4D Equivalence Theorem and Gauge Symmetry on an Orbifold

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We investigate high-energy behavior of scattering amplitudes in extra dimensional gauge theory where the gauge symmetry is broken by the boundary conditions. We study, in particular, the 5D $SU(5)$ grand unified theory whose 5th-dimensional coordinate is compactified on $S^1/Z_2$. We pay attention to the gauge symmetry compatible with the boundary conditions on an orbifold and present the BRST formalism of the 4D theory that is obtained through integration of the 5D theory along the extra dimension. We derive the 4D equivalence theorem on the basis of the Slavnov-Taylor identities. We also calculate the amplitudes of the process including four massive gauge bosons in the external lines and compare them with those for the connected reactions in which the gauge fields are replaced by the corresponding would-be Nambu-Goldstone (NG) fields. We explicitly confirm that the equivalence theorem holds.

§1. Introduction

It is well-known that in 4D gauge theories with explicit gauge symmetry breakings the tree-level amplitudes of four massive gauge bosons in the external lines exhibit bad high-energy behavior, $\sim O(E^4/m^4)$ and $\sim O(E^2/m^2)$, which breaks the unitarity at tree level. (Unitarity at the tree level requires that the tree amplitude of $n$ external particles at high-energy increase with the energy no more rapidly than $O(E^{4-n})$. If this bound holds, we sometimes say that ‘unitarity is maintained’.) Contrastingly, when the gauge bosons obtain their masses through the Higgs mechanism, the power-law behavior $\sim O(E^4/m^4)$ and $\sim O(E^2/m^2)$ is canceled by the contribution of the Higgs bosons.¹⁻⁴ The cancellation is guaranteed by the equivalence theorem which states that the amplitude of massive gauge bosons in the external lines is the same, up to some constant factor, as that of the connected reaction in which the gauge fields are replaced by the corresponding would-be Nambu-Goldstone (NG) fields.⁵⁻¹⁰

With the above considerations, it is natural to inquire what the situation is regarding unitarity in the extra-dimensional gauge theories in which the gauge symmetries are broken by the boundary conditions. In the higher-dimensional gauge theory, the reduction of gauge symmetry is realized through the boundary conditions of the extra-dimensional coordinate (see, for example, Ref. 11)). In this case,
the gauge bosons corresponding to the broken gauge symmetries acquire masses not through the Higgs mechanism but as the Kaluza-Klein (KK) states with masses $n/R$, where $R$ is the compactification scale, with a positive integer $n$. It has been shown in Refs. 13)–15) that higher-dimensional gauge theories preserve unitarity in the sense that the power law behavior $\sim O(E^4/m^4)$ or $\sim O(E^2/m^2)$ is canceled at the tree level. Related discussions are given in Refs. 16)–22).

In a previous paper we studied the unitarity bounds of the extra-dimensional gauge theory in which the gauge symmetry is broken by nontrivial boundary conditions. We calculated the amplitudes of the process including four massive gauge bosons in the external lines in the framework of the 5D Standard Model (SM) and the $SU(5)$ GUT whose 5th-dimensional coordinate is compactified on $S^1/Z_2$. We showed that the $O(E^4/m^4)$ and $O(E^2/m^2)$ power-law behavior of the amplitude vanishes and that the broken gauge theory realized through the orbifolding preserves unitarity at high energy. It has been noted that the structure of the interactions among KK states is crucial for preserving unitarity. Such calculations have been done in the unitary gauge. In those calculations, the 5th gauge field is gauged away and absorbed into the longitudinal component of the 4D gauge field through the appropriate gauge transformation compatible with the boundary conditions on the $S^1/Z_2$ orbifold.

In this paper we examine the structure of the gauge symmetry compatible with the boundary conditions. We invoke the BRST formalism in the ’t Hooft-Feynman gauge and demonstrate that the 4D equivalence theorem holds at the tree level as in the SM, considering the fact that the 5th gauge field is a NG field. We also confirm this theorem with certain amplitudes by explicit calculation of the Feynman diagrams.

This paper is organized as follows. In §2 we study the gauge transformation on the $S^1/Z_2$ orbifold. In §3 we present the BRST formalism for the 4D theory which is obtained through integration of the 5D theory. In §4 we derive the 4D equivalence theorem and note that the 5th gauge field is the would-be NG field. In §5 we present the amplitudes of the processes including four massive gauge bosons in the external lines and compare them with those for the connected reactions in which the gauge fields are replaced by the corresponding would-be NG fields. The final section is devoted to summary and discussion.

### §2. Gauge symmetry on orbifold

We begin by specifying the system to be studied. We consider the 5D gauge theory with the gauge field existing in the bulk. We denote the 5th-dimensional coordinate by $y$, which is compactified on an $S^1/Z_2$ orbifold. Under the $Z_2$ parity transformation, $y \rightarrow -y$, the gauge fields $A_\nu(x^\mu, y) \ (\nu = 0 \ldots 3)$ and $A_5(x^\mu, y)$ transform as

$$A_\nu(x^\mu, y) \rightarrow A_\nu(x^\mu, -y) = PA_\nu(x^\mu, y)P^{-1},$$

$$A_5(x^\mu, y) \rightarrow A_5(x^\mu, -y) = -PA_5(x^\mu, y)P^{-1},$$

where $P$ is the parity operator. The gauge transformation is consistent with the boundary conditions on the $S^1/Z_2$ orbifold.
where $P$ is the operator of the $Z_2$ transformation. Two walls at $y = 0$ and $\pi R$ are fixed points under the $Z_2$ transformation. The physical space can be taken to be the interval $0 \leq y \leq \pi R$. Here we take $Z_2$ as $P = 1$, so that the mode expansion of $A_\nu$ ($A_5$) for the 5th-dimensional coordinate is given by series of cosine (sine) functions. In addition, we consider the nontrivial boundary conditions $T : y \rightarrow y + 2\pi R$, where the parity (reflection) operator $P'$ about $y = \pi R$ is given by $P' = TP$.

The gauge symmetry is broken by the nontrivial parity operator $P'$ in the gauge group basis. In the case of the 5D $SU(3)_W$ theory, the $Z_2$ parity operator, $P' = \text{diag}(1,1,-1)$, realizes the gauge reduction $SU(3)_W \rightarrow SU(2)_L \times U(1)_Y$. In the case of the 5D $SU(5)$ theory, the $Z_2$ parity operator, $P' = \text{diag}(-1,-1,-1,1,1)$, realizes the gauge reduction $SU(5) \rightarrow SU(3)_c \times SU(2)_L \times U(1)_Y$.

