Application of the Perturbative Renormalization Group Method Based on the Lie Group to Pulse Dynamics

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Pulse dynamics are studied using the perturbative renormalization group method based on the Lie group. The method is explained with several examples, and its similarities and dissimilarities with other methods are discussed. An interacting pulse is composed of a superposition of a single pulse solution (that is, the zero-th order solution) and its corrections. In the present method, the corrections are assumed to be expansions in terms of the zero eigenfunctions of a linear operator, whose coefficients are time-dependent variables. Unlike in other methods, in the present method, the time-dependence of the position of a pulse is introduced a posteriori in order to avoid the divergence of the corrections obtained in the naive perturbation expansion. The time-dependence of the position of a pulse is easily obtained by taking the inner product with the zero eigenfunction of the adjoint operator.

It is shown that the present method based on the Lie group is a first-order approximation of interacting pulses. However, the truncated renormalization transformation reproduces the results obtained with other methods. It is necessary to use the zero-th order solutions of the linear operator and its adjoint operator in order to perform actual calculations.

§1. Introduction

When we observe pattern formations, we often notice that they develop with certain spatio-temporal scales. In order to describe such spatio-temporal evolution, it is necessary to know how these scales are chosen in the system. The essential part of the evolution equation determines these unique scalings, while other terms can be treated as perturbations. When we describe spatio-temporal patterns with nonlinear equations, there appear slow and/or large-scale variables, which can be characterized by small parameters. For different choices of the orders of magnitudes of such small parameters, there appear different types of patterns. It is necessary to derive the time evolution equation in order to understand the mechanism of the pattern formation. For this purpose, the reductive perturbation method and the multiple scale method have been proposed.\textsuperscript{1),2)} However, in these methods, it is necessary to know the resultant forms of the solution a priori. The key procedures of these methods are to find the transformations of variables by making physically reasonable predictions and trial and error. For this reason, these methods can be difficult to apply for naive researchers. Another method applied to such problems is the renormalization group approach. The renormalization transformation eliminates the divergence appearing in results obtained from a naive perturbation expansion, and one can obtain the asymptotic behavior of the system. In this method, the secular terms are removed by renormalizing them into integral constants, and the time-dependences of the renormalized variables are obtained from the renormalization group (RG) equations.
Unlike the situation in the reductive perturbation method and multiple scale method, it is not necessary to know the orders of the small parameters in the renormalization transformation. This is a great advantage of the RG method.

1.1. Renormalization group method

The RG approach has a long history in theoretical physics. There are many studies on its methods, and several reduction methods for ordinary differential equations have been proposed. We here review only the works related to nonlinear physics and our present study.

The Illinois group, Chen, Goldenfeld, Oono and their collaborators, proposed simple and systematic procedures to obtain the asymptotic behavior of solutions of differential equations using perturbative RG.\footnote{3)–6)} When a solution is naively expanded with respect to small parameters, it is often the cases that the corrections include secular terms. Then, in order to avoid the resulting divergence, the integral constants are renormalized to temporally or spatially dependent variables. Their RG method is a very powerful tool to reduce nonlinear evolution equations, and it has been applied to many systems, including reaction diffusion systems, the Kuramoto-Sivashinsky equation and the Swift-Hohenberg equation.\footnote{7), 8)}

Although these RG methods are simple, the underlying mathematical structure and their relevance were unclear. Kunihiro clearly explained the relevance and usefulness of the RG equation for global analysis by introducing a geometrical interpretation of the RG approach as a so-called envelope method.\footnote{9)} In that theory, it is assumed that the integral constants can depend on arbitrary values of the initial times and positions. The RG equation can then be interpreted as an envelope equation that determines the slow motion of the variables in the system.

Although the symmetry of the RG has been studied for many years,\footnote{10)–12)} the original exact concept of symmetry is modified to an approximate meaning of the semi-group.\footnote{13), 14)} A renormalization transformation based on the Lie group has been proposed by Russian researchers.\footnote{15)} Their renormalization transformation is a kind of functional self-similarity transformation based on modern group analysis. They proposed general schemes to construct the RG symmetry for boundary value problems and elucidated the Lie group structure in the RG method. Although the method is standard in Lie group theory, it is difficult to find the Lie or Lie-Bäcklund basis of the Lie symmetries for several cases. In order to overcome this difficulty, Goto, Masutomi and Nozaki constructed a representation of a translational group without invoking Lie symmetries.\footnote{16)} In their paper, they presented the recipes to obtain the asymptotic RG equation from autonomous and non-autonomous nonlinear systems.

