Reversing Symmetries in a Two-Parameter Family of Area-Preserving Maps

K. Zare\(^1\) and K. Tanikawa\(^2\)

\(^1\)Department of Mathematics, Texas State University, San Marcus, Tx 78666, USA
\(^2\)National Astronomical Observatory of Japan, Mitaka 181-8588, Japan

(Received September 1, 2004)

The integrable twist map is used to derive an invertible two-parameter family of area-preserving maps specified by two arbitrary functions of a real variable. It has been shown that the integrable case has infinitely many reversing symmetries (not necessarily all involutory). Almost all of these symmetries are destroyed under addition of non-integrable terms. However some involutory reversing symmetries survive if certain restrictions on the functions are imposed. An involutory reversing symmetry exists in three different cases. In the first two cases, the symmetry lines are continuous curves leading to infinitely many symmetric periodic orbits. The third case is an interesting example of an involutory reversing symmetry in which the corresponding symmetry lines are isolated points leading to none or few symmetric periodic orbits. The combined restrictions of the first two cases lead to a family of maps with double reversing symmetries. Furthermore there exists a subset of this family (identified by an additional restriction) with quadruple reversing symmetries.

§1. Introduction

The area preserving maps have been studied extensively in the past hundred years. This special attention is partly motivated by the area preserving property of the Poincaré maps in the Hamiltonian systems. This property has been used in the researches of Poincaré\(^1\) and Birkhoff\(^2\)–\(^4\) on the restricted three body problem. In addition Birkhoff\(^2\) has shown that the Poincaré map for this problem has an involutory reversing symmetry. This Birkhoff reversing symmetry exists for a sub-class of two-degrees of freedom Hamiltonian systems,\(^5\) and it may be used to obtain an efficient method of searching for the symmetric periodic orbits.\(^6\) In the recent years, use of a more general definition\(^7\) has led to the reversing symmetries that are not necessarily involutory. The reversing symmetry may be used to study the global dynamics such as the locations and the distribution of periodic points, the distribution of stable and chaotic regions, and the identifications of scattering regions. For examples from classical mechanics, see Ref. 5).

In this paper, first a two-parameter family of area preserving mappings is introduced and then the sub-classes which possess a trivial reversing symmetry are identified. We begin in §2 with a two-parameter family of area preserving maps. This family is originated from the integrable twist map and it will reduce to some well-known classes of area preserving maps.\(^8\),\(^9\) The reversing symmetries of these maps are identified in §3, using the general definition mentioned earlier. Finally in §4 the corresponding symmetric periodic points are discussed.
§ 2. A two-parameter family of area-preserving maps

Let us consider a map of an orientable surface such as \( \mathbb{R}^2 \), \( S^1 \times \mathbb{R} \), or \( S^2 \). One of the simplest examples of such a map is the monotone twist map

\[
\begin{pmatrix}
  x_1 \\
  y_1
\end{pmatrix}
= f(x, y) = \begin{pmatrix}
  x + \omega(y) \\
  y
\end{pmatrix},
\]

(1)

where \( \omega \) is a monotone function. We note that the mapping (1) is (i) area preserving (i.e., \( \det Df = 1 \)), (ii) invertible with a simple inverse

\[
f^{-1}(x, y) = \begin{pmatrix}
  x - \omega(y) \\
  y
\end{pmatrix},
\]

(2)

and (iii) integrable (i.e., \( y = c \) is an invariant curve for any constant \( c \)).

Now let us introduce a family of mapping

\[
\begin{pmatrix}
  x_1 \\
  y_1
\end{pmatrix}
= f(x, y, \varepsilon_1) = \begin{pmatrix}
  x + \omega(y) + \varepsilon_1 g(x, y) \\
  y + \varepsilon_1 h(x, y)
\end{pmatrix},
\]

(3)

where \( \varepsilon_1 \) is a parameter and \( g \) and \( h \) are arbitrary functions. Note that \( f(x, y, 0) \) is the monotone twist map mentioned earlier.