Hereafter, a ($\dot{a}$) is used to denote unbroken (broken) gauge-indices. The gauge fields are expanded as

$$A_\nu^a(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2^n n! \pi R}} A_\nu^{a(n)}(x^\mu) \cos \frac{ny}{R},$$  

$$A_\nu^{\dot{a}}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} A_\nu^{\dot{a}(n+\frac{1}{2})}(x^\mu) \cos \frac{(n + \frac{1}{2})y}{R},$$  

$$A_5^{\dot{a}}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} A_5^{\dot{a}(n+\frac{1}{2})}(x^\mu) \sin \frac{(n + \frac{1}{2})y}{R},$$  

$$A_5^a(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} A_5^{a(n+1)}(x^\mu) \sin \frac{(n + 1)y}{R}.$$

The 5D Lagrangian on the orbifold is given by

$$\mathcal{L}_5 = -\frac{1}{4}(F_{\mu \nu}^a)^2 - \frac{1}{4}(F_{\mu \nu}^{\dot{a}})^2 - \frac{1}{2}(F_{\mu 5}^{\dot{a}})^2 - \frac{1}{2}(F_{\mu 5}^a)^2.$$  

Here, we have

$$F_{\mu \nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_5(f^{abc} A_\mu^b A_\nu^c + f^{\dot{a}\dot{b}\dot{c}} A_\mu^{\dot{b}} A_\nu^{\dot{c}}),$$  

$$F_{\mu \nu}^{\dot{a}} = \partial_\mu A_\nu^{\dot{a}} - \partial_\nu A_\mu^{\dot{a}} - g_5(f^{\dot{a}\dot{b}\dot{c}} A_\mu^{\dot{b}} A_\nu^{\dot{c}} + f^{abc} A_\mu^b A_\nu^c),$$  

$$F_{\mu 5}^{\dot{a}} = \partial_\mu A_5^{\dot{a}} - \partial_5 A_\mu^{\dot{a}} - g_5(f^{\dot{a}\dot{b}\dot{c}} A_\mu^{\dot{b}} A_5^{\dot{c}} + f^{abc} A_\mu^b A_5^c),$$  

$$F_{\mu 5}^a = \partial_\mu A_5^a - \partial_5 A_\mu^a - g_5(f^{abc} A_\mu^b A_5^c + f^{\dot{a}\dot{b}\dot{c}} A_\mu^{\dot{b}} A_5^{\dot{c}}),$$

with $g_5$ being the 5D gauge coupling and $f^{abc}$, $f^{\dot{a}\dot{b}\dot{c}}$, $f^{\dot{a}\dot{b}\dot{c}}$ and $f^{abc}$ the structure constants.

The above 5D Lagrangian is invariant under the 5D gauge transformation generated by

$$\delta A_\mu^a = \partial_\mu e_a^a - g_5(f^{abc} A_\mu^b e_c^c + f^{\dot{a}\dot{b}\dot{c}} A_\mu^{\dot{b}} e^{\dot{c}}),$$  

$$\delta A_\mu^{\dot{a}} = \partial_\mu e_{\dot{a}}^{\dot{a}} - g_5(f^{\dot{a}\dot{b}\dot{c}} A_\mu^{\dot{b}} e^{\dot{c}} + f^{abc} A_\mu^b e_c^c),$$
\[ \delta A^a_5 = \partial_5 \epsilon^a - g_5 (f^{abc} A^b_5 \epsilon^c + f^{\dot{a} \dot{b} \dot{c}} \dot{A}^\dot{b}_5 \dot{\epsilon}^{\dot{c}}), \]  
\[ \delta A^a_5 = \partial_5 \epsilon^a - g_5 (f^{abc} A^b_5 \epsilon^c + f^{\dot{a} \dot{b} \dot{c}} \dot{A}^\dot{b}_5 \dot{\epsilon}^{\dot{c}}), \]

where the 5D gauge functions \( \epsilon^a(x^\mu, y) \) and \( \dot{\epsilon}^{\dot{a}}(x^\mu, y) \) are given by

\[ \epsilon^a(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2^{3n} \pi R}} \epsilon^{a(n)}(x^\mu) \cos \frac{ny}{R}, \]
\[ \dot{\epsilon}^{\dot{a}}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \dot{\epsilon}^{a(n + \frac{1}{2})}(x^\mu) \cos \left( \frac{(n + \frac{1}{2})y}{R} \right). \]

It is noted that the above transformation is compatible with the boundary conditions on the orbifold.

The 5D gauge transformation can be rewritten in terms of the KK modes as

\[ \delta A^{a(0)}_\mu = \partial_\mu \epsilon^{a(0)} - g_4 \sum_{m=1}^{\infty} \left[ f^{abc} A^{b(m-1)}_\mu \epsilon^c(m-1) + f^{\dot{a} \dot{b} \dot{c}} \dot{A}^{\dot{b}(m-\frac{1}{2})}_\mu \dot{\epsilon}^{\dot{c}(m-\frac{1}{2})} \right], \]
\[ \delta A^{a(n)}_\mu = \partial_\mu \epsilon^{a(n)} - g_4 f^{abc} \left[ A^{b(0)}_\mu \epsilon^c(n) + A^{b(n)}_\mu \epsilon^c(0) \right] 
\quad - \frac{g_4}{\sqrt{2}} f^{\dot{a} \dot{b} \dot{c}} \left[ \sum_{m=1}^{n-1} A^{b(n-m)}_\mu \epsilon^c(m) + \sum_{m=1}^{\infty} A^{b(n+m)}_\mu \epsilon^c(m) \right] 
\quad + \sum_{m=n+1}^{\infty} A^{b(m-n)}_\mu \epsilon^c(m) \]
\[ \delta A^{a(n + \frac{1}{2})}_\mu = \partial_\mu \epsilon^{a(n + \frac{1}{2})} - g_4 \left[ f^{abc} A^{b(n - \frac{1}{2})}_\mu \epsilon^c(0) + f^{\dot{a} \dot{b} \dot{c}} A^{\dot{b}(0)}_\mu \dot{\epsilon}^{\dot{c}(n - \frac{1}{2})} \right] 
\quad - \frac{g_4}{\sqrt{2}} f^{\dot{a} \dot{b} \dot{c}} \left[ \sum_{m=1}^{n-1} A^{\dot{b}(n-m + \frac{1}{2})}_\mu \dot{\epsilon}^{\dot{c}(m - \frac{1}{2})} + \sum_{m=1}^{\infty} A^{\dot{b}(n+m - \frac{1}{2})}_\mu \dot{\epsilon}^{\dot{c}(m - \frac{1}{2})} \right] 
\quad + \sum_{m=n+1}^{\infty} A^{\dot{b}(m-n + \frac{1}{2})}_\mu \dot{\epsilon}^{\dot{c}(m - \frac{1}{2})} \]
\[ \delta A_{5}^{(n-\frac{1}{2})} = -\frac{n-\frac{1}{2}}{R} \epsilon \hat{a}(n-\frac{1}{2}) - g_{4} f^{\hat{a}bc} A_{5}^{b(\frac{n-1}{2})} \epsilon^{c(0)} \]

\[ - \frac{g_{4}}{\sqrt{2}} f^{\hat{a}bc} \left[ \sum_{m=1}^{n-1} A_{5}^{b(n-m-\frac{1}{2})} \epsilon^{c(m)} + \sum_{m=1}^{\infty} A_{5}^{b(n+m-\frac{1}{2})} \epsilon^{c(m)} \right] - \sum_{m=n}^{\infty} A_{5}^{b(m-n+\frac{1}{2})} \epsilon^{c(m)} \]

\[ - \frac{g_{4}}{\sqrt{2}} f^{\hat{a}bc} \left[ \sum_{m=1}^{n-1} A_{5}^{b(n-m)} \epsilon^{c(m-\frac{1}{2})} + \sum_{m=1}^{\infty} A_{5}^{b(n+m)-1} \epsilon^{c(m-\frac{1}{2})} \right] - \sum_{m=n+1}^{\infty} A_{5}^{b(n-m)} \epsilon^{c(m-\frac{1}{2})} \]

\[ \delta A_{5}^{a(n)} = -\frac{n}{R} \epsilon^{a(n)} - g_{4} f^{abc} A_{5}^{b(n)} \epsilon^{c(0)} \]

\[ - \frac{g_{4}}{\sqrt{2}} f^{abc} \left[ \sum_{m=1}^{n-1} A_{5}^{b(n-m)} \epsilon^{c(m)} + \sum_{m=1}^{\infty} A_{5}^{b(n+m)} \epsilon^{c(m)} \right] - \sum_{m=n+1}^{\infty} A_{5}^{b(n-m)} \epsilon^{c(m)} \]

\[ - \frac{g_{4}}{\sqrt{2}} f^{abc} \left[ \sum_{m=1}^{n-1} A_{5}^{b(n-m+\frac{1}{2})} \epsilon^{c(m-\frac{1}{2})} + \sum_{m=1}^{\infty} A_{5}^{b(n+m-\frac{1}{2})} \epsilon^{c(m-\frac{1}{2})} \right] - \sum_{m=n+1}^{\infty} A_{5}^{b(n-m+\frac{1}{2})} \epsilon^{c(m-\frac{1}{2})} \]

\[ \delta A_{5}^{a(n)} = -\frac{n}{R} \epsilon^{a(n)} - g_{4} f^{abc} A_{5}^{b(n)} \epsilon^{c(0)} \]

\[ - \frac{g_{4}}{\sqrt{2}} f^{abc} \left[ \sum_{m=1}^{n-1} A_{5}^{b(n-m)} \epsilon^{c(m)} + \sum_{m=1}^{\infty} A_{5}^{b(n+m)} \epsilon^{c(m)} \right] - \sum_{m=n+1}^{\infty} A_{5}^{b(n-m)} \epsilon^{c(m)} \]

where \( n \) and \( m \) are positive integers and \( g_{4} \) is the 4D gauge coupling related to the 5D gauge coupling, \( g_{5} \), as \( g_{4} = g_{5}/\sqrt{2\pi R} \).

### §3. 4D Lagrangian and BRST formalism

The 4D Lagrangian is obtained by integrating the 5D Lagrangian and it is composed of the kinetic term \( \mathcal{L}_{\text{KE}} \) and the interaction term \( \mathcal{L}_{\text{INT}} \):

\[ \mathcal{L}_{4} = \int_{0}^{2\pi R} dy \mathcal{L}_{5} = \mathcal{L}_{\text{KE}} + \mathcal{L}_{\text{INT}}. \]

This Lagrangian is invariant under the 4D gauge transformation defined by Eqs. (18)–(22). The kinetic term \( \mathcal{L}_{\text{KE}} \) is given by

\[ \mathcal{L}_{\text{KE}} = -\frac{1}{4} \left[ \partial_{\mu} A_{\nu}^{a(0)} - \partial_{\nu} A_{\mu}^{a(0)} \right]^{2} - \frac{1}{4} \sum_{n=1}^{\infty} \left[ \partial_{\mu} A_{\nu}^{a(n)} - \partial_{\nu} A_{\mu}^{a(n)} \right]^{2} - \frac{1}{2} \sum_{n=1}^{\infty} \left[ \partial_{\mu} A_{5}^{a(n)} + M_{n} A_{\mu}^{a(n)} \right]^{2} \]
\[-\frac{1}{4} \sum_{n=1}^{\infty} \left( \partial_{\mu} \bar{A}_{\nu}^{(n-\frac{1}{2})} - \partial_{\nu} \bar{A}_{\mu}^{(n-\frac{1}{2})} \right)^2 \]

\[-\frac{1}{2} \sum_{n=1}^{\infty} \left( \partial_{\mu} \bar{A}_{5}^{(n-\frac{1}{2})} + M_{n-\frac{1}{2}} \bar{A}_{\mu}^{(n-\frac{1}{2})} \right)^2, \quad (24)\]

where $M_n = n/R$ and $M_{n-\frac{1}{2}} = \left(n - \frac{1}{2}\right)/R$ are the masses of the KK vector bosons.

In the following, we consider the 4D theory defined by the Lagrangian $L_4$ and impose an $R_\xi$ gauge-fixing in the form:

\[L_{GF} = -\frac{1}{2\xi_0} \left[ F^{a(0)} \right]^2 - \frac{1}{2\xi_n} \left[ F^{a(n)} \right]^2 - \sum_{n=1}^{\infty} \frac{1}{2\xi_{n-\frac{1}{2}}} \left[ F^{\bar{a}(n-\frac{1}{2})} \right]^2, \quad (25)\]

where $\xi_0$, $\xi_n$ and $\xi_{n-\frac{1}{2}}$ are arbitrary gauge parameters and

\[F^{a(0)} = \partial_{\mu} A^{a(0)}_{\mu}, \quad (26)\]

\[F^{a(n)} = \partial_{\mu} A^{a(n)}_{\mu} + \xi_n M_n A^{a(n)}_{5}, \quad (27)\]

\[F^{\bar{a}(n-\frac{1}{2})} = \partial_{\mu} A^{\bar{a}(n-\frac{1}{2})}_{\mu} + \xi_{n-\frac{1}{2}} M_{n-\frac{1}{2}} A^{\bar{a}(n-\frac{1}{2})}_{5}. \quad (28)\]

The above gauge-fixing term eliminates the kinetic term mixing between $A^{a(n)}_{\mu}$ ($A^{\bar{a}(n-\frac{1}{2})}_{\mu}$) and $A^{a(n)}_{5}$ ($A^{\bar{a}(n-\frac{1}{2})}_{5}$), and we can identify the $A^{a(n)}_{5}$ and $A^{\bar{a}(n-\frac{1}{2})}_{5}$ modes as the would-be NG fields. The modes $A^{a(n)}_{5}$ and $A^{\bar{a}(n-\frac{1}{2})}_{5}$ have the gauge-dependent masses $M^{2}_{5(n)} = \xi_n M^2_n$ and $M^{2}_{5(n-\frac{1}{2})} = \xi_{n-\frac{1}{2}} M^2_{n-\frac{1}{2}}$, respectively. In a previous paper\(^{15}\) we employed the unitary gauge, $\xi_n = \infty$ and $\xi_{n-\frac{1}{2}} = \infty$, where $A^{a(n)}_{5}$ and $A^{\bar{a}(n-\frac{1}{2})}_{5}$ decouple because $M^{2}_{5(n)} \rightarrow \infty$ and $M^{2}_{5(n-\frac{1}{2})} \rightarrow \infty$. The appropriate Faddeev-Popov ghost term is

\[L_{FP} = -\sum_{n=0}^{\infty} \bar{\eta}^{a(n)} \delta_B F^{a(n)} - \sum_{n=1}^{\infty} \bar{\eta}^{\bar{a}(n-\frac{1}{2})} \delta_B F^{\bar{a}(n-\frac{1}{2})}. \quad (29)\]

The BRST transformation of the gauge fields is given by the right-hand sides of Eqs. (18)–(22) with the gauge functions $\epsilon^{a(0)}$, $\epsilon^{a(n)}$ and $\epsilon^{\bar{a}(n-\frac{1}{2})}$ replaced by the ghost fields $\eta^{a(0)}$, $\eta^{a(n)}$ and $\eta^{\bar{a}(n-\frac{1}{2})}$, respectively. The BRST transformation of the anti-ghost fields is given by

\[\delta_B \eta^{a(n)} = -\frac{1}{\xi_n} F^{a(n)}, \quad (30)\]

\[\delta_B \eta^{\bar{a}(n-\frac{1}{2})} = -\frac{1}{\xi_{n-\frac{1}{2}}} F^{\bar{a}(n-\frac{1}{2})}. \quad (31)\]

\(^{15}\) If we consider the 4D theory should be defined by the gauge-fixed 5D Lagrangian, there appears only one gauge parameter.
§4. 4D equivalence theorem

The equivalence theorem constitutes the assertion that the scattering amplitude of the massive gauge bosons with longitudinal polarization $\lambda = L$ is equal, up to some constant factor, to that of the corresponding would-be NG bosons in the high-energy limit. The interactions among would-be NG fields and the 4D vector fields consist of the cubic vertices with only one derivative and the quartic vertices with no derivatives. Power-counting reveals that the tree-level scattering amplitude of would-be NG bosons in the external line is $O(1)$ at high energy. Thus unitarity is automatically preserved. In this section we show that this equivalence results from the 4D gauge invariance of the theory. The basic line of reasoning here is the same as that given in Refs. 6), 7) and 10).

First we note that the scattering state must satisfy the BRST invariance (the physical state condition), $Q_B|\text{in/out}\rangle = 0$. Hence for the BRST transformation of a field $\Phi$, $\delta_B \Phi = [Q_B, \Phi]$, the equation $\langle B, \text{out}|\delta_B \bar{\eta}^a(n)|C, \text{in}\rangle = 0$ holds. Taking as $\Phi$ the anti-ghost $\bar{\eta}$, we obtain the equations

\begin{align}
\langle B, \text{out}|\delta_B \bar{\eta}^a(n)|C, \text{in}\rangle &= 0, \\
\langle B', \text{out}|\delta_B \bar{\eta}^{a(n-\frac{1}{2})}|C', \text{in}\rangle &= 0.
\end{align}

From these equations and the BRST transformations Eqs. (30) and (31), we obtain the Slavnov-Taylor identities

\begin{align}
\langle B, \text{out}|F^a(n)|C, \text{in}\rangle &= 0, \\
\langle B', \text{out}|F^{a(n-\frac{1}{2})}|C', \text{in}\rangle &= 0.
\end{align}

Adopting the 't Hooft-Feynman gauge

$$\xi_n = \xi_{n-\frac{1}{2}} = 1$$

in the gauge-fixing terms in Eqs. (27) and (28), we obtain the relations

\begin{align}
\langle B, \text{out} | \partial_\mu A^a(n) + M_n A_5^a(n)|C, \text{in}\rangle &= 0, \\
\langle B', \text{out} | \partial_\mu \tilde{A}^{a(n-\frac{1}{2})} + M_{n-\frac{1}{2}} \tilde{A}^{a(n-\frac{1}{2})} |C', \text{in}\rangle &= 0.
\end{align}

From the above relations in Eqs. (37) and (38), we find

\begin{align}
-i \frac{p_\mu}{M_n} \hat{S}[C \rightarrow B + A^a(n)(p, \lambda)] &= S[C \rightarrow B + A_5^a(n)(p)], \\
-i \frac{p_\mu}{M_{n-\frac{1}{2}}} \hat{S}[C' \rightarrow B' + \tilde{A}^{a(n-\frac{1}{2})}(p, \lambda)] &= S[C' \rightarrow B' + \tilde{A}^{a(n-\frac{1}{2})}(p)],
\end{align}

where

$$\epsilon^\mu(p, \lambda) \hat{S}[C \rightarrow B + A^a(n)(p, \lambda)] = 0.$$
and
\[ \epsilon^\mu(p, \lambda) \hat{S}[C' \rightarrow B' + A^{n-\frac{1}{2}}_\mu(p, \lambda)] \]

\[ \epsilon^\mu(p, \lambda) \hat{S}[C' \rightarrow B' + A^{n-\frac{1}{2}}_\mu(p, \lambda)] \]

\[ -i S[C \rightarrow B + A^{a(n)}_\mu(p, \lambda)] = S[C \rightarrow B + A^{a(n)}_5(p)] + O \left( \frac{M_n}{E} \right) \]

\[ -i S[C' \rightarrow B' + A^{\hat{a}(n-\frac{1}{2})}_\mu(p, \lambda)] = S[C' \rightarrow B' + A^{\hat{a}(n-\frac{1}{2})}_5(p)] + O \left( \frac{M_{n-\frac{1}{2}}}{E} \right) \].

These relations represent the equivalence theorem for the case of the amplitude with only one external longitudinal vector.

For the amplitude with many external vectors, we start with the identity for positive integers \(m, l\) and \(n\),

\[ 0 = \langle B''', \text{out} | \delta_B T[\tilde{\eta}^{a(m)}(x_1)F^{b(l)}(x_2)\cdots F^{\tilde{a}(n-\frac{1}{2})}(x_j)\cdots]|C''\rangle, \text{in} \]

\[ = \langle B''', \text{out} | T[\delta_B \tilde{\eta}^{a(m)}(x_1)F^{b(l)}(x_2)\cdots F^{\tilde{a}(n-\frac{1}{2})}(x_j)\cdots]|C''\rangle, \text{in} \]

\[ - \langle B'', \text{out} | T[\tilde{\eta}^{a(m)}(x_1)\delta_B F^{b(l)}(x_2)\cdots F^{\tilde{a}(n-\frac{1}{2})}(x_j)\cdots]|C''\rangle, \text{in} \]

\[ - \cdots \]

\[ - \langle B'', \text{out} | T[\tilde{\eta}^{a(m)}(x_1)F^{b(l)}(x_2)\cdots \delta_B F^{\tilde{a}(n-\frac{1}{2})}(x_j)\cdots]|C''\rangle, \text{in} \]

\[ - \cdots \].

(49)
This leads to the Slavnov-Taylor identity,

\[ 0 = \langle B''', \text{out} | T[F^{a(m)}(x_1)F^{b(l)}(x_2)\cdots F^{\hat{a}(n-\frac{1}{2})}(x_j)\cdots] | C'', \text{in} \rangle \]
\[ -i\delta^{ab}\delta^{ml}\delta(x_1 - x_2)\langle B''', \text{out} | T[\cdots F^{\hat{a}(n-\frac{1}{2})}(x_j)\cdots] | C'', \text{in} \rangle - \cdots. \]  

(50)

This implies that for the connected transition matrix element we have

\[ \langle B''', \text{out} | T[F^{a(m)}(x_1)\cdots F^{\hat{a}(n-\frac{1}{2})}(x_j)\cdots] | C'', \text{in} \rangle_{\text{con}} = 0. \]  

(51)

Then, by the inductive argument given in Ref. 10), we obtain, at sufficiently high energies, the relation

\[ (-i)\cdots(-i)S[C''\rightarrow B'' + A^{a(m)}_{\mu}(p, \lambda = L) + \cdots + A^{\hat{a}(n-\frac{1}{2})}_{\mu}(p', \lambda = L) + \cdots] \]
\[ = S[C''\rightarrow B'' + A^{a(m)}_{5}(p) + \cdots + A^{\hat{a}(n-\frac{1}{2})}_{5}(p') + \cdots] \]
\[ + O\left[\frac{M_m}{E}\right] + \cdots + O\left[\frac{M_{n-\frac{1}{2}}}{E'}\right] + \cdots. \]  

(52)

This is essentially the equivalence theorem. It should be noted that the right-hand side of (52) is gauge-dependent and is an unphysical amplitude. For a precise formulation of the equivalence theorem in the standard model including loop corrections and discussion of gauge-dependence, see Refs. 6) and 10).

§5. Amplitudes of four massive gauge bosons in the GUT on an orbifold

In a previous paper\textsuperscript{15}) we calculated the amplitudes of a process including four massive gauge bosons in the external lines in the framework of the 5D SU(5) GUT whose 5th-dimensional coordinate is compactified on \(S^1/Z_2\). We showed that the power-law behavior \(\sim O(E^4/m^4)\) and \(\sim O(E^2/m^2)\) in the amplitude vanishes and that the broken gauge theory realized through the orbifolding preserves unitarity at high energy. Here we present the results of the calculations of the Feynman diagrams up to \(O(1)\) and compare them with those for connected reactions in which the gauge fields are replaced by the corresponding would-be NG fields.

First, we present the notation for a massive vector boson in the external line. In the center-of-mass frame, we take the initial momenta to be

\[ p_1 = E\left(1, 0, 0, \sqrt{1 - \frac{m^2}{E^2}}\right) \]  

(53)

and

\[ p_2 = E\left(1, 0, 0, -\sqrt{1 - \frac{m^2}{E^2}}\right) \]  

(54)

and the final momenta to be

\[ k_1 = E\left(1, \sqrt{1 - \frac{m^2}{E^2}} \sin \theta, 0, \sqrt{1 - \frac{m^2}{E^2}} \cos \theta\right) \]  

(55)
and

$$k_2 = E \left( 1, -\sqrt{1 - \frac{m^2}{E^2}} \sin \theta, 0, -\sqrt{1 - \frac{m^2}{E^2}} \cos \theta \right), \quad (56)$$

where $m$ is the gauge boson mass. Then the longitudinal polarization vectors become

$$\epsilon^\mu(p_1, \lambda = L) = \frac{E}{m} \left( \sqrt{1 - \frac{m^2}{E^2}}, 0, 0, 1 \right), \quad (57)$$

$$\epsilon^\mu(p_2, \lambda = L) = \frac{E}{m} \left( \sqrt{1 - \frac{m^2}{E^2}}, 0, 0, -1 \right), \quad (58)$$

$$\epsilon^\mu(k_1, \lambda = L) = \frac{E}{m} \left( \sqrt{1 - \frac{m^2}{E^2}}, \sin \theta, 0, \cos \theta \right), \quad (59)$$

and

$$\epsilon^\mu(k_2, \lambda = L) = \frac{E}{m} \left( \sqrt{1 - \frac{m^2}{E^2}}, -\sin \theta, 0, -\cos \theta \right). \quad (60)$$

Before examining the amplitudes of four massive gauge bosons in the orbifold model, we briefly discuss those in the 4D SM, where the $W$ and $Z$ gauge bosons acquire mass through the Higgs mechanism. For the process $W^+W^- \rightarrow W^+W^-$, there are five diagrams: (1a) $s$-channel photon and $Z$ exchange, (1b) $t$-channel photon and $Z$ exchange, (1c) quadrilinear vertex, (1d) $s$-channel Higgs exchange and (1e) $t$-channel Higgs exchange. As shown in a previous paper\textsuperscript{15} the power-law behavior \( \sim O(E^4/m^4) \) in the amplitude vanishes due to the cancellation among the three diagrams (1a)–(1c). Furthermore, the behavior \( \sim O(E^2/m^2) \) vanishes due to the cancellation among all the diagrams (1a)–(1e). In Table I we summarize the previous results, adding the new results for the $O(1)$ amplitudes.

By contrast, the amplitudes for the connected reactions in which the gauge fields $W$ are replaced by the corresponding would-be NG fields $G$ are obtained as follows.

For the process $G^+G^- \rightarrow G^+G^-$, there are three $O(1)$ diagrams: (2a) $s$-channel

<table>
<thead>
<tr>
<th>Diagram</th>
<th>$\frac{ig^2 E_{\mu}^+}{m}$</th>
<th>$\frac{ig^2 E_{\mu}^+}{m}$</th>
<th>$\frac{ig^2}{m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1a)</td>
<td>$-4 \cos \theta$</td>
<td>$- \cos \theta$</td>
<td>$3 \cos \theta - \frac{1}{4 \cos^2 \theta} \cos \theta$</td>
</tr>
<tr>
<td>(1b)</td>
<td>$3 - 2 \cos \theta - \cos^2 \theta$</td>
<td>$- \frac{1}{2} + \frac{1}{2} \cos \theta$</td>
<td>$- \frac{1}{2} - \frac{1}{2} \cos \theta + \frac{1}{4 \cos^2 \theta} \cos \theta$</td>
</tr>
<tr>
<td>(1c)</td>
<td>$-3 + 6 \cos \theta + \cos^2 \theta$</td>
<td>$2 - 6 \cos \theta$</td>
<td>$- \frac{1}{2} - \frac{1}{2} \cos \theta - \frac{1}{4 \cos^2 \theta}$</td>
</tr>
<tr>
<td>(1d)</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$1 - \frac{1}{2 \cos \theta}$</td>
</tr>
<tr>
<td>(1e)</td>
<td>$\frac{1}{2} - \frac{1}{2} \cos \theta$</td>
<td>$- \frac{1}{2} - \frac{1}{2} \cos \theta - \frac{1}{4 \cos^2 \theta}$</td>
<td></td>
</tr>
</tbody>
</table>
Table II. The coefficients of the amplitude of $G^+ G^- \to G^+ G^-$ in (2a)–(2c) for the 4D SM.

<table>
<thead>
<tr>
<th></th>
<th>$ig^2 E^4/m^2 \times$</th>
<th>$ig^2 E^2/m^2 \times$</th>
<th>$ig^2 \times$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2a)</td>
<td>$-3 - 2 \cos \theta - \cos^2 \theta$</td>
<td>$8 \cos \theta$</td>
<td>$1 + \cos \theta + 2 \cos^2 \theta$</td>
</tr>
<tr>
<td>(2b)</td>
<td>$3 + 6 \cos \theta + \cos^2 \theta$</td>
<td>$2 - 6 \cos \theta$</td>
<td>$1 - \cos \theta$</td>
</tr>
<tr>
<td>(2c)</td>
<td>$-4$</td>
<td>$-4$</td>
<td>$4 - \frac{m_z^2}{m^2}$</td>
</tr>
</tbody>
</table>

Table III. The coefficients of the amplitude of $XX^* \to XX^*$ in (3a)–(3e) for the 4D SU(5) GUT. Both $O(E^4/m^4)$ and $O(E^2/m^2)$ contributions are canceled among (3a)–(3e).

<table>
<thead>
<tr>
<th></th>
<th>$ig^2 E^4/m^2 \times$</th>
<th>$ig^2 E^2/m^2 \times$</th>
<th>$ig^2 \times$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3a)</td>
<td>$-4 \cos \theta$</td>
<td>$-3 \cos \theta$</td>
<td>$\frac{1 + \cos \theta + 2 \cos^2 \theta}{1 - \cos \theta}$</td>
</tr>
<tr>
<td>(3b)</td>
<td>$3 + 6 \cos \theta + \cos^2 \theta$</td>
<td>$2 - 6 \cos \theta$</td>
<td>$-\frac{m_z^2}{m^2}$</td>
</tr>
<tr>
<td>(3c)</td>
<td>$-4$</td>
<td>$-4$</td>
<td>$4 - \frac{m_z^2}{m^2}$</td>
</tr>
</tbody>
</table>

4D Equivalence Theorem and Gauge Symmetry on an Orbifold

Next, the amplitudes for the connected reactions in which the gauge fields $X \rightarrow X$ exchange, (2b) $t$-channel photon and $Z$ exchange, (2c) quadrilinear vertex. In Table II we give the results for the $O(1)$ amplitudes. As expected from the equivalence theorem, the scattering amplitude of the gauge fields $W^+W^- \rightarrow W^+W^-$ is equal to that of the corresponding would-be NG fields $G^+G^- \rightarrow G^+G^-$ to $O(m/E)$.

Next, let us consider the amplitudes of four massive gauge bosons in the 4D SU(5) GUT, where the $X$ and $Y$ gauge bosons acquire mass through the Higgs mechanism. For the process $XX^* \rightarrow XX^*$, there are five diagrams: (3a) $s$-channel $A_3$, $A_8$, $W_3$, $B$ exchange, (3b) $t$-channel $A_3$, $A_8$, $W_3$, $B$ exchange, (3c) quadrilinear vertex, (3d) $s$-channel $\Sigma_3$, $\Sigma_8$, $\Sigma_{W_3}$, $\Sigma_B$ exchange and (3e) $t$-channel $\Sigma_3$, $\Sigma_8$, $\Sigma_{W_3}$, $\Sigma_B$ exchange. Here, $A_3$, $A_8$, $W_3$ and $B$ represent the diagonal elements of the gauge fields of SU(5), corresponding to $SU(3)_c$, $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$ components, respectively. Similarly, $\Sigma_3$, $\Sigma_8$, $\Sigma_{W_3}$ and $\Sigma_B$ represent the diagonal elements of the adjoint Higgs fields of SU(5). As shown in a previous paper\(^{15}\) the power-law behavior $\sim O(E^4/m^4)$ vanishes due to the cancellation among the three diagrams (3a)–(3e) and the behavior $\sim O(E^2/m^2)$ vanishes due to the cancellation among the four diagrams (3b)–(3e). In Table III we summarize the results, including the new results for the $O(1)$ amplitudes. In the table there appears an “averaged mass” of the Higgs defined as $m^2_{\Sigma} = \frac{1}{4}(m^2_{\Sigma_3} + \frac{1}{3}m^2_{\Sigma_8} + m^2_{\Sigma_{W_3}} + \frac{5}{3}m^2_{\Sigma_B})$.

Next, the amplitudes for the connected reactions in which the gauge fields $XX^*$ are replaced by the corresponding would-be NG fields $G$ are obtained as follows. For the process $GG^* \rightarrow GG^*$, there are three $O(1)$ diagrams: (4a) $s$-channel $A_3$, $A_8$, $W_3$, $B$ exchange, (4b) $t$-channel $A_3$, $A_8$, $W_3$, $B$ exchange, (4c) quadrilinear vertex. In Table IV we give the results for the $O(1)$ amplitudes. It is noted that the coefficient of the amplitude (4c) is given, in terms of the quartic coupling constants $\lambda_1$ and $\lambda_2$, as $-i(\lambda_1 + \frac{1}{2}\lambda_2)$. This is rewritten in the form of the mass ratio $-2ig^2 \frac{m^2_{\Sigma}}{m^2}$. As expected from the equivalence theorem, the scattering amplitude of the gauge fields $XX^* \rightarrow XX^*$ is equal to that of the corresponding would-be NG fields $GG^* \rightarrow GG^*$.
The coupling of the four point vertex in the process (5e) is given by

$$W_{U}$$

change, (5b) $A_{O}$

Table V. The coefficients of the amplitude of $X^{(1/2)}X^{(1/2)} \rightarrow X^{(1/2)}X^{(1/2)}$ in (5a)–(5e) for the 5D SU(5) GUT. Both $O(E^{4}/m^{4})$ and $O(E^{2}/m^{2})$ contributions are canceled among (5a)–(5e).

<table>
<thead>
<tr>
<th></th>
<th>$ig^{2}E_{m}^{3}$</th>
<th>$ig^{2}E_{m}^{7}$</th>
<th>$ig^{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5a)</td>
<td>$-4 \cos \theta$</td>
<td>$-2 \cos \theta$</td>
<td>$3 \cos \theta$</td>
</tr>
<tr>
<td>(5b)</td>
<td>$-2 \cos \theta$</td>
<td>$-2 \cos \theta$</td>
<td>$-\frac{4}{3} \cos \theta$</td>
</tr>
<tr>
<td>(5c)</td>
<td>$-2 \cos \theta - \cos^{2} \theta$</td>
<td>$8 \cos \theta$</td>
<td>$1 + \cos \theta + 2 \cos^{2} \theta$</td>
</tr>
<tr>
<td>(5d)</td>
<td>$\frac{1}{2} \cos \theta - \frac{1}{2} \cos^{2} \theta$</td>
<td>$-3 + 3 \cos \theta$</td>
<td>$\frac{1}{2} \cos \theta + 1 - \frac{1}{2} \cos \theta$</td>
</tr>
<tr>
<td>(5e)</td>
<td>$-\frac{1}{3} + 9 \cos \theta + \frac{1}{2} \cos^{2} \theta$</td>
<td>$3 - 9 \cos \theta$</td>
<td>$-1 - \cos \theta$</td>
</tr>
</tbody>
</table>

to $O(m/E)$.

Finally, we examine the 5D SU(5) theory with the $Z_{2}$ parity operator, $P' = \text{diag}(-1, -1, -1, 1, 1)$, which realizes the gauge reduction $SU(5) \rightarrow SU(3)c \times SU(2)L \times U(1)Y$. For the scattering process of the $n=1/2$ mode of X bosons, $X^{(1/2)}X^{(1/2)} \rightarrow X^{(1/2)}X^{(1/2)}$, there are five diagrams: (5a) s-channel $A_{3}^{(0)}, A_{2}^{(0)}, W_{3}^{(0)}, B^{(0)}$ exchange, (5b) s-channel $A_{3}^{(1)}, A_{8}^{(1)}, W_{3}^{(1)}, B^{(1)}$ exchange, (5c) t-channel $A_{3}^{(0)}, A_{8}^{(0)}, W_{3}^{(0)}, B^{(0)}$ exchange, (5d) t-channel $A_{3}^{(1)}, A_{8}^{(1)}, W_{3}^{(1)}, B^{(1)}$ exchange and (5e) quadrilinear vertex. Here, $A_{3}^{(0)}, A_{8}^{(0)}, W_{3}^{(0)}, B^{(0)}$ and $A_{3}^{(1)}, A_{8}^{(1)}, W_{3}^{(1)}, B^{(1)}$ denote the zero modes and the KK excited modes, respectively. The power-law behavior $\sim O(E^{4}/m^{4})$ and $\sim O(E^{2}/m^{2})$ vanishes due to the cancellation among the diagrams (5a)–(5e). In Table V we summarize the results up to $O(1)$.

In the above calculations, we note that the 5D gauge coupling, $g_{5}$, has mass dimension $-1/2$, which is related to the 4D gauge coupling, $g_{4}$, as $g_{4} = g_{5}/\sqrt{2\pi R}$. Therefore the couplings of $A_{3}^{(0)} - X^{(1/2)} - X^{(1/2)}$, $A_{8}^{(0)} - X^{(1/2)} - X^{(1/2)}$, $W_{3}^{(0)} - X^{(1/2)} - X^{(1/2)}$, and $B^{(0)} - X^{(1/2)} - X^{(1/2)}$ are

$$g_{5} \int_{0}^{2\pi R} dy \left( \frac{1}{\sqrt{\pi R}} \cos \frac{y}{2R} \right)^{2} \frac{1}{\sqrt{2\pi R}} = g_{4},$$

while the couplings of $A_{3}^{(1)} - X^{(1/2)} - X^{(1/2)}$, $A_{8}^{(1)} - X^{(1/2)} - X^{(1/2)}$, $W_{3}^{(1)} - X^{(1/2)} - X^{(1/2)}$ and $B^{(1)} - X^{(1/2)} - X^{(1/2)}$ are

$$g_{5} \int_{0}^{2\pi R} dy \left( \frac{1}{\sqrt{\pi R}} \cos \frac{y}{2R} \right)^{2} \left( \frac{1}{\sqrt{\pi R}} \cos \frac{y}{R} \right) = \frac{g_{4}}{\sqrt{2}}.$$

The coupling of the four point vertex in the process (5e) is given by

$$ig_{5}^{2} \int_{0}^{2\pi R} dy \left( \frac{1}{\sqrt{\pi R}} \cos \frac{y}{2R} \right)^{4} = \frac{3}{2} ig_{4}^{2}.$$
Table VI. The coefficients of the amplitude of $X_5^{(1/2)} X_5^{(1/2)*} \to X_5^{(1/2)} X_5^{(1/2)*}$ in (6a)–(6d) for the 5D $SU(5)$ GUT.

<table>
<thead>
<tr>
<th>$(6a),(6b)$</th>
<th>$(6c),(6d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ig^2 E^3 / m^2 \times$</td>
<td>$ig^2 E^3 / m^2 \times$</td>
</tr>
<tr>
<td>$-\frac{3}{2} \cos \theta$</td>
<td>$\frac{3}{2} \sin \theta$</td>
</tr>
</tbody>
</table>

This implies that the amplitude of $(5c)$ is larger than that of $(3c)$ by a factor of $3/2$. The power-law behavior $\sim O(E^4/m^4)$ and $\sim O(E^2/m^2)$ is canceled although there are no Higgs contributed diagrams as in the 4D $SU(5)$ GUT [(3d),(3e)]. Table V suggests that the KK-modes play an important role in preserving unitarity, as noted in a previous paper.$^{15}$ They are responsible for the cancellation of the power-law behavior $\sim O(E^2/m^2)$ and $\sim O(E^4/m^2)$, whereas the Higgs scalar in the spontaneous breaking theories is responsible for the $O(E^2/m^2)$ cancellation.

Similarly, the amplitudes for the connected reactions in which the 4D gauge fields $X^{(1/2)}$ are replaced by the 5th gauge fields $X_5^{(1/2)}$ are obtained as follows. For the process $X_5^{(1/2)} X_5^{(1/2)*} \to X_5^{(1/2)} X_5^{(1/2)*}$, there are four $O(1)$ diagrams $O(1)$: (6a) $s$-channel $A_3^{(0)}$, $A_8^{(0)}$, $W_3^{(0)}$, $B^{(0)}$ exchange, (6b) $s$-channel $A_3^{(1)}$, $A_8^{(1)}$, $W_3^{(1)}$, $B^{(1)}$ exchange, (6c) $t$-channel $A_3^{(0)}$, $A_8^{(0)}$, $W_3^{(0)}$, $B^{(0)}$ exchange and (6d) $t$-channel $A_3^{(1)}$, $A_8^{(1)}$, $W_3^{(1)}$, $B^{(1)}$ exchange. It is noted that the couplings of $A_3^{(0)} - X_5^{(1/2)} - X_5^{(1/2)*}$, $A_8^{(0)} - X_5^{(1/2)} - X_5^{(1/2)*}$, $W_3^{(0)} - X_5^{(1/2)} - X_5^{(1/2)*}$ and $B^{(0)} - X_5^{(1/2)} - X_5^{(1/2)*}$ are

$$g_5 \int_{0}^{2\pi R} dy \left( \frac{1}{\sqrt{\pi R}} \sin \frac{y}{2R} \right)^2 \frac{1}{\sqrt{2\pi R}} = g_4,$$

while the couplings of $A_3^{(1)} - X_5^{(1/2)} - X_5^{(1/2)*}$, $A_8^{(1)} - X_5^{(1/2)} - X_5^{(1/2)*}$, $W_3^{(1)} - X_5^{(1/2)} - X_5^{(1/2)*}$ and $B^{(1)} - X_5^{(1/2)} - X_5^{(1/2)*}$ are

$$g_5 \int_{0}^{2\pi R} dy \left( \frac{1}{\sqrt{\pi R}} \sin \frac{y}{2R} \right)^2 \left( \frac{1}{\sqrt{\pi R}} \cos \frac{y}{R} \right) = -g_4 \sqrt{2}.$$

In Table VI we give the results for the $O(1)$ amplitudes. As expected from the 4D equivalence theorem mentioned in the previous section, the scattering amplitude of the 4D gauge fields $X^{(1/2)} X^{(1/2)*} \to X^{(1/2)} X^{(1/2)*}$ is equal to that of the 5th gauge fields (the corresponding would-be NG-like fields) $X_5^{(1/2)} X_5^{(1/2)*} \to X_5^{(1/2)} X_5^{(1/2)*}$ to $O(m/E)$ corrections.

§6. Summary and discussion

In this paper we have investigated the high-energy behavior of the tree-level scattering amplitudes of massive gauge bosons in the 5D orbifold model compactified on $S^1/Z_2$.

The Feynman diagram with massive vector bosons in the external lines, in general, gives an $O(E^4/m^4)$ and/or $O(E^2/m^2)$ contribution to the amplitude and this energy-dependence could cause the violation of the unitarity bound. If the mass of
the gauge boson comes from the spontaneous breaking, it is well-known that these unitarity-violating $O(E^4/m^4)$ and $O(E^2/m^2)$ contributions are canceled among diagrams and furthermore that the $O(1)$ amplitude is the same as the scattering amplitude of the corresponding would-be NG fields up to some constant factor. This equivalence between the amplitudes results from the gauge invariance of the theory. In the orbifold model, by contrast, the symmetry breaking occurs through nontrivial boundary conditions and the boundary conditions themselves do not respect the symmetry. In this sense it can be said that the violation of symmetry for the orbifold model is added by-hand. Thus it is a nontrivial problem to determine whether the unitarity bound is maintained and the equivalence theorem holds. We have discussed this issue in this paper.

We first noted in §2 that the 4D theory written in terms of the KK gauge and scalar fields is invariant under the 4D gauge transformation that mixes the infinite number of KK modes. Then, in §3, we employed this 4D gauge symmetry in the BRST formalism and in §4 derived the Slavnov-Taylor identities among amplitudes. This, in turn, gives the equivalence theorem. Furthermore in §5 we calculated the $O(1)$ amplitude in the 5D $SU(5)$ orbifold model and in the 4D $SU(5)$ (ordinary) GUT model. The results explicitly confirm the equivalence theorem.

Some comments related to unitarity are in order. Unitarity is closely connected to renormalizability: It appears that renormalizable theories preserve unitarity (at least known renormalizable theories preserve unitarity). However, the 4D gauge theory we have studied is not a renormalizable theory because it consists of infinitely-many KK fields whose contribution to the loop diagrams yields divergence. The sum of the KK fields in the 4D theory corresponds to momentum integration along the extra-dimensional direction in the 5D theory. The non-renormalizability of our 4D theory originates from that of the 5D theory. Then, what are the implications of the tree-level unitarity that we have shown to hold in this paper? There are two alternative scenarios. First, it is possible that the orbifold theory is a low energy effective theory applicable to energies less than some scale $M$ and the infinitely high KK tower should be truncated at this scale $M$. In this case the loop diagrams no longer diverge owing to this KK tower truncation, which corresponds in the 5D theory to the momentum cutoff along the extra-dimensional direction. The second possible scenario is to modify the orbifold theory so that it is applicable to all energies. If we consider some physical object such as D-branes situated on the orbifold fixed-point, the quantum fluctuations of this object bring about the modification of our orbifold model. Such fluctuations are expected to yield a damping factor for higher KK modes and cause the sum over all KK modes to converge. This implies that the theory may become renormalizable. This line of thought is worthy of further research.

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References


Note added: After this paper was submitted, the authors became aware of Ref. 26) where a related issue is discussed.