Recently, Ei, Fujii and Kunihiro proposed a systematic procedure to construct the invariant manifolds for differential equations.\footnote{17)} They treated generic vector equations, which consist of a linear operator $A$ possessing zero eigenvalues and perturbative nonlinear terms. They showed how to construct the invariant manifold as the initial value of the evolution equations using the RG method. Their formulation supports the foundation of the envelope method proposed by Kunihiro. Their procedures are slightly different in the cases that $A$ is diagonalizable and non-
diagonalizable. These are referred to as the semi-simple case and the Jordan cell case, respectively. In their approach, the correction terms are decomposed into two components by projection operators denoted $P$ and $Q$. The initial value is also perturbatively expanded, each term is determined such that the perturbation expansion be valid. When $A$ has semi-simple zero eigenvalues, the correction terms are obtained from a formal integral with an initial value. We here note that there appear secular terms in the perturbed solution. In order to eliminate the divergent (‘dangerous’) terms, the initial value is chosen such that the divergent terms are canceled out. In their theory, the integral constants are assumed to depend on the arbitrary initial time, and their evolution equation is obtained as the RG equation. The above-described procedures are carried out order by order, and through them the invariant manifolds as the initial value are derived. When the linear operator $A$ has a Jordan cell, although the procedures are similar, the dimension of the invariant manifold is greater than that of the unperturbed invariant manifold.

1.2. Pulse dynamics

Pulse dynamics has been actively studied. The key point in deriving the interacting pulse solutions is the treatment of the arbitrary constants in the system, such as the position of the pulse and the traveling speed. Even if these constants are arbitrary in a single pulse solution, perturbative terms and interactions among pulses play important roles in determining the asymptotic speed and the relative positions of pulses. Ei and Ohta proposed a systematic method to study the dynamics of interacting pulses in one dimension.\(^{18}\) In that method, the equations of motion for the position of a pulse are obtained from a non-secularity condition, that is, a kind of solvability condition. Their theory was extended to the pulse dynamics of the two-dimensional Benney equation by Ogawa and Liu.\(^{19}\) Pulses in the one-dimensional Benney equation repel each other, so that all pulses are separated by equal distance asymptotically. They derived the temporal evolution equation for the position of a pulse in two dimensions, and found a non-trivial stable configuration of many traveling pulses. This theoretical result is supported by their numerical simulations. Ei, Fujii and Kunihiro applied their RG prescription to the case of pulse interactions in the time-dependent Ginzburg-Landau (TDGL) equation and the Korteweg-de Vries (KdV) equation. They successfully derived the pulse dynamics from the RG equation.

Thus, as described above, there have been several systematic methods applied to the study of pulse dynamics. In spite of this fact, we here propose a different method. In this paper, we apply the reformulated RG method based on the Lie group proposed by Goto, Masutomi and Nozaki\(^{16}\) to pulse dynamics. Although some results presented here have already been obtained with other methods, we derive similar results and compare our method with other methods. In the next section, we briefly review the present method.
§2. Method

The procedures employed in the theory proposed by Goto et al. are the following. First, the secular series solution of a perturbed system is obtained using the naive perturbation expansion with respect to a small parameter. Second, the integral constants are determined, and the renormalization transformation is introduced in order to avoid the divergence. It is assumed that all secular terms are renormalized to integral constants. It is found that the renormalization transformation can be rewritten in the form of the translational group. The asymptotic representation of the Lie group in terms of renormalized integral constants is obtained. The generator of the translational group yields an RG equation.

In order to apply their Lie group approach to pulse dynamics, it is necessary to find arbitrary constants in a single pulse solution and a small parameter in the system. For autonomous systems, which possess translational symmetry, we note that the position of the pulse is usually an arbitrary constant. However, when there are other pulses, the nonlinear term in the evolution equation causes an interaction among the pulses. Our main assumption is that the width of the interface of these solutions is sufficiently smaller than the distance between the pulses. This implies that interactions among pulses are small, and therefore we can treat them as a perturbation of the superposition of independent single pulse solutions. We can find a small parameter related to the interaction, which is usually a monotonically decreasing function of the distance between pulses.

The present scheme for the treatment of the pulse dynamics is the following. The interacting pulse solution is written as the sum of a superposition of a single pulse solutions (which corresponds to the zero-th order solution) and its corrections. We here assume that the corrections can be expanded in terms of zero eigenfunctions of a linear operator. Using the naive perturbation expansion, we note that there appear secular terms in the corrections. In order to avoid the resulting divergence, the arbitrary constants in the system are renormalized to time-dependent variables. In the present method, all secular terms are assumed to be renormalized to arbitrary constants. By comparing the interacting pulse solution obtained from the naive perturbation expansion with the Taylor expansion of the renormalized solution, we find correspondences between the coefficients of the renormalization transformation and the secular terms. Thus we can construct the renormalization transformation using the secular terms. The renormalization transformation is then rewritten in the form of the translational group, and we are thereby able to obtain an asymptotic representation of the Lie group. The generator of the Lie group yields the time dependence of the position of the pulses.

In the next section, we present examples demonstrating the above prescription. We consider the TDGL equation, the KdV equation (including the Benney equation) and reaction diffusion systems in the context of pulse dynamics in one dimension.
§3. Models

3.1. Interaction of asymmetric kink solutions

We first treat the interaction of an asymmetric kink pair. Let us consider the TDGL equation

$$\frac{\partial u}{\partial t} = \epsilon^2 \frac{\partial^2 u}{\partial x^2} - \frac{1}{2} (u - a)(u - 1)(u + 1) + \beta \frac{\partial u^2}{\partial x},$$

(3.1.1)

where $\epsilon$ is a small positive constant, and the last term breaks the spatial symmetry of the solution. Here we consider the case $\beta \geq 0$, $-1 < a < 1$. When $a \neq 0$, (3.1.1) has asymmetric traveling kink solutions. In the moving frame represented by the coordinate $z = x - vt$, these two kinds of traveling kink solutions are $u^{(\pm)}(z) = \tanh \theta^{(\pm)} z$, where $\theta^{(\pm)} = \frac{1}{2\epsilon^2} \beta \pm \sqrt{\beta^2 + \epsilon^2}$, and the corresponding traveling speeds are $v^{(\pm)} = \frac{1}{2\epsilon} a$, respectively. The function $u^{(\pm)}(z)$ are called the kink and anti-kink solutions, respectively.

In the case $a = 0$, (3.1.1) is

$$\frac{\partial u}{\partial t} = \epsilon^2 \frac{\partial^2 u}{\partial x^2} - \frac{1}{2} u(u - 1)(u + 1) + \beta \frac{\partial u^2}{\partial x}. $$

(3.1.2)

This equation has two kinds of stationary asymmetric kink solutions, $u_{0,\pm}(x) = \tanh(k_{\pm} x)$, where $k_{\pm} = \frac{1}{2\epsilon^2} \beta \pm \sqrt{\beta^2 + \epsilon^2}$. Due to the symmetry breaking term, $\beta \partial_x u^2$, the width of these kink and anti-kink solutions are different. They are respectively approximated by $\sim (1/|k_{\pm}'|)$. It is noted that $(1/|k_-|) \geq (1/|k_+|)$. We assume that there is an asymmetric kink and anti-kink pair, whose positions are denoted by $x_1$ and $x_2$ (where $x_1 < x_2$), and that the distance between two kinks is sufficiently large compared to the width of the interface, that is, $|x_2 - x_1| \gg (1/|k_-|)$.

The nonlinear terms in (3.1.2) introduce the interactions between kinks, and we treat them as a perturbation, because the distance $|x_2 - x_1|$ is assumed to be large.

Now we consider the dynamics of asymmetric kinks in the above system. For the case of symmetric kinks ($\beta = 0$), although these kink and anti-kink solutions have the same stability, it is known that these kink pair attracts each other, and they eventually merge.$^{18,20}$ Contrastingly, for the case of asymmetric kinks, it seems that the dynamics have not yet been thoroughly studied. From the point of view described above, we study the dynamical properties of the kinks using the RG method based on the Lie group.

In order to describe the interacting kinks solution of (3.1.2), we write the zero-th order solution such as $\bar{U}(x) = u_{0,0_+}(x - x_1) + u_{0,0_-}(x - x_2) - 1 \equiv U_+ + U_- - 1$. We remark that the functions $U_\pm$ satisfy $\epsilon^2 U_{\pm,xx} + f(U_{\pm}) + \beta(U_{\pm}^2)_x = 0$, where the suffix $x$ denotes differentiation with respect to $x$ and $f(U) = \frac{1}{2} U(1 - U^2)$. With the above choice, $\bar{U}$ has a rectangular shape with two interfaces, one at $x_1$ and one at $x_2$.

We write the interacting kinks solution as

$$u(x, t) = \bar{U}(x) + B(x, t)$$

$$= \bar{U}(x) + \sum_{l=1}^{\infty} \delta l b_l(x, t)$$
Here we use the approximations $L$ be expanded in terms of the zero eigenfunctions of $\lambda$. Thus we obtain the time-evolution equation for $b_1(x, t)$ up to $O(\delta)$;

$$[\partial_t - \mathcal{L}](\delta b_1) = \left[-f(U_+) - f(U_-) + f(\bar{U})\right]$$

$$+ \beta \partial_x [\bar{U}^2 - U_+^2 - U_-^2],$$

where we have defined the linear operator $\mathcal{L} = \epsilon^2 \partial_x^2 + f'(U) + 2\beta \bar{U} \partial_x + 2\beta(\partial_x \bar{U})$. Here we use the approximations $f'(U) \sim f'(U_+)$ near $x = x_1$, and $f'(\bar{U}) \sim f'(U_-)$ near $x = x_2$, because the distance between two kinks is large compared with the widths of the interfaces, so that we can regard $U_{+,x}$ and $U_{-,x}$ to be spatially well localized. Thus, the form of $\mathcal{L}$ can be simplified near the kink positions $x_1$ and $x_2$ as $\mathcal{L}_\pm = \epsilon^2 \partial_x^2 + f'(U_\pm) + 2\beta U_{\pm} \partial_x + 2\beta(\partial_x U_{\pm})$, respectively. