \mathcal{N}/U(1) \) sector.

\(^*\) The \( p = 0 \) case is an exception. In that case, the on-shell condition can be solved by setting \( j = -q \), with \( m \) arbitrary.
If we expand \( \bar{\Sigma} \) and \( \Sigma \) as

\[
\bar{\Sigma} = \bar{\tau} + \theta^+ \bar{\zeta}_+ + \theta^- \bar{\zeta}_- + \bar{\theta}^+ \bar{\epsilon}_+ + \bar{\theta}^- \bar{\epsilon}_-
\]

\[
- \theta^+ \bar{\theta}^+ \bar{\lambda}_{++} + \theta^- \bar{\theta}^- \bar{\lambda}_{--} + \bar{\theta}^+ \bar{\theta}^- \bar{\lambda}_{+-} - \theta^+ \bar{\theta}^- \bar{\lambda}_{-+} - \theta^+ \theta^- \sigma + \bar{\theta}^+ \bar{\theta}^- \bar{\lambda}
\]

\[
+ \theta^- \bar{\theta}^+ \bar{\eta}^- + \theta^+ \bar{\theta}^+ \bar{\eta}^+ - \theta^+ \theta^- \bar{\xi}^+ + \theta^+ \theta^- \bar{\xi}^- + \theta^+ \theta^- \bar{\xi}^+ + \theta^+ \theta^- \bar{\xi}^-
\]

\[
\bar{\Sigma} = \tau + \theta^+ \epsilon_+ + \theta^- \epsilon_+ + \bar{\theta}^+ \zeta_+ + \bar{\theta}^- \zeta_-
\]

\[
- \theta^+ \bar{\theta}^+ \lambda_{++} + \theta^- \bar{\theta}^- \lambda_{--} + \bar{\theta}^+ \bar{\theta}^- \lambda_{+-} - \theta^+ \bar{\theta}^- \lambda_{-+} - \theta^+ \theta^- \lambda + \bar{\theta}^+ \bar{\theta}^- \sigma
\]

\[
+ \theta^- \bar{\theta}^+ \xi^+ - \theta^+ \bar{\theta}^+ \xi^- - \theta^+ \theta^- \eta^+ + \theta^+ \theta^- \eta^- + \theta^+ \theta^- \bar{\eta}^+
\]

\[
(5.18)
\]

and \( \Phi \) as (5.6a), the gauge transformations of the component fields for \( Q_N \neq 0 \) are given by

\[
\delta \phi = - \bar{\lambda},
\]

\[
\delta \bar{\psi}_- = - \bar{\eta}^+ - \nabla^+ \bar{\epsilon}_+ + (\nabla^3 + q) \bar{\epsilon}_-, \quad \delta \psi_+ = 0,
\]

\[
\delta \bar{\psi}_+ = \bar{\eta}^- + \nabla^- \epsilon_+ + (\nabla^3 - q) \bar{\epsilon}_+, \\
\delta \psi_- = 0,
\]

\[
\delta v_{++} = - \nabla^- \bar{\lambda}, \\
\delta v_3 = - \nabla^3 \bar{\lambda}, \\
\delta v_{--} = - \nabla^+ \bar{\lambda}, \\
\delta v_Y = - q \bar{\lambda}, \\
\delta \varphi = 0,
\]

\[
\delta \omega = - \bar{f} + \nabla^+ \bar{\lambda}_{++} - \nabla^- \bar{\lambda}_{--} + (\nabla^3 - q - 1) \bar{\lambda}_{++} + (\nabla^3 + q + 1) \bar{\lambda}_{+-} + (K_{(m+1)} + \Delta_N) \bar{\tau}, \\
\delta \chi^- = 0,
\]

\[
\delta \bar{\chi}^+ = - \nabla^+ \bar{\eta}^- - (\nabla^3 + p - q - 1) \bar{\eta}^+ - (K_{(m)} + \Delta_N) \bar{\epsilon}_-, \\
\delta \chi^+ = 0,
\]

\[
\delta \bar{\chi}^- = \nabla^- \bar{\eta}^+ + (\nabla^3 + q + 1) \bar{\eta}^- + (K_{(m+1)} + \Delta_N) \bar{\epsilon}_+, \\
\delta d = (K_{(m)} + \Delta_N) \bar{\lambda}. \\
(5.19)
\]

We can fix these invariances by taking the gauge conditions

\[
\phi = \omega = \bar{\psi}_+ = \bar{\psi}_- = 0. \quad (5.20)
\]

The off-shell supermultiplet \( V_M(m) = (v_{++}(m - 1), v_3(m), v_{--}(m + 1), v_Y(m), \varphi(m + 1), d(m); \psi_+(m), \psi_-(m + 1), \chi^+(m + 1), \chi^-(m), \bar{\chi}^+(m), \bar{\chi}^-(m - 1)) \) includes six degrees of freedom for both bosons and fermions. The equations of motion (5.13) therefore reduce to

\[
d(m) = \varphi(m + 1) = 0,
\]
\( (K_{(m)} + \Delta_N) \begin{pmatrix} v_{++}(m-1) \\ v_3(m) \\ v_{--}(m+1) \\ v_Y(m) \end{pmatrix} = 0, \)

\[(m - 1 + j)v_{++}(m - 1) - 2 \left( m - \frac{k}{2} \right) v_3(m) + (m + 1 - j)v_{--}(m + 1) + 2qv_Y(m) = 0, \]

\( \begin{pmatrix} m - \frac{k}{2}p - q + 1 & m + j \\ m + 1 - j & m - \frac{k}{2}p + q \end{pmatrix} \begin{pmatrix} \chi^+(m+1) \\ \chi^-(m) \end{pmatrix} - 2k\Delta_N \begin{pmatrix} \psi^-(m+1) \\ \psi^+(m) \end{pmatrix} = 0, \)

\( \begin{pmatrix} m - \frac{k}{2}p + q & -(m + j) \\ -(m + 1 - j) & m - \frac{k}{2}p - q + 1 \end{pmatrix} \begin{pmatrix} \psi^-(m+1) \\ \psi^+(m) \end{pmatrix} - \begin{pmatrix} \chi^+(m+1) \\ \chi^-(m) \end{pmatrix} = 0, \)

\( \begin{pmatrix} m - \frac{k}{2}p + q & m - 1 + j \\ m - j & m - \frac{k}{2}p - q - 1 \end{pmatrix} \begin{pmatrix} \chi^+(m) \\ \chi^-(m-1) \end{pmatrix} = 0. \)

These are the equations of motion for the massive vector supermultiplet on \( AdS_3 \times S^1. \)

The space-time \( R \)-charge of the vector field \( (v_{++}(m-1), v_3(m), v_{--}(m+1), v_Y(m)) \) is \( Q_{s-t} = s + kp, \) which coincides with the spectrum of the second series (with \( s \neq 0 \)) of the space-time chiral primaries in Ref. 11). The equations of motion (5.21) can be solved by setting \( j = 1 - m = q + 1 - \frac{k}{2}p,^{*} \) which leads the on-shell supermultiplet \( (v_{++}(m-1), v_3(m), v_{--}(m+1); \bar{\chi}^-(m-1), \chi^-(m), \chi^+(m+1)) \) to be the space-time chiral \( \Delta_{s-t} = \frac{Q_{s-t}}{2}. \)

The gauge invariances (5.16) for the case \( Q_N = \Delta_N = 0 \) written in the component fields

\[
\begin{align*}
\delta \phi &= - \lambda_{+-} + \lambda, \\
\delta \bar{\psi}_- &= - \bar{\eta}^+ - \nabla^{++} \bar{\epsilon}_+ + (\nabla^3 + q)\bar{\epsilon}_-, \\
\delta \psi_+ &= - \eta^-, \\
\delta \bar{\psi}_+ &= \bar{\eta}^- + \nabla^{--} \epsilon_- - (\nabla^3 - q)\epsilon_+, \\
\delta \psi_- &= \eta^+, \\
\delta v_{++} &= - \nabla^{--} \bar{\lambda}, \\
\delta v_3 &= - \nabla^3 \bar{\lambda}, \\
\delta v_{--} &= - \nabla^{++} \bar{\lambda}, \\
\delta v_Y &= - q\bar{\lambda}, \\
\delta \omega &= - f + \nabla^{++} \bar{\lambda}_{++} - \nabla^{--} \bar{\lambda}_{--} + (\nabla^3 - q - 1)\bar{\lambda}_{++} + (\nabla^3 + q + 1)\bar{\lambda}_{--} + K_{(m+1)} \bar{\tau}, \\
\delta \varphi &= - f, \\
\delta \chi^- &= 0, \\
\delta \bar{\chi}^+ &= - \nabla^{++} \bar{\eta}^- - (\nabla^3 - q - 1)\bar{\eta}^+ - K_{(m)} \bar{\epsilon}_-, \\
\end{align*}
\]

\(^{*}\) The value of \( m \) is also not fixed by the equations of motion for \( p = 0. \)
\[ \delta \chi^+ = 0, \]
\[ \delta \chi^- = \nabla^+ \bar{\eta} + (\nabla^3 + q + 1) \bar{\eta} + \mathcal{K}_{(m+1)} \bar{\epsilon}^+, \]
\[ \delta d = - \mathcal{K}_{(m)} \bar{\epsilon}_-. \]

(5.22)

can be fixed by taking the gauge
\[ \phi = \omega = \varphi = \psi_\pm = \bar{\psi}_\pm = 0. \]

(5.23)

The equations of motion (5.13) are given in this gauge by
\[
\mathcal{K}_{(m)} \begin{pmatrix} v_{++}(m-1) \\ v_3(m) \\ v_{--}(m+1) \\ v_Y(m) \end{pmatrix} + \begin{pmatrix} m-j \\ m-k \frac{p}{2} \\ m+j \\ q \end{pmatrix} d(m) = 0,
\]
\[
d(m) = - \frac{1}{2} (m-1+j) v_{++}(m-1) + (m+1-j) v_{--}(m+1) - 2 \left( m - \frac{k}{2} p \right) v_3 + 2q v_Y(m),
\]
\[
\begin{pmatrix} m+1-kp \\ m+1-j \end{pmatrix} \begin{pmatrix} m+j \\ m \end{pmatrix} \begin{pmatrix} \chi^+(m+1) \\ \chi^-(m) \end{pmatrix} = 0,
\]
\[
\begin{pmatrix} m \\ m-j \end{pmatrix} \begin{pmatrix} m+1+j \\ m-1-kp \end{pmatrix} \begin{pmatrix} \bar{\chi}^+(m) \\ \bar{\chi}^-(m-1) \end{pmatrix} = 0,
\]

(5.24)

which are still invariant under the residual gauge transformation
\[
\delta \begin{pmatrix} v_{++}(m-1) \\ v_3(m) \\ v_{--}(m+1) \\ v_Y(m) \end{pmatrix} = - \begin{pmatrix} m-j \\ m-k \frac{p}{2} \\ m+j \\ q \end{pmatrix} \bar{\lambda}(m).
\]

(5.25)

The equations (5.24) thus should be interpreted as the Maxwell and Dirac equations on \( AdS_3 \times S^1 \). The off-shell superfield \( \mathbf{V}(m) = (v_{++}(m-1), v_3(m), v_{--}(m+1), v_Y(m), d(m); \bar{\chi}^-(m-1), \bar{\chi}^+(m), \chi^-(m), \chi^+(m+1)) \) represents the massless vector supermultiplet. The on-shell physical fields \( (v_{++}(m-1), v_{--}(m+1); \bar{\chi}^-(m-1), \chi^+(m+1)) \) with \( j = 1 - m = 1 \) provide the missing state \( s = 0 \) in the second series. It should be stressed that the physical spectra differ significantly in the cases \( s = 0 \) and \( s \neq 0 \). The former (latter) is described by the massive (massless) vector supermultiplet with \( 3+3 \) \( (2+2) \) on-shell degrees of freedom for bosons and fermions.

\section*{§6. Space-time superconformal symmetry}

To this point we have investigated the hybrid formalism of the superstring on \( AdS_3 \times S^1 \), preserving the manifest space-time \( N = 2 \) superconformal symmetry. We have also identified the two series of the space-time chiral primaries and clarified their supersymmetry structure. They are described by the chiral and the vector supermultiplets, but the structure of the vector supermultiplet depends on whether
it is massive or massless, as is well known. The massless vector supermultiplet has fewer on-shell physical degrees of freedom due to the gauge invariance.

In this section, we present explicit forms of the supersymmetry transformations on these supermultiplets by taking the Wess-Zumino (WZ)-like gauges (5.8), (5.20) and (5.23), for which the supersymmetry structure is easy to understand, because the numbers of fields needed to realize off-shell supersymmetry are minimal. We must distinguish two cases here, whether \( p = 0 \) or \( p \neq 0 \), because the manners in which the off-shell supersymmetry is realized differ significantly in these cases. For vanishing light-cone momentum \( p = 0 \), all of the infinite numbers of supersymmetries are realized on the supermultiplet. For non-vanishing light-cone momentum \( p \neq 0 \), by contrast, the space-time symmetry becomes a spectrum generating symmetry, since the spectral flow operation does not commute with the world-sheet Hamiltonian \( L_0 \). Only two of the supersymmetries are closed on the supermultiplet, and the others generate new physical states with different masses.

Let us begin by investigating the first series, \( l = -1 \), found to form the chiral supermultiplet \( \Phi(m) = (\varphi(m), d(m-1); \chi^{+}(m), \chi^{-}(m-1)) \) in the WZ-like gauge (5.8). For \( p = 0 \), the entire \( N = 2 \) superconformal symmetry is realized on the supermultiplet, and the component fields are transformed as

\[
\begin{align*}
\delta^+_{r} \varphi(m) &= 0, \\
\delta^+_{r} d(m-1) &= \left( m - 1 - 2r + \left( \frac{1}{2} + r \right) j + \left( \frac{1}{2} - r \right) q \right) \chi^-(m - r - 3/2) \\
&\quad - \left( m - 1 - 2r - \left( \frac{1}{2} - r \right) j - \left( \frac{1}{2} + r \right) q \right) \chi^+(m - r - 1/2), \\
\delta^+_{r} \chi^+(m) &= - \left( m - r - \frac{1}{2} + \left( \frac{1}{2} + r \right) j + \left( \frac{1}{2} - r \right) q \right) \varphi(m - r - 1/2), \\
\delta^+_{r} \chi^-(m-1) &= \left( m - r - \frac{1}{2} - \left( \frac{1}{2} - r \right) j - \left( \frac{1}{2} + r \right) q \right) \varphi(m - r - 1/2), \\
\delta^-_{r} \varphi(m) &= - \left( \frac{1}{2} - r \right) \chi^-(m - r - 1/2) + \left( \frac{1}{2} + r \right) \chi^+(m - r + 1/2), \\
\delta^-_{r} d(m-1) &= 0, \\
\delta^-_{r} \chi^+(m) &= \left( \frac{1}{2} - r \right) d(m - r - 1/2), \\
\delta^-_{r} \chi^-(m-1) &= \left( \frac{1}{2} + r \right) d(m - r - 1/2).
\end{align*}
\]

These relations are induced from the actions of \( G^\pm_n \), (3.8b) and (3.8c). We can confirm that they actually satisfy the \( N = 2 \) superconformal algebra with vanishing central charge \( c = 6kp = 0 \):

\[
\begin{align*}
[\delta^L_n, \delta^L_n] &= -(l - n)\delta^L_{n+l}, & [\delta^L_n, \delta^\pm_{n,r}] &= - \left( \frac{n}{2} - r \right) \delta^\pm_{n+r}, \\
[\delta^L_n, \delta^T_n] &= n\delta^T_{n+l}, & [\delta^T_n, \delta^\pm_{n,r}] &= \mp \delta^\pm_{n+r}.
\end{align*}
\]
\[
\{ \delta^+_r, \delta^-_s \} = \delta^c_{r+s} + \frac{1}{2}(r-s)\delta^T_{r+s},
\]

where \( \delta^c_{n} \) and \( \delta^T_{n} \) are bosonic transformations induced from the actions of \( \mathcal{L}_n \) (3.8a) and \( \mathcal{I}_n \) (3.8d), whose explicit forms are given in Appendix C.

For \( p \neq 0 \), however, only two supersymmetries, \( \delta^\pm_{\frac{1}{2}} \), are closed on the chiral supermultiplet:

\[
\begin{align*}
\delta^+_{\frac{1}{2}} \varphi(m) &= 0, \\
\delta^+_{\frac{1}{2}} d(m-1) &= - (m-j)\chi^+(m) - \left( m - \frac{k}{2}p + q \right) \chi^-(m-1), \\
\delta^-_{\frac{1}{2}} d(m-1) &= 0, \\
\delta^-_{\frac{1}{2}} \chi^+(m) &= 0, \\
\delta^-_{\frac{1}{2}} \chi^-(m-1) &= (m-j)\varphi(m), \\
\left( \delta^c_0 - \frac{1}{2}\delta^T_0 \right) \Phi(m) &= - \left( m - \frac{k}{2}p + q \right) \Phi(m).
\end{align*}
\] (6.3)

These satisfy the three-dimensional (one bosonic and two fermionic) subalgebra of (6.2) defined by the single non-trivial relation

\[
\left\{ \delta^+_{\frac{1}{2}}, \delta^-_{\frac{1}{2}} \right\} = \delta^c_0 - \frac{1}{2}\delta^T_0.
\] (6.4)

The right-hand side vanishes on shell, because the on-shell fields are space-time chiral primaries.

The two cases for \( l = 0 \), i.e. \( Q_N \neq 0 \) and \( Q_N = 0 \), have similar structures. For \( p = 0 \) in the \( Q_N \neq 0 \) case, the superconformal transformations on the massive vector supermultiplet \( V_M(m) = (v^+_{++}(m-1), v^+_3(m), v^-_{--}(m+1), v_Y(m), \varphi(m+1), d(m); \psi_+(m), \psi_-(m+1), \chi^+(m+1), \chi^-(m), \bar{\chi}^+(m), \bar{\chi}^-(m-1)) \) are given by

\[
\begin{align*}
\delta^+_rv^+_{++}(m-1) &= - \left( \frac{1}{2} + r \right) \chi^-(m-r-1/2) \\
&\quad - \left( \frac{1}{2} - r \right) (m-j)\psi_+(m-r-1/2) \\
\delta^+_rv^+_3(m) &= \frac{1}{2} \left( \frac{1}{2} - r \right) \chi^-(m-r-1/2) + \frac{1}{2} \left( \frac{1}{2} + r \right) \chi^+(m-r+1/2)
\end{align*}
\]
\[
\begin{align*}
&- \left(\frac{1}{2} - r\right) m \psi_+ (m - r - 1/2) - \left(\frac{1}{2} + r\right) m \psi_-(m - r + 1/2), \\
&\delta_r^+ v_-(m + 1) = - \left(\frac{1}{2} - r\right) \chi^+(m - r + 1/2) \\
&\quad - \left(\frac{1}{2} - r\right) (m + j) \psi_+(m - r - 1/2) \\
&\quad - \left(\frac{1}{2} + r\right) (m + j) \psi_-(m - r + 1/2), \\
&\delta_r^+ v_Y(m) = - \frac{1}{2} \left(\frac{1}{2} - r\right) \chi^-(m - r - 1/2) + \frac{1}{2} \left(\frac{1}{2} + r\right) \chi^+(m - r + 1/2) \\
&\quad - \left(\frac{1}{2} - r\right) q \psi_+(m - r - 1/2) - \left(\frac{1}{2} + r\right) q \psi_-(m - r + 1/2), \\
&\delta_r^+ \varphi(m + 1) = 0, \\
&\delta_r^+ d(m) = \left(\frac{1}{2} - r\right) (-j(j - 1) + q(q + 1)) \psi_+(m - r - 1/2) \\
&\quad + \left(\frac{1}{2} + r\right) (-j(j - 1) + q(q + 1)) \psi_-(m - r + 1/2), \\
&\delta_r^+ \psi_+(m) = - \left(\frac{1}{2} + r\right) \varphi(m - r + 1/2), \\
&\delta_r^+ \psi_-(m + 1) = \left(\frac{1}{2} - r\right) \varphi(m - r + 1/2), \\
&\delta_r^+ \chi^+(m + 1) = 0, \\
&\delta_r^+ \chi^-(m) = 0, \\
&\delta_r^+ \chi^+(m) = - \left(\frac{1}{2} - r\right) d(m - r - 1/2) \\
&\quad - \left(\frac{1}{2} - r\right) (m - j + 1) v_+(m - r - 3/2) \\
&\quad - \left(\frac{1}{2} + r\right) (m - j + 1) (v_3 - v_Y)(m - r - 1/2) \\
&\quad + \left(\frac{1}{2} - r\right) (m - 1 - q) (v_3 + v_Y)(m - r - 1/2) \\
&\quad + \left(\frac{1}{2} + r\right) (m - 1 - q) v_-(m - r + 1/2), \\
&\delta_r^+ \chi^-(m - 1) = - \left(\frac{1}{2} + r\right) d(m - r - 1/2) \\
&\quad + \left(\frac{1}{2} - r\right) (m + q) v_+(m - r - 3/2) \\
&\quad + \left(\frac{1}{2} + r\right) (m + q) (v_3 - v_Y)(m - r - 1/2)
\end{align*}
\]
\[
\delta_r v_{++}(m-1) = -\left(\frac{1}{2} + r\right) (m - j)(v_3 + \nu_Y)(m - r - 1/2)
\]
\[
\delta_r v_3(m) = \left(\frac{1}{2} + r\right) \chi^+(m - r + 1/2) + \frac{1}{2} \left(\frac{1}{2} - r\right) \chi^-(m - r - 1/2),
\]
\[
\delta_r v_{--}(m + 1) = -\left(\frac{1}{2} - r\right) \chi^+(m - r + 1/2),
\]
\[
\delta_r \nu_Y(m) = -\left(\frac{1}{2} + r\right) \chi^+(m - r + 1/2) + \frac{1}{2} \left(\frac{1}{2} - r\right) \chi^-(m - r - 1/2),
\]
\[
\delta_r \varphi(m + 1) = -\left(\frac{1}{2} - r\right) \chi^-(m - r + 1/2) + \left(\frac{1}{2} + r\right) \chi^+(m - r + 3/2)
\]
\[
+ \left(\frac{1}{2} + r\right) \left(\frac{m - r + 1}{2} + j\right)
\]
\[
+ \left(\frac{1}{2} - r\right) \left(\frac{m - r + 3}{2} - q\right) \psi_+(m - r + 1/2)
\]
\[
- \left(\frac{1}{2} - r\right) \left(\frac{m - r + 3}{2} - j\right)
\]
\[
+ \left(\frac{1}{2} + r\right) \left(\frac{m - r + 1}{2} + q\right) \psi_-(m - r + 3/2),
\]
\[
\delta_r d(m) = \left(\frac{1}{2} - r\right) \left(\frac{m - r + 1}{2} - j\right)
\]
\[
+ \left(\frac{1}{2} + r\right) \left(\frac{m - r + 1}{2} + q\right) \chi^+(m - r + 1/2)
\]
\[
+ \left(\frac{1}{2} + r\right) \left(\frac{m - r - 1}{2} + j\right)
\]
\[
+ \left(\frac{1}{2} - r\right) \left(\frac{m - r - 1}{2} - q\right) \chi^-(m - r - 1/2),
\]
\[
\delta_r \psi_+(m) = \left(\frac{1}{2} - r\right) v_{++}(m - r - 1/2) - \left(\frac{1}{2} + r\right) (v_3 + \nu_Y)(m - r + 1/2),
\]
\[
\delta_r \psi_-(m + 1) = \left(\frac{1}{2} + r\right) v_{--}(m - r + 3/2) + \left(\frac{1}{2} - r\right) (v_3 - \nu_Y)(m - r + 1/2),
\]
\[
\delta_r \chi^+(m + 1) = \left(\frac{1}{2} - r\right) d(m - r + 1/2)
\]

\[
+ \left( \frac{1}{2} - r \right) \left( m - r + \frac{3}{2} - j \right) \\
+ \left( \frac{1}{2} + r \right) (m + q) v_{--}(m - r + 3/2)
\]
\[
- \left( \frac{1}{2} + r \right) \left( m - r + \frac{1}{2} + j \right) \\
+ \left( \frac{1}{2} - r \right) \left( m - 2r - q \right) (v_3 + v_Y)(m - r + 1/2)
\]
\[
- \left( \frac{1}{2} - r \right)^2 (v_3 - v_Y)(m - r + 1/2) - \left( \frac{1}{4} - r^2 \right) v_{++}(m - r - 1/2),
\]
\[
\delta_r \chi^-(m) = \left( \frac{1}{2} + r \right) d(m - r + 1/2) \\
+ \left( \frac{1}{2} + r \right) \left( m - r - \frac{1}{2} + j \right) \\
+ \left( \frac{1}{2} - r \right) \left( m + 1 - q \right) v_{++}(m - r - 1/2)
\]
\[
- \left( \frac{1}{2} - r \right) \left( m - r + \frac{1}{2} - j \right) \\
+ \left( \frac{1}{2} + r \right) \left( m - 2r + 1 + q \right) (v_3 - v_Y)(m - r + 1/2) \\
+ \left( \frac{1}{2} + r \right)^2 (v_3 + v_Y)(m - r + 1/2) + \left( \frac{1}{4} - r^2 \right) v_{--}(m - r + 3/2),
\]
\[
\delta_r \chi^+(m) = 0, \\
\delta_r \chi^-(m - 1) = 0.
\]

These also satisfy the \( N = 2 \) superconformal algebra (6.2). In this case, the two supersymmetry transformations on the \( p \neq 0 \) fields become

\[
\delta_{-\frac{1}{2}} v_{++}(m - 1) = - (m - j) \psi_+(m),
\]
\[
\delta_{-\frac{1}{2}} v_{3}(m) = \frac{1}{2} \chi^-(m) - (m - \frac{k}{2} p) \psi_+(m),
\]
\[
\delta_{-\frac{1}{2}} v_{--}(m + 1) = - \chi^+(m + 1) - (m + j) \psi_+(m),
\]
\[
\delta_{-\frac{1}{2}} v_Y(m) = - \frac{1}{2} \chi^-(m) - q \psi_+(m),
\]
\[
\delta_{-\frac{1}{2}} \phi(m + 1) = 0,
\]
\[
\delta_{-\frac{1}{2}} d(m) = k \mathcal{K}(m) \psi_+(m),
\]
\[
\begin{align*}
\delta_{-\frac{1}{2}}^{+}\psi_{+}(m) &= 0, \\
\delta_{-\frac{1}{2}}^{-}\psi_{-}(m + 1) &= \varphi(m + 1), \\
\delta_{-\frac{1}{2}}^{+}\chi_{+}(m + 1) &= 0, \\
\delta_{-\frac{1}{2}}^{-}\chi_{-}(m) &= 0, \\
\delta_{-\frac{1}{2}}^{+}\bar{\chi}_{+}(m) &= -d(m) - (m - 1 + j)v_{++}(m - 1) \\
&\quad + \left(m - \frac{k}{2}p - q - 1\right)(v_{3} + v_{Y})(m), \\
\delta_{-\frac{1}{2}}^{-}\bar{\chi}_{-}(m - 1) &= \left(m - \frac{k}{2}p + q\right)v_{++}(m - 1) - (m - j)(v_{3} + v_{Y})(m - 1), \\
\delta_{\frac{1}{2}}^{-}v_{++}(m - 1) &= -\bar{\chi}_{-}(m - 1), \\
\delta_{\frac{1}{2}}^{+}v_{+}(m) &= \frac{1}{2}\bar{\chi}_{+}(m), \\
\delta_{\frac{1}{2}}^{-}v_{-}(m + 1) &= 0, \\
\delta_{\frac{1}{2}}^{+}\varphi(m + 1) &= \chi_{+}(m + 1) + (m + j)\psi_{+}(m) - \left(m - \frac{k}{2}p + q\right)\psi_{-}(m + 1), \\
\delta_{\frac{1}{2}}^{-}d(m) &= (m - 1 + j)\bar{\chi}_{-}(m - 1) + \left(m - \frac{k}{2}p + q\right)\bar{\chi}_{+}(m), \\
\delta_{\frac{1}{2}}^{+}\psi_{+}(m) &= (v_{3} + v_{Y})(m), \\
\delta_{\frac{1}{2}}^{-}\psi_{-}(m + 1) &= v_{-}(m + 1), \\
\delta_{\frac{1}{2}}^{+}\chi_{+}(m + 1) &= -(m + j)(v_{3} + v_{Y})(m) + \left(m - \frac{k}{2}p + q\right)v_{-}(m + 1), \\
\delta_{\frac{1}{2}}^{-}\chi_{-}(m) &= d(m) + (m - 1 + j)v_{++}(m - 1) \\
&\quad - \left(m - \frac{k}{2}p + q\right)(v_{3} - v_{Y})(m) + (v_{3} + v_{Y})(m), \\
\delta_{\frac{1}{2}}^{+}\bar{\chi}_{+}(m) &= 0, \\
\delta_{\frac{1}{2}}^{-}\bar{\chi}_{-}(m - 1) &= 0, \\
\left(\delta_{0}^{L} - \frac{1}{2}\delta_{0}^{T}\right) V_{M}(m) &= -\left(m - \frac{k}{2}p + q\right) V_{M}(m)v_{++}(m - 1). 
\end{align*}
\]
tions for $p = 0$ are given by

$$\delta^+ v_{++}(m - 1) = -\left(\frac{1}{2} + r\right) \chi^-(m - r - 1/2),$$

$$\delta^+ v_3(m) = \frac{1}{2} \left(\frac{1}{2} - r\right) \chi^-(m - r - 1/2) + \frac{1}{2} \left(\frac{1}{2} + r\right) \chi^+(m - r + 1/2),$$

$$\delta^+ v_{--}(m + 1) = -\left(\frac{1}{2} - r\right) \chi^+(m - r + 1/2),$$

$$\delta^+ v_Y(m) = -\frac{1}{2} \left(\frac{1}{2} - r\right) \chi^-(m - r - 1/2) + \frac{1}{2} \left(\frac{1}{2} + r\right) \chi^+(m - r + 1/2),$$

$$\delta^+ d(m) = 0,$$

$$\delta^+ \chi^+(m + 1) = 0,$$

$$\delta^+ \chi^-(m) = 0,$$

$$\delta^+ \bar{\chi}^+(m) = -\left(\frac{1}{2} - r\right) d(m - r - 1/2)$$

$$-\left(\frac{1}{2} - r\right) (m - 1 + j) v_{++}(m - r - 3/2)$$

$$-\left(\frac{1}{2} + r\right) (m - 1 + j)(v_3 - v_Y)(m - r - 1/2)$$

$$+\left(\frac{1}{2} - r\right) (m - 1)(v_3 + v_Y)(m - r - 1/2)$$

$$+\left(\frac{1}{2} + r\right) (m - 1)v_{--}(m - r + 1/2),$$

$$\delta^+ \bar{\chi}^-(m - 1) = -\left(\frac{1}{2} + r\right) d(m - r - 1/2)$$

$$+\left(\frac{1}{2} - r\right) m v_{++}(m - r - 3/2)$$

$$+\left(\frac{1}{2} + r\right) m(v_3 - v_Y)(m - r - 1/2)$$

$$-\left(\frac{1}{2} - r\right) (m - j)(v_3 + v_Y)(m - r - 1/2)$$

$$-\left(\frac{1}{2} + r\right) (m - j)v_{--}(m - r + 1/2),$$

$$\delta^- v_{++}(m - 1) = -\left(\frac{1}{2} + r\right) \bar{\chi}^-(m - r - 1/2),$$

$$\delta^- v_3(m) = \frac{1}{2} \left(\frac{1}{2} + r\right) \bar{\chi}^+(m - r + 1/2) + \frac{1}{2} \left(\frac{1}{2} - r\right) \bar{\chi}^-(m - r - 1/2),$$

$$\delta^- v_{--}(m + 1) = -\left(\frac{1}{2} - r\right) \bar{\chi}^+(m - r + 1/2),$$
\[
\delta_r v_Y(m) = -\frac{1}{2} \left(\frac{1}{2} + r\right) \bar{\chi}^+(m - r + 1/2) + \frac{1}{2} \left(-\frac{r}{2} - r\right) \bar{\chi}^-(m - r - 1/2),
\]
\[
\delta_r d(m) = \left(\frac{m - \frac{1}{2} - r}{2} (j - 1)\right) \bar{\chi}^+(m - r + 1/2)
+ \left(\frac{m + \frac{1}{2} + r}{2} (j - 1)\right) \bar{\chi}^-(m - r - 1/2),
\]
\[
\delta_r \chi^+(m + 1) = \left(-\frac{1}{2} - r\right) d(m - r + 1/2)
- \left(\frac{m + \frac{1}{2} + r}{2} \right) \left(m - r + \frac{1}{2} + j\right)
+ \left(\frac{m + \frac{1}{2} + r}{2} \right) \left(m - 2r\right) (v_3 + v_Y)(m - r + 1/2)
+ \left(\frac{m + \frac{1}{2} + r}{2} \right) \left(m - r + 3/2\right)
- \left(\frac{m + \frac{1}{2} + r}{2} \right) \left(v_3 + v_Y\right)(m - r + 1/2) - \left(\frac{1}{4} - r^2\right) v_{++}(m - r - 1/2),
\]
\[
\delta_r \chi^-(m) = \left(\frac{m + \frac{1}{2} + r}{2} \right) d(m - r + 1/2)
+ \left(\frac{m + \frac{1}{2} + r}{2} \right) \left(m - r - \frac{1}{2} + j\right)
+ \left(\frac{m + \frac{1}{2} + r}{2} \right) \left(m + 1\right) v_{++}(m - r - 1/2)
- \left(\frac{m + \frac{1}{2} + r}{2} \right) \left(m - r + 1/2 - j\right)
+ \left(\frac{m + \frac{1}{2} + r}{2} \right) \left(m - 2r\right) (v_3 + v_Y)(m - r + 1/2)
+ \left(\frac{m + \frac{1}{2} + r}{2} \right) \left(m - r + 3/2\right)
- \left(\frac{m + \frac{1}{2} + r}{2} \right) \left(\frac{1}{4} - r^2\right) v_{--}(m - r + 3/2),
\]
\[
\delta_r \bar{\chi}^+(m) = 0,
\delta_r \bar{\chi}^-(m - 1) = 0,
\]
which satisfy the \(N = 2\) superconformal algebra (6.2) up to the gauge transformation (5.25), e.g.
\[
\{\delta_r^+, \delta_s^-\} = \delta_r^+ \delta_s^- + \frac{1}{2} (r - s) \delta_{r+s}^+ + \delta(\bar{\lambda}). 
\]
The gauge parameter $\bar{\lambda}$ is field dependent and given by

$$
\bar{\lambda}(m) = - \left( \frac{1}{2} - r \right) \left( \frac{1}{2} + s \right) (v_3 + v_Y)(m - r - s) \\
- \left( \frac{1}{2} + r \right) \left( \frac{1}{2} + s \right) v_-(m - r - s + 1) \\
- \left( \frac{1}{2} - r \right) \left( \frac{1}{2} - s \right) v_+(m - r - s - 1) \\
- \left( \frac{1}{2} + r \right) \left( \frac{1}{2} - s \right) (v_3 - v_Y)(m - r - s).
$$

(6.9)

The two manifest supersymmetries for $p \neq 0$ are now

$$
\delta^+_{\frac{1}{2}} v_+(m - 1) = 0,
\delta^+_{\frac{1}{2}} v_3(m) = \frac{1}{2} \chi^-(m),
\delta^+_{\frac{1}{2}} v_-(m + 1) = - \chi^+(m + 1),
\delta^+_{\frac{1}{2}} v_Y(m) = - \frac{1}{2} \chi^-(m),
\delta^+_{\frac{1}{2}} d(m) = 0,
\delta^+_{\frac{1}{2}} \chi^+(m + 1) = 0,
\delta^+_{\frac{1}{2}} \chi^-(m) = 0,
\delta^+_{\frac{1}{2}} \bar{\chi}^+(m) = - d(m) - (m - 1 + j)v_+(m - 1) \\
+ \left( m - \frac{k}{2}p - q - 1 \right) (v_3 + v_Y)(m),
\delta^+_{\frac{1}{2}} \bar{\chi}^-(m - 1) = - (m - j)(v_3 + v_Y)(m) + \left( m - \frac{k}{2}p + q \right) v_+(m - 1),
\delta^+_{\frac{1}{2}} v_+(m - 1) = - \bar{\chi}^-(m - 1),
\delta^+_{\frac{1}{2}} v_3(m) = \frac{1}{2} \bar{\chi}^+(m),
\delta^+_{\frac{1}{2}} v_-(m + 1) = 0,
\delta^+_{\frac{1}{2}} v_Y(m) = - \frac{1}{2} \bar{\chi}^+(m),
\delta^+_{\frac{1}{2}} d(m) = (m - 1 + j)\bar{\chi}^-(m - 1) + \left( m - \frac{k}{2}p + q \right) \bar{\chi}^+(m),
\delta^+_{\frac{1}{2}} \chi^+(m + 1) = - (m + j)(v_3 + v_Y)(m) + \left( m - \frac{k}{2}p + q \right) v_-(m + 1),
\delta^+_{\frac{1}{2}} \chi^-(m) = d(m) + (m - 1 + j)v_+(m - 1)
$$
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\[-\left(m - \frac{k}{2}p + q\right) (v_3 - v_Y)(m) + (v_3 + v_Y)(m),\]

\[\delta^\frac{1}{2} \chi^+(m) = 0,\]

\[\delta^\frac{1}{2} \chi^-(m - 1) = 0,\]

\[\left(\delta^c_0 - \frac{1}{2} \delta^T_0\right) V(m) = -m V(m),\]

(6.10)

where we have used \(q = \frac{k}{2}p\), obtained from the \(U(1)\) constraint. They also satisfy the sub-algebra of the modified supersymmetry (6.8),

\[\left\{\delta^+, \delta^-\right\} = \delta^c_0 - \frac{1}{2} \delta^T_0 + \delta(\bar{\lambda}),\]

(6.11)

with the gauge parameter

\[\bar{\lambda}(m) = -(v_3 + v_Y)(m).\]

(6.12)

These two supersymmetries are anti-commutative on shell up to the gauge transformation.

\section*{7. Summary and discussion}

In this paper, we have studied superstrings on \(AdS_3 \times S^1\) using a hybrid formalism. The description was obtained through a field redefinition from the world-sheet fields in the RNS formalism.\(^{10,11}\) We found that the space-time supersymmetry is manifestly preserved and closed off shell. The physical spectrum was investigated and identified with two series of space-time chiral primaries found from the analysis in the RNS formalism.\(^{11}\) While the first series is simply represented by an \(AdS\) analog of the chiral supermultiplet, the second series is described by different supermultiplets in the two cases \(Q_N = 0\) and \(Q_N \neq 0\). The former (latter) is the massless (massive) vector supermultiplet with two (three) on-shell physical degrees of freedom for both bosons and fermions.

The supersymmetries on \(AdS_3 \times S^1\) can be enlarged to the boundary \(N = 2\) superconformal symmetry. The entire infinite-dimensional symmetry is realized on the vanishing light-cone momentum, \(p = 0\), states. This results from the fact that the mass spectrum is degenerated with respect to the momentum \(m\). For non-vanishing light-cone momentum, \(p \neq 0\), however, this degeneracy is lifted, and the superconformal symmetry becomes the spectrum generating symmetry. Only two of them are closed on a supermultiplet. These two form a simple subalgebra whose right-hand side vanishes on shell. The other symmetries generate new physical states with different masses.

The Penrose limit of \(AdS_3 \times S^1\) gives an NS-NS plane wave background.\(^{18}\) Hybrid superstrings propagating on this background have already been studied.\(^{5}\) It is interesting to compare the results of the two models and trace, for example, the transition of the physical spectra in this limit. Such an analysis for superstrings...
on $AdS_3 \times S^3$ was recently given within the RNS formalism.\textsuperscript{19) We can carry out a similar analysis for hybrid superstrings on $AdS_3 \times S^1$, keeping all the supersymmetries manifest. We hope to report the results of such analysis in the future.

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Appendix A

The Space-Time $N = 2$ Superconformal Symmetry and the $sl(1|2)$ Current Algebra

In this appendix, we rewrite the space-time $N = 2$ superconformal generators (3.8) in terms of the currents of the global $sl(1|2)$ symmetry.

First, we introduce the local (holomorphic) currents for the global subalgebra $sl(1|2)$ of the $N = 2$ superconformal algebra as

$$L_{\pm 1} = - \oint \frac{dz}{2\pi i} J_{\pm \pm}, \quad L_0 = - \oint \frac{dz}{2\pi i} J^3, \quad I_0 = \oint \frac{dz}{2\pi i} J^Y,$$

$$G_{\pm 1/2}^+ = \oint \frac{dz}{2\pi i} Q_\pm^+, \quad G_{\pm 1/2}^- = \oint \frac{dz}{2\pi i} Q_\pm^-; \quad (A.1)$$

where

$$J_{\pm \pm} = - e^{\mp \beta i(X_0 + X_1)} \left( \frac{1}{\beta} i \partial X^1 \mp \frac{1}{Q} \partial \phi_L \right) \pm \Theta^\pm \mathcal{P}_\mp \pm \bar{\Theta}^\pm \bar{\mathcal{P}}_\mp,$$

$$J_3 = \frac{1}{\beta} i \partial X^0 + \frac{1}{2} (\Theta^+ \mathcal{P}_+ - \Theta^- \mathcal{P}_- + \Theta^+ \bar{\mathcal{P}}_+ - \Theta^- \bar{\mathcal{P}}_-),$$

$$J^Y = 2(\frac{1}{Q} \partial \tilde{Y} - i \partial \rho) + (\Theta^+ \mathcal{P}_+ + \Theta^- \mathcal{P}_- - \Theta^+ \bar{\mathcal{P}}_+ - \Theta^- \bar{\mathcal{P}}_-),$$

$$Q_\pm = \mathcal{P}_\mp + \bar{\Theta}^\pm e^{\mp \beta i(X_0 + X_1)} \left( \frac{1}{\beta} i \partial X^1 \mp \frac{1}{Q} \partial \phi_L \right)$$

$$- \bar{\Theta}^\pm (\frac{1}{\beta} i \partial X^0 \pm \frac{1}{Q} i \partial \tilde{Y} \mp i \partial \rho \pm \Theta^\pm \mathcal{P}_\pm \mp \bar{\Theta}^\mp \bar{\mathcal{P}}_\mp)$$

$$\pm \Theta^\pm \bar{\Theta}^\mp \mathcal{P}_\mp \mp \frac{2}{Q^2} \partial \bar{\Theta}^\pm,$$  \quad (A.2)

$$\bar{Q}_\pm = \bar{\mathcal{P}}_\pm.$$

These currents satisfy the affine Lie superalgebra $sl(1|2)^{(1)}$ at level $k$:

$$J^{++}(z)J^{--}(w) \sim \frac{k}{(z-w)^2} - \frac{2J^3(w)}{z-w}, \quad J^3(z)J^{\pm \pm}(w) \sim \frac{\pm J^{\pm \pm}(w)}{z-w},$$

$$J^3(z)J^3(w) \sim \frac{-k/2}{(z-w)^2}, \quad J^Y(z)J^Y(w) \sim \frac{2(k-2)}{(z-w)^2}.$$
\[ J^{++}(z)Q^-(w) \sim -\frac{Q^+(w)}{z-w}, \quad J^{+-}(z)Q^-(w) \sim -\frac{Q^+(w)}{z-w}, \]
\[ J^{--}(z)Q^+(w) \sim \frac{Q^-(w)}{z-w}, \quad J^{--}(z)\bar{Q}^+(w) \sim \frac{Q^-(w)}{z-w}, \]
\[ J^3(z)Q^\pm(w) \sim \pm\frac{1}{2} \frac{Q^\pm(w)}{z-w}, \quad J^3(z)\bar{Q}^\pm(w) \sim \pm\frac{1}{2} \frac{\bar{Q}^\pm(w)}{z-w}, \]
\[ J^Y(z)Q^\pm(w) \sim -\frac{Q^\pm(w)}{z-w}, \quad J^Y(z)\bar{Q}^\pm(w) \sim \frac{Q^\pm(w)}{z-w}, \]
\[ Q^+(z)\bar{Q}^+(w) \sim -\frac{J^{++}(w)}{z-w}, \quad Q^-(z)\bar{Q}^-(w) \sim -\frac{J^{--}(w)}{z-w}, \]
\[ \bar{Q}^+(z)Q^-(w) \sim \frac{k}{(z-w)^2} - \frac{(J^3 - \frac{1}{2}J^Y)(w)}{z-w}, \]
\[ \bar{Q}^-(z)Q^+(w) \sim \frac{k}{(z-w)^2} - \frac{(J^3 + \frac{1}{2}J^Y)(w)}{z-w}. \]

The \( N = 2 \) superconformal generators can be written in terms of these currents as\(^{13}\)

\[
\mathcal{L}_n = \oint \frac{dz}{2\pi i} \left( -\frac{1}{2}n(n+1)\gamma^{n-1}J^{++} + (n^2 - 1)\gamma^nJ^3 - \frac{1}{2}n(n-1)\gamma^{n+1}J^{--} \right) \otimes,
\]
\[
\mathcal{G}^+_r = \oint \frac{dz}{2\pi i} \left( r + \frac{1}{2} \right) \gamma^{r-\frac{1}{2}}\bar{Q}^+ - \left( r - \frac{1}{2} \right) \gamma^{r+\frac{1}{2}}\bar{Q}^- \otimes,
\]
\[
\mathcal{G}^-_r = \oint \frac{dz}{2\pi i} \left( r + \frac{1}{2} \right) \gamma^{r-\frac{1}{2}}Q^+ - \left( r - \frac{1}{2} \right) \gamma^{r+\frac{1}{2}}Q^- - \left( r^2 - \frac{1}{4} \right) (\Theta^+ + \Theta^-) \gamma^{r-\frac{3}{2}}(J^{++} - 2J^3 + \gamma^2J^{--}) \otimes,
\]
\[
\mathcal{I}_n = \oint \frac{dz}{2\pi i} \left( \gamma^nJ^Y + n\gamma^{n-1}(\Theta^+ + \Theta^-)(\mathcal{P}_- - \mathcal{P}_+) - (\Theta^+ + \Theta^-)(\bar{\mathcal{P}}_- - \bar{\mathcal{P}}_+) \right) \otimes. \tag{A.4}
\]

where \( \otimes \) denotes normal ordering with respect to the coefficient fields and the \( sl(1|2) \) currents.

**Appendix B**

*The Similarity Transformation to Real Variables*

The hybrid fields given in the text are analogs of chiral coordinates. They are convenient for actual calculations but they possess some nontrivial hermiticity properties. We can obtain the corresponding real variables with proper hermiticity.
by carrying out the similarity transformation generated by

\[ R = \oint \frac{dz}{2\pi i} \left( \frac{1}{2} \Theta^+ \Theta + e^{bi(X^0 + X^1)} \left( \frac{1}{\beta} i \partial X^1 + \frac{1}{Q} \partial \phi_L \right) - \frac{1}{2} \Theta^+ \Theta - \left( \frac{1}{\beta} i \partial X^0 - \frac{1}{Q} i \partial \bar{Y} + i \partial \rho - \Theta^- \bar{P}_- + \Theta^+ \bar{P}_+ \right) \right. \]

\[ \left. - \frac{1}{2} \Theta^- \Theta^+ \left( \frac{1}{\beta} i \partial X^0 + \frac{1}{Q} i \partial \bar{Y} - i \partial \rho + \Theta^+ \bar{P}_+ - \Theta^- \bar{P}_- \right) + \frac{1}{2} \Theta^- \Theta - e^{-bi(X^0 + X^1)} \left( \frac{1}{\beta} i \partial X^1 - \frac{1}{Q} \partial \phi_L \right) \right. \]

\[ \left. + \frac{1}{2} \Theta^+ \Theta^- \Theta^+ \left( \frac{1}{Q} i \partial \bar{Y} - i \partial \rho \right) + \frac{1}{2} \left( \frac{1}{Q^2} - 1 \right) \Theta^+ \Theta^- \partial (\Theta^+ \Theta^-) \right) . \]

(B-1)

Under this transformation, \( L_n \) and \( I_n \) are invariant, but the supercharges \( G_r^\pm \) are changed into the symmetric forms with proper hermiticity:

\[ \hat{G}_r^+ = \oint \frac{dz}{2\pi i} \left( \left( r + \frac{1}{2} \right) \gamma^{r-\frac{1}{2}} \left( \bar{P}_- + \frac{1}{2} \Theta^- \gamma \left( \frac{1}{\beta} i \partial X^1 - \frac{1}{Q} \partial \phi_L \right) \right) - \frac{1}{2} \Theta^+ \left( \frac{1}{\beta} i \partial X^0 - \frac{1}{Q} i \partial \bar{Y} + i \partial \rho - \Theta^- \bar{P}_- - \left( r - \frac{3}{2} \right) \Theta^+ \bar{P}_+ + \left( r - \frac{1}{2} \right) \Theta^- \bar{P}_- \right) \right. \]

\[ \left. - \frac{1}{2} \left( r + \frac{1}{2} \right) \Theta^- \Theta^+ \bar{P}_- + \frac{1}{4} \Theta^+ \Theta^- \Theta^+ \left( \frac{1}{Q} i \partial \bar{Y} - i \partial \rho \right) \right. \]

\[ \left. - \frac{1}{4} \left( \frac{1}{Q^2} - 1 \right) \partial (\Theta^+ \Theta^-) \Theta^+ + \frac{1}{2} \Theta^+ \right) \]

\[ \left. \left( r - \frac{1}{2} \right) \gamma^{r+\frac{1}{2}} \left( \bar{P}_+ + \frac{1}{2} \Theta^+ \gamma^{-1} \left( \frac{1}{\beta} i \partial X^1 + \frac{1}{Q} \partial \phi_L \right) \right) - \frac{1}{2} \Theta^- \left( \frac{1}{\beta} i \partial X^0 + \frac{1}{Q} i \partial \bar{Y} - i \partial \rho + \Theta^+ \bar{P}_+ + \left( r + \frac{1}{2} \right) \Theta^+ \bar{P}_+ - \left( r + \frac{3}{2} \right) \Theta^- \bar{P}_- \right) \right. \]

\[ \left. - \frac{1}{2} \left( r - \frac{1}{2} \right) \Theta^- \Theta^+ \bar{P}_+ - \frac{1}{4} \Theta^+ \Theta^- \Theta^- \left( \frac{1}{Q} i \partial \bar{Y} - i \partial \rho \right) \right. \]

\[ \left. + \frac{1}{4} \left( \frac{1}{Q^2} - 1 \right) \partial (\Theta^+ \Theta^-) \Theta^- - \frac{1}{2} \Theta^- \right) \]

\[ \left. - \frac{1}{2} \left( r^2 - \frac{1}{4} \right) \left( \gamma^{r-\frac{3}{2}} \Theta^+ \Theta^+ \bar{P}_- - \gamma^{r+\frac{3}{2}} \Theta^- \Theta^- \bar{P}_+ \right) \right) , \]

\[ = \oint \frac{dz}{2\pi i} \times \left( \left( r + \frac{1}{2} \right) \gamma^{r-\frac{1}{2}} \hat{Q}^+ + \left( r - \frac{1}{2} \right) \gamma^{r+\frac{1}{2}} \hat{Q}^- \right. \]

\[ \left. - \frac{1}{2} \left( r^2 - \frac{1}{4} \right) \gamma^{r-\frac{3}{2}} (\Theta^+ + \gamma \Theta^-) (J^{++} - 2 \gamma J^3 + \gamma^2 J^-) \right) , \]
\[ \hat{G}_r^+ = \oint \frac{dz}{2\pi i} \left( (r + \frac{1}{2}) \gamma^{r-\frac{1}{2}} (p_- + \frac{1}{2} \Theta^- \gamma \left( \frac{1}{\beta} i\partial X^1 - \frac{1}{Q} \partial \phi_L \right) \right) \]

\[ - \frac{1}{2} \Theta^+ \left( \frac{1}{\beta} i\partial X^0 + \frac{1}{Q} i\partial \hat{Y} - i\partial \rho - \left( r \frac{-3}{2} \right) \Theta^+ p_+ + \left( r \frac{-1}{2} \right) \Theta^- - \Theta^+ q_- + \Theta^- q_- \right) \]

\[ + \frac{1}{2} \left( r + \frac{1}{2} \right) \Theta^+ \Theta^- p_- - \frac{1}{4} \Theta^+ \Theta^+ \Theta^+ \left( \frac{1}{Q} i\partial \hat{Y} - i\partial \rho \right) \]

\[ - \frac{1}{4} \left( \frac{1}{Q^2} - 1 \right) \Theta^+ \Theta^+ (\Theta^+ q_-) + \frac{1}{2} \partial \Theta^+ \]

\[ - \left( r \frac{-1}{2} \right) \gamma^{r+\frac{1}{2}} \left( p_+ + \frac{1}{2} \Theta^+ \gamma^{-1} \left( \frac{1}{\beta} i\partial X^1 + \frac{1}{Q} \partial \phi_L \right) \right) \]

\[ - \frac{1}{2} \Theta^+ \left( \frac{1}{\beta} i\partial X^0 - \frac{1}{Q} i\partial \hat{Y} + i\partial \rho + \left( r \frac{-1}{2} \right) \Theta^+ p_+ - \left( r \frac{1}{2} \right) \Theta^- - \Theta^+ q_+ \right) \]

\[ + \frac{1}{2} \left( r \frac{-1}{2} \right) \Theta^- \Theta^+ q_+ + \frac{1}{4} \Theta^- \Theta^- q^- \left( \frac{1}{Q} i\partial \hat{Y} - i\partial \rho \right) \]

\[ + \frac{1}{4} \left( \frac{1}{Q^2} - 1 \right) \Theta^- \Theta^- q^- - \frac{1}{2} \partial \Theta^- \]

\[ + \frac{1}{2} \left( r^2 - \frac{1}{4} \right) \left( \gamma^{r-\frac{3}{2}} \Theta^+ \Theta^+ q_+ - \gamma^{r+\frac{3}{2}} \Theta^- \Theta^- q_+ \right) \]

\[ = \oint \frac{dz}{2\pi i} \left( (r + \frac{1}{2}) \gamma^{r-\frac{1}{2}} q^+ - \left( r \frac{-1}{2} \right) \gamma^{r+\frac{1}{2}} q^- \right) \]

\[ - \frac{1}{2} \left( r^2 - \frac{1}{4} \right) \gamma^{r-\frac{3}{2}} \left( \Theta^+ + \gamma \Theta^- \right) \left( J^{++} - 2 \gamma J^3 + \gamma^2 J^- \right) \]

(B.2)

Here, the transformed fermionic currents (\( \hat{Q}^\pm \), \( \hat{\hat{Q}}^\pm \)) also have symmetric forms:

\[ \hat{Q}^\pm = p_+ + \frac{1}{2} \Theta^\pm e^{\pm \beta (X^0 + X^1)} \left( \frac{1}{\beta} i\partial X^1 \mp \frac{1}{Q} \partial \phi_L \right) \]

\[ - \frac{1}{2} \Theta^\pm \left( \frac{1}{\beta} i\partial X^0 \pm \frac{1}{Q} i\partial \hat{Y} \mp i\partial \rho \pm \Theta^\pm p_\pm \mp \Theta^\pm \hat{p}_\pm \right) \]

\[ \mp \frac{1}{2} \Theta^\pm \Theta^\pm p_\pm - \frac{1}{4} \Theta^\pm \Theta^\pm \Theta^\pm \left( \frac{1}{Q} i\partial \hat{Y} - i\partial \rho \right) \]

\[ - \frac{1}{4} \left( \frac{1}{Q^2} - 1 \right) \Theta^\pm \partial (\Theta^\pm \Theta^\pm) \mp \frac{1}{Q^2} \partial \Theta^\pm, \]

\[ \hat{\hat{Q}}^\pm = \hat{p}_+ + \frac{1}{2} \Theta^\pm e^{\pm \beta (X^0 + X^1)} \left( \frac{1}{\beta} i\partial X^1 \mp \frac{1}{Q} \partial \phi_L \right) \]

\[ - \frac{1}{2} \Theta^\pm \left( \frac{1}{\beta} i\partial X^0 \mp \frac{1}{Q} i\partial \hat{Y} \mp i\partial \rho \mp \Theta^\pm p_\pm \pm \Theta^\pm \hat{p}_\pm \right) \]

\[ \mp \frac{1}{2} \Theta^\pm \Theta^\pm \hat{p}_\pm + \frac{1}{4} \Theta^\pm \Theta^\pm \Theta^\pm \left( \frac{1}{Q} i\partial \hat{Y} - i\partial \rho \right) \]
\[-\frac{1}{4} \left( \frac{1}{Q^2} - 1 \right) \partial \left( \Theta^\pm \Theta^\mp \right) \Theta^\pm \mp \frac{1}{Q^2} \partial \Theta^\pm. \tag{B.3} \]

We note that the new currents \((J^\pm, J^3, J^Y, \hat{Q}^\pm, \hat{\bar{Q}}^\pm)\) satisfy the same \(sl(1|2)\) current superalgebra, \((A.3)\).\(^\ast\)

The similarity transformation also acts on the world-sheet \(N = 2\) superconformal generators. The bosonic generators \(T\) and \(I\) are invariant, but the fermionic \(G^\pm\) are transformed as

\[
G^+ = \frac{Q}{\sqrt{2}} e^{-i\beta \hat{Q}^0} T \hat{d}^+ \hat{d}^- + \hat{G}^+_N/U(1),
\]

\[
G^- = \frac{Q}{\sqrt{2}} e^{i\beta \hat{Q}^0} T \hat{d}^+ \hat{d}^- + \hat{G}^-_N/U(1), \tag{B.4}
\]

where

\[
T = 1 + \frac{1}{2} \Theta^+ \Theta^- - \frac{1}{2} \Theta^+ \Theta^- + \frac{1}{4} \Theta^+ \Theta^- \Theta^+ \Theta^-,
\]

\[
\hat{d}^+ = P^+ - \frac{1}{2} \Theta^+ e^{i\beta \hat{Q}^0} \left( \frac{1}{\beta} i \partial X^1 \pm \frac{1}{Q} \partial \phi_L \right)
\]

\[
+ \frac{1}{2} \Theta^\pm \left( \frac{1}{\beta} i \partial X^0 \pm \frac{1}{Q} i \partial Y \mp i \partial \rho \pm \Theta^\pm P^\pm \mp \Theta^\mp P^\mp \right)
\]

\[
+ \frac{3}{4} \left( \frac{1}{Q^2} - 1 \right) \Theta^\pm \partial (\Theta^\pm \Theta^\mp) - \frac{1}{2Q^2} \partial (\Theta^\pm \Theta^\pm \Theta^\mp),
\]

\[
\hat{d}^- = \bar{P}^+ - \frac{1}{2} \Theta^+ e^{i\beta \hat{Q}^0} \left( \frac{1}{\beta} i \partial X^1 \pm \frac{1}{Q} \partial \phi_L \right)
\]

\[
+ \frac{1}{2} \Theta^\mp \left( \frac{1}{\beta} i \partial X^0 \pm \frac{1}{Q} i \partial Y \mp i \partial \rho \pm \Theta^\pm P^\pm \mp \Theta^\mp P^\mp \right)
\]

\[
+ \frac{3}{4} \left( \frac{1}{Q^2} - 1 \right) \Theta^\mp \partial (\Theta^\pm \Theta^\mp) + \frac{1}{2Q^2} \partial (\Theta^\pm \Theta^\pm \Theta^\mp). \tag{B.5}
\]

The normal ordering denoted by \(\times\times\) in \((B.2)\) and \((B.4)\) is that with respect to the new currents.

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**Appendix C**

**Bosonic Transformation Laws of the Component Fields**

In this appendix, we present the bosonic transformation laws \((\delta L^a_n, \delta I^a_n)\) of the component fields in the WZ-like gauge.

\(^\ast\) This can be easily seen from the fact that the original and the new currents are related by the similarity transformation generated by \(R + R'\), where \(R' = \frac{1}{Q^2} \oint \partial \hat{Q}^0 (\Theta^+ \partial \Theta^- - \Theta^- \partial \Theta^+)\).
For \( l = -1 \), the bosonic transformations on the chiral supermultiplet \((\varphi, d; \chi^\pm)\) are given by

\[
\begin{align*}
\delta_n^L \varphi(m) &= -(m + n(j - 1)) \varphi(m - n), \\
\delta_n^L d(m - 1) &= -(m - 1 + n(j - 1)) d(m - n - 1), \\
\delta_n^L \chi^+(m) &= -\left( m + n(j - 1) + \frac{1}{2}(n^2 - 1) \right) \chi^+(m - n) \\
&\quad - \frac{n}{2}(n - 1) \chi^-(m - n - 1), \\
\delta_n^L \chi^-(m - 1) &= -\left( m - 1 + n(j - 1) - \frac{1}{2}(n^2 - 1) \right) \chi^-(m - n - 1) \\
&\quad + \frac{n}{2}(n + 1) \chi^+(m - n), \\
\delta_n^T \varphi(m) &= 2q \varphi(m - n), \\
\delta_n^T d(m - 1) &= (2d + 2)d(m - n - 1), \\
\delta_n^T \chi^+(m) &= (2q + n + 1) \chi^+(m - n) + n\chi^-(m - n - 1), \\
\delta_n^T \chi^-(m - 1) &= (2q - n + 1) \chi^-(m - n - 1) - n\chi^+(m - n). \\
\end{align*}
\]

(C.1)

For \( l = 0 \) and \( Q_N \neq 0 \), the off-shell fields are the massive vector supermultiplet \((v_{\pm\pm}, v_3, v_Y, \varphi, d; \psi_{\pm}, \chi^\pm, \bar{\chi}^\pm)\), and their transformations are given by

\[
\begin{align*}
\delta_n^L v_{++}(m - 1) &= -(m - 1 + n(j - 1) - n^2 + 1) v_{++}(m - n - 1) \\
&\quad - n(n + 1)v_3(m - n), \\
\delta_n^L v_3(m) &= -(m + n(j - 1)) v_3(m - n) \\
&\quad + \frac{1}{2}n(n - 1)v_{++}(m - n - 1) - \frac{1}{2}n(n + 1)v_{--}(m - n + 1), \\
\delta_n^L v_{--}(m + 1) &= -(m + 1 + n(j - 1) + n^2 - 1) v_{--}(m - n + 1) \\
&\quad + n(n - 1)v_3(m - n), \\
\delta_n^L v_Y(m) &= -(m + n(j - 1)) v_Y(m - n), \\
\delta_n^L \varphi(m + 1) &= -(m + 1 + n(j - 1)) \varphi(m - n + 1), \\
\delta_n^L d(m) &= -(m + n(j - 1)) d(m - n), \\
\delta_n^L \psi_+(m) &= -\left( m + n(j - 1) - \frac{1}{2}(n^2 - 1) \right) \psi_+(m - n) \\
&\quad - \frac{1}{2}n(n + 1)\psi_-(m - n + 1), \\
\delta_n^L \psi_-(m + 1) &= -\left( m + 1 + n(j - 1) + \frac{1}{2}(n^2 - 1) \right) \psi_-(m - n + 1) \\
&\quad + \frac{1}{2}n(n - 1)\psi_+(m - n), \\
\delta_n^L \chi^+(m + 1) &= -\left( m + 1 + n(j - 1) + \frac{1}{2}(n^2 - 1) \right) \chi^+(m - n + 1).
\end{align*}
\]
−1/2n(n − 1)χ−(m − n),

\[ \delta_n^L \chi^-(m) = - \left( m + n(j - 1) - 1/2(n^2 - 1) \right) \chi^-(m - n) + 1/2n(n + 1)\chi^+(m - n + 1), \]

\[ \delta_n^L \chi^+(m) = - \left( m + n(j - 1) + 1/2(n^2 - 1) \right) \chi^+(m - n) - 1/2n(n - 1)\chi^-(m - n - 1), \]

\[ \delta_n^L \bar{\chi}^-(m - 1) = - \left( m - 1 + n(j - 1) - 1/2(n^2 - 1) \right) \bar{\chi}^-(m - n - 1) + 1/2n(n + 1)\bar{\chi}^+(m - n). \]

We can easily confirm that the supersymmetry transformations (6.1) and (6.5) actually satisfy the algebra (6.2) by using these bosonic transformation laws.

For the compactification independent case, \( l = 0 \) and \( Q_N = 0 \), the transformations on the massless vector supermultiplet \((v_\pm, v_3, v_Y, d, \chi^\pm, \bar{\chi}^\pm)\) are obtained as

\[ \delta_n^L v_{++}(m - 1) = - (m - 1 + n(j - 1) - n^2 + 1)v_{++}(m - n - 1) - n(n + 1)v_3(m - n), \]

\[ \delta_n^L v_3(m) = - (m + n(j - 1))v_3(m - n) + 1/2n(n - 1)v_{++}(m - n - 1) - 1/2n(n + 1)v_{--}(m - n + 1), \]

\[ \delta_n^L v_{--}(m + 1) = - (m + n(j - 1) + n^2 - 1)v_{--}(m - n + 1) + n(n - 1)v_3(m - n), \]
\[ \delta_n^C v_Y(m) = - (m + n(j - 1)) v_Y(m - n), \]
\[ \delta_n^C d(m) = - (m + n(j - 1)) d(m - n), \]
\[ \delta_n^C \chi^+(m + 1) = - (m + 1 + n(j - 1) + \frac{1}{2} (n^2 - 1)) \chi^+(m - n + 1) - \frac{1}{2} n(n - 1) \chi^-(m - n), \]
\[ \delta_n^C \chi^-(m) = - (m + n(j - 1) - \frac{1}{2} (n^2 - 1)) \chi^-(m - n) + \frac{1}{2} n(n + 1) \chi^+(m - n + 1), \]
\[ \delta_n^C \bar{\chi}^+(m) = - (m + n(j - 1) + \frac{1}{2} (n^2 - 1)) \bar{\chi}^+(m - n) - \frac{1}{2} n(n - 1) \bar{\chi}^-(m - n - 1), \]
\[ \delta_n^C \bar{\chi}^-(m - 1) = - (m - 1 + n(j - 1) - \frac{1}{2} (n^2 - 1)) \bar{\chi}^-(m - n - 1) + \frac{1}{2} n(n + 1) \bar{\chi}^+(m - n), \]
\[ \delta_n^T v_{++}(m - 1) = 2 n v_Y(m - n), \]
\[ \delta_n^T v_3(m) = 2 n v_Y(m - n), \]
\[ \delta_n^T v_{--}(m + 1) = 2 n v_Y(m - n), \]
\[ \delta_n^T v_Y(m) = 2 n v_3(m - n) - n v_{++}(m - n - 1) + n v_{--}(m - n + 1), \]
\[ \delta_n^T d(m) = 0, \]
\[ \delta_n^T \chi^+(m + 1) = (n - 1) \chi^+(m - n + 1) + n \chi^-(m - n), \]
\[ \delta_n^T \chi^-(m) = - (n + 1) \chi^-(m - n) - n \chi^+(m - n + 1), \]
\[ \delta_n^T \bar{\chi}^+(m) = - (n - 1) \bar{\chi}^+(m - n) + n \bar{\chi}^-(m - n - 1), \]
\[ \delta_n^T \bar{\chi}^-(m - 1) = (n + 1) \bar{\chi}^-(m - n - 1) + n \bar{\chi}^+(m - n). \] (C.3)

We can confirm that the algebra (6.8) holds for the transformations (6.7).

References