The mapping (3) is in general non-integrable, however, it is possible to maintain the area-preservation and invertability. It is area-preserving (i.e., \( \det Df = 1 \)) if and only if

\[
g_x + h_y - \omega'(y)h_x + \varepsilon_1(g_xh_y - h_xg_y) = 0.
\]

(4)

For Eq. (4) to be satisfied, it is sufficient to let

\[
g_x + h_y - \omega'(y)h_x = 0,
\]

(5a)

and

\[
g_xh_y - h_xg_y = 0.
\]

(5b)

Equation (5b) implies \( g = \phi(h) \), and consequently Eq. (5a) reduces to

\[
(\phi'(h) - \omega'(y))h_x + h_y = 0,
\]

(6)

with the following characteristic equation:

\[
dx/dy = \phi'(h) - \omega'(y).
\]

(7)

Equation (7) may be integrated analytically if \( \phi(h) = \varepsilon_2 h \), where \( \varepsilon_2 \) is a constant parameter.

This leads to the characteristic curves

\[
x + \omega(y) - \varepsilon_2 y = \text{constant},
\]

(8)

and to a two-parameter family of area-preserving maps

\[
\begin{pmatrix}
  x_1 \\
  y_1
\end{pmatrix}
= f(x, y, \varepsilon_1, \varepsilon_2) = \begin{pmatrix}
  x + \omega(y) + \varepsilon_1 \varepsilon_2 h(x + \omega(y) - \varepsilon_2 y) \\
  y + \varepsilon_1 h(x + \omega(y) - \varepsilon_2 y)
\end{pmatrix}.
\]

(9)
We note that $f$ is invertible with the following inverse:

$$f^{-1}(x, y, \varepsilon_1, \varepsilon_2) = \left(\frac{x - \omega(y - \varepsilon_1 h(x - \varepsilon_2 y)) - \varepsilon_1 \varepsilon_2 h(x - \varepsilon_2 y)}{y - \varepsilon_1 h(x - \varepsilon_2 y)}\right).$$  \hspace{1cm} (10)

**Remark:** The mapping (9) may be reduced to the extended standard map, if $\varepsilon_1 = \varepsilon_2 = 1$, $\omega(y) = y$, and $h$ is an odd function. The mapping (9) may also be written as $z_1 = z + \omega(y_1)$, $y_1 = y + \varepsilon_1 h(z)$ where $z = x + \omega(y) - \varepsilon_2 y$. This is the generalized standard form given in Refs. 9) and 10). The symmetries of these maps where the functions are restricted to polynomials are discussed in Ref. 10).

### §3. Reversing symmetries

An invertible map $\rho_0$ is called a reversing symmetry of an invertible map $f$, whenever $f^{-1}_\rho_0 = \rho_0 f$, from which we have by induction

$$f^{-n}_\rho_0 = \rho_0 f^n.$$  \hspace{1cm} (11)

Denoting the inverse of $\rho_0$ by $\varphi_0$, we have

$$\varphi_0 f^{-n} = f^n \varphi_0,$$  \hspace{1cm} (12)

which shows that $\varphi_0$ is also a reversing symmetry. We may rewrite Eqs. (11) and (12) as

$$f^{-m}_\rho_i = \rho_i f^m, \quad \varphi_i f^{-m} = f^m \varphi_i,$$  \hspace{1cm} (13)

where

$$\rho_i = \rho_0 f^i = f^{-i}_\rho_0, \quad \varphi_i = f^i \varphi_0 = \varphi_0 f^{-i}.$$  \hspace{1cm} (14)

Equation (13) implies that $\rho_i$ and $\varphi_i$ are also reversing symmetries of $f$. Therefore corresponding to the fundamental symmetries $\rho_0$ and $\varphi_0$, there exist two bi-infinite sequences of reversing symmetries, namely

$$\{\ldots, \rho_{-i}, \ldots, \rho_{-1}, \rho_0, \rho_1, \ldots, \rho_i, \ldots\},$$  \hspace{1cm} (15a)

and

$$\{\ldots, \varphi_{-i}, \ldots, \varphi_{-1}, \varphi_0, \varphi_1, \ldots, \varphi_i, \ldots\}.$$  \hspace{1cm} (15b)

We may easily verify that

$$\varphi_{-i} \rho_i = \rho_i \varphi_{-i} = \rho_{-i} \varphi_i = \varphi_i \rho_{-i} = \text{id},$$  \hspace{1cm} (16)

and

$$\varphi_{-i} \rho_{i+k} = \rho_{i-k} \varphi_{-i} = \rho_{-i} \varphi_{i-k} = \varphi_{i+k} \rho_{-i} = f^k.$$  \hspace{1cm} (17)

It follows from (16) that (15b) is the reversing sequence of a sequence obtained by inverting the members of (15a).

A symmetry line of a reversing symmetry $\rho_i$ is defined by

$$L_i = \{p | \rho_i(p) = p\}.$$  \hspace{1cm} (18)
Consequently corresponding to the sequence (15a), there exists a sequence of symmetry lines
\[ \cdots, L_{-i}, \cdots, L_{-1}, L_0, L_1, \cdots, L_i, \cdots. \]  
(19)
Note that the reversing sequence is the sequence of symmetry lines for (15b), as it follows from (16).

Now let \( p \in L_0 \), then using Eq. (17) we have
\[ f^k(p) = \rho_{-2k} \varphi_k(p) = \varphi_k \rho_0(p) = \varphi_k(p), \]  
(20)
which implies that \( f^k(p) \in L_{-2k} \), if \( p \in L_0 \). Similarly we may show that \( f^k(p) \in L_{-2k+1} \), if \( p \in L_1 \). Therefore the sequence (19) may be divided into two proper subsequences (i.e., invariant under \( f \)), namely
\[ \cdots, L_{-4}, L_{-2}, L_0, L_2, L_4, \cdots \]  
(21)
and
\[ \cdots, L_{-3}, L_{-1}, L_1, L_3, L_5, \cdots \]  
(22)
The orbit of a point \( p \in L_0 \) remains in the sequence (21), and Eqs. (11) and (12) become
\[ f^{-n}(p) = \rho_0 f^n(p), \]  
\[ f^m(p) = \varphi_0 f^{-n}(p). \]  
This means that the backward orbit of \( p \) is \( \rho_0 \) symmetric to the forward orbit of \( p \), and the forward orbit of \( p \) is \( \varphi_0 \) symmetric to the backward orbit of \( p \). Therefore \( \rho_0 \) and \( \varphi_0 \) work in the opposite directions. One is mapping the future to the past and the other the past to the future. Similarly the orbit of a point \( p \in L_1 \) remains in the sequence (22), and
\[ f^{-m}(p) = \rho_1 f^m(p) = \rho_0 f^{m+1}, \]
\[ f^m(p) = \varphi_1 f^{-m}(p) = \varphi_0 f^{-m-1}, \]  
which implies again \( \rho_0 \) and \( \varphi_0 \) symmetries in the opposite directions.

In the special case where \( \rho_0 \) is an involutory reversing symmetry (i.e., \( \varphi_0 = \rho_0 \)), we have \( \varphi_{-i} = \varphi_0 f^i = \rho_0 f^i = \rho_i \). Therefore (15b) is simply the reversing sequence, and Eqs. (16) and (17) become
\[ \rho_i^2 = \text{id} \quad \text{and} \quad f^k = \rho_i \rho_{i+k}. \]  
(23)
In this case, the above sense of direction is lost and there is only one fundamental symmetry \( \rho_0 \).

3.1. Reversing symmetries of the twist map

Now we investigate the reversing symmetries of the simple integrable map given by Eq. (1). Using Eqs. (1), (2), and (11), we obtain
\[ U_0(x + \omega(y), y) = U_0(x, y) - \omega(V_0(x, y)), \] 
\[ V_0(x + \omega(y), y) = V_0(x, y), \]  
(24)
where \( U_0(x, y) \) and \( V_0(x, y) \) denote the components of \( \rho_0 \). The general solution of (24) may be written as
\[ U_0(x, y) = -\tau(y) \omega(V_0(x, y))x + \sum A_k(y) e^{i2k \pi \tau(y)x}, \] 
\[ V_0(x, y) = \sum B_k(y) e^{i2k \pi \tau(y)x}, \]  
(25)
where \( \tau(y) = 1/\omega(y) \), \( A_k(y) = \overline{A}_{-k}(y) \), and \( B_k(y) = \overline{B}_{-k}(y) \). This solution implies that the integrable twist map (1) has infinitely many reversing symmetries.

These reversing symmetries, in general, are not involutory. However, we may identify three involutory sub-class as follows. Let us assume that in (25) \( A_0(y) \) and \( B_0(y) \) are the only non-zero functions. Then the reversing symmetry (25) is involutory if

\[
\omega(B_0(y))A_0(B_0(y)) = \omega(y)A_0(y), \quad B_0(B_0(y)) = y. \tag{26}
\]

Equation (26) has three trivial solutions: (a) \( B_0(y) = y \), (b) \( B_0(y) = -y \), \( \omega(y) \) and \( A_0(y) \) are odd functions, and (c) \( B_0(y) = -y \), \( \omega(y) \) and \( A_0(y) \) are even functions.

This leads to the following involutory reversing symmetries:

\[
\rho_0 = \left( \begin{array}{c} -x + A_0(y) \\ y \end{array} \right), \tag{27a}
\]

\[
\rho_0 = \left( \begin{array}{c} x + A_0(y) \\ -y \end{array} \right), \tag{27b}
\]

\[
\omega \text{ and } A_0 \text{ are odd functions, and}
\]

\[
\rho_0 = \left( \begin{array}{c} -x + A_0(y) \\ -y \end{array} \right). \tag{27c}
\]

\( \omega \) and \( A_0 \) are even functions.

3.2. Reversing symmetries of mapping (9)

Here we search for the reversing symmetries in the two-parameter family of maps introduced in §2. Using Eqs. (9) – (11), we have

\[
W_0(x_1, y_1) = W_0(x, y) - \omega(V_0(x_1, y_1)),
V_0(x_1, y_1) = V_0(x, y) - \varepsilon_1 h(W_0(x, y)), \tag{28}
\]

where \( W_0(x, y) = U_0(x, y) - \varepsilon_2 V_0(x, y) \). Equation (28) possesses three trivial solutions:

(a) \( W_0(x, y) = -x - \omega(y) + \varepsilon_2 y, \quad V_0(x, y) = y, \quad h(-t) = -h(t) \),

(b) \( W_0(x, y) = x + \omega(y) - \varepsilon_2 y, \quad V_0(x, y) = -y, \quad \omega(-t) = -\omega(t), \) and

(c) \( W_0(x, y) = -x - \omega(y) + \varepsilon_2 y, \quad V_0(x, y) = -y, \quad h(-t) = h(t), \omega(-t) = \omega(t) \).

This leads to the following involutory reversing symmetries:

\[
\rho_0 = \left( \begin{array}{c} -x + 2\varepsilon_2 y - \omega(y) \\ y \end{array} \right), \tag{29a}
\]

where \( h \) is an odd function,

\[
\rho_0 = \left( \begin{array}{c} x - 2\varepsilon_2 y + \omega(y) \\ -y \end{array} \right). \tag{29b}
\]
where $\omega$ is an odd function, and

$$\rho_0 = \begin{pmatrix} -x - \omega(y) \\ -y \end{pmatrix},$$

(29c)

where $h$ and $\omega$ are even functions. Note that with the proper selection of $A_0(y)$, Eqs. (27) and (29) are identical. Therefore these particular reversing symmetries of mapping (1) survive under perturbation introduced in (9).

Equation (29) implies that the mapping (9) has

1. no trivial reversing symmetry if $h$ and $\omega$ are neither odd nor both even,
2. one trivial reversing symmetry if $h$ is odd, or if $\omega$ is odd, or if $h$ and $\omega$ are both even, and
3. double reversing symmetry if $h$ and $\omega$ are both odd.

In the case (3), the double reversing symmetries $\rho_0$ and $\rho_0^*$ are respectively given by (29a) and (29b) and we have

$$\rho_0^* = -\rho_0, \quad \text{and} \quad f^k(-p) = -f^k(p).$$

(30)

Remarks: 1. In the special case of the extended standard maps (i.e., $\varepsilon_1 = \varepsilon_2 = 1$, $\omega(y) = y$), Eqs. (29a) and (29b) reduce to the double reversing symmetries discussed in Ref. 8).

2. If $h$ and $\omega$ are both odd and $\omega(t) = \mp \varepsilon_1 h(t)$, then Eq. (28) possesses two additional trivial solutions: $W_0(x, y) = y$, $V_0(x, y) = \pm x \mp \varepsilon_2 y$, and $W_0(x, y) = -y$, $V_0(x, y) = \mp x \pm \varepsilon_2 y$, leading to

$$\rho_0 = \begin{pmatrix} \pm \varepsilon_2 x + (1 \mp \varepsilon_2^2)y \\ \pm x \mp \varepsilon_2 y \end{pmatrix}$$

and

$$\rho_0 = \begin{pmatrix} \mp \varepsilon_2 x - (1 \mp \varepsilon_2^2)y \\ \mp x \pm \varepsilon_2 y \end{pmatrix}.$$  

Therefore in this case the mapping (9) has quadruple reversing symmetry. The particular choice of $\varepsilon_2 = 0$ and $\omega(t) = a \sin t$ yields the quadruply reversible non-twist map discussed in Ref. 11).

§4. Periodic points

In this section we consider the mapping (9) with $\varepsilon_1 = \varepsilon_2 = 1$. The periodic points are the solutions of the following simultaneous equations:

(a) Fixed points

$$\omega(y) = 0, \quad h(x - y) = 0.$$  

(31a)

(b) Period two points

$$\omega(y) + \omega(y - h(x - y)) = 0, \quad h(x - y) + h(x - y + \omega(y)) = 0.$$  

(31b)
(c) Period three points
\[
\omega(y) + \omega(y - h(x - y)) + \omega(y + h(x - y + \omega(y))) = 0,
\]
\[
h(x - y) + h(x - y + \omega(y)) + h(x - y - \omega(y - h(x - y))) = 0. \quad (31c)
\]

In the case that we have a reversing symmetry $\rho_0$, it follows from (16) and (17) that a necessary and sufficient condition for a periodic point of period $k$ is
\[
\rho_{i+k}(p) = \rho_i(p). \quad (32)
\]
This implies that for a periodic point $p$, the sequence (15a) is a periodic sequence. In particular, Eq. (32) is satisfied whenever
\[
\rho_{i+k}(p) = p, \quad \text{and} \quad \rho_i(p) = p. \quad (33)
\]
In this case $p \in L_{i+k} \cap L_i$, and we refer to $p$ as a symmetric periodic point. The symmetric periodic points with even period are the intersections of members of the sequence (21), or (22). The symmetric periodic points with odd period are the intersections of members of (21) with members of (22).

In the case of double reversing symmetry, using Eqs. (23) and (30) we obtain
\[
\rho^*_i \rho_i = \text{id}, \quad \text{and} \quad f^k = -\rho^*_i \rho_{i+k}. \quad (34)
\]
Here for $p \in L_i^* \cap L_{i+k}$, we have $f^k(p) = -p$ and $f^{2k}(p) = f^k(-p) = -f^k(p) = p$. Therefore in this case, the intersections of symmetry lines of different symmetries are doubly symmetric periodic points with even period. To find the symmetry lines, we obtain the reversing symmetries $\rho_i$ from Eq. (14) and we use definition (18). The intersections of these lines lead to symmetric periodic points

4.1. Symmetric periodic points if $h$ is odd

If we assume that $h$ is an odd function, the mapping (9) has a fundamental reversing symmetry given by Eq. (29a), and the symmetry lines are
\[
L_{-2} : x - y - (1/2)\omega(y + h(-x + y)) = 0, \quad L_{-1} : x - y = 0,
\]
\[
L_0 : x - y + (1/2)\omega(y) = 0, \quad L_1 : x - y + \omega(y) = 0,
\]
\[
L_2 : x - y + \omega(y) + (1/2)\omega(y + h((x - y + \omega(y)))). \quad (35)
\]
(a) Symmetric fixed points
\[
L_0 \cap L_1 : \quad x - y = 0, \quad \omega(y) = 0. \quad (36a)
\]
(b) Symmetric period two points
\[
L_0 \cap L_2 : \quad x - y + (1/2)\omega(y) = 0, \quad \omega(y) + \omega(y + h((1/2)\omega(y))) = 0. \quad (36b)
\]
The intersection of odd symmetry lines $L_{-1} \cap L_1$ reduces to (36a) and therefore contains only the fixed points.
(c) Symmetric period three points
\[
L_{-1} \cap L_2 : \quad x - y = 0, \quad \omega(y) + (1/2)\omega(y + h(\omega(y))) = 0. \quad (36c)
\]
Note that these points are on $L_{-1}$ and they map into points on $L_0$ and $L_1$ in the forward and backward directions respectively.
4.2. Symmetric periodic points if \( \omega \) is odd

Now assuming \( \omega \) is an odd function, we have the reversing symmetry given by (29b) and the symmetry lines are

\[
\begin{align*}
L_{-2}^* & : y - h(x - y) = 0, & L_{-1}^* & : y - (1/2)h(x - y) = 0, & L_0^* & : y = 0, \\
L_1^* & : y + (1/2)h(x - y + \omega(y)) = 0, & L_2^* & : y + h(x - y + \omega(y)) = 0.
\end{align*}
\]

(a) Symmetric fixed points

\( L_0^* \cap L_1^* : y = 0, h(x) = 0. \) (38a)

(b) Symmetric period two points

\( L_{-1}^* \cap L_1^* : y - (1/2)h(x - y) = 0, h(x - y) + h(x - y + \omega(y)) = 0. \) (38b)

The intersection of even symmetry lines \( L_0^* \cap L_2^* \) reduces to (38a) and it contains the fixed points only.

(c) Symmetric period three points

\( L_{-1}^* \cap L_2^* : y - (1/2)h(x - y) = 0, h(x - y) + 2h(x - y + \omega(y)) = 0. \) (38c)

Note again that these points are on \( L_{-1}^* \) and the other two points in their orbits are on \( L_0^* \) and \( L_1^* \).

4.3. Symmetric periodic points if \( h \) and \( \omega \) are both even

In the case that \( h \) and \( \omega \) are even, we have the reversing symmetry given by (29c) and the symmetry lines are

\[
\begin{align*}
L_{-2}^{**} & : x = h((1/2)\omega(0)) + (1/2)\omega(0), & y = h((1/2)\omega(0)), \\
L_{-1}^{**} & : x = (1/2)h(0), & y = (1/2)h(0), \\
L_0^{**} & : x = -(1/2)\omega(0), & y = 0, \\
L_1^{**} & : x = -(1/2)h(0) - \omega((1/2)h(0)), & y = -(1/2)h(0), \\
L_2^{**} & : x = -(1/2)\omega(0) - h((1/2)\omega(0)) - \omega(h(1/2)\omega(0)), & y = -h((1/2)\omega(0)).
\end{align*}
\]

(39)

In this case the symmetry lines are isolated points rather than curves. These symmetry points in general are not periodic unless additional restrictions on \( h \) and \( \omega \) are imposed.

For example (a) if \( h(0) = \omega(0) = 0 \), then \( x = y = 0 \) is a symmetric fixed point, and (b) if \( h(0) + 2h((1/2)\omega(0)) = 0, \omega(0) + 2\omega((1/2)h(0)) = 0 \), then \( x = y = (1/2)h(0) \) is a symmetric period three point.

4.4. Symmetric periodic points if \( h \) and \( \omega \) are both odd

This is the case of double reversing symmetries and among the symmetric periodic points discussed in §§4.1 and 4.2 we may identify a set of doubly symmetric periodic points with even period.

(a) Fixed points

\( L_0 \cap L_0^* : x = 0, y = 0. \) (40a)
The origin is the only doubly symmetric fixed point.

(b) Period two points

\[ L_0 \cap L_1^* : \quad x - y + \frac{1}{2} \omega(y) = 0, \quad y + \frac{1}{2} h((1/2)\omega(y)) = 0. \] (40b)

The intersection \( L_0^* \cap L_1 \) contains only the fixed point.

(c) Period four points

\[
\begin{align*}
L_2 \cap L_0^* : & \quad y = 0, \quad x + \frac{1}{2} \omega(h(x)) = 0, \\
L_{-1} \cap L_1^* : & \quad x - y = 0, \quad y + \frac{1}{2} h((1/2)\omega(y)) = 0.
\end{align*}
\] (40c)

Remarks: 1. There exists an involutory reversing symmetry if \( h \) and \( \omega \) are both even. However the corresponding symmetry lines are isolated points leading to very few symmetric orbits and in particular if any very few symmetric periodic orbits.

2. In the case that \( h \) and \( \omega \) are both odd if either (i) \( \omega(t) = -\varepsilon_1 h(t) \) or (ii) \( \omega(t) = \varepsilon_1 h(t) \), the mapping (9) has quadruple reversing symmetry. The symmetry lines of the additional symmetries are

(i) \( (L_0)_a : x = (1+\varepsilon_2)y, \quad (L_0^*)_a : x = (-1+\varepsilon_2)y \) and

(ii) \( (L_0)_a : y = x = 0, \quad (L_0^*)_a : y = x = 0 \).

Note that the symmetry lines are continuous lines in (i) and isolated points in (ii).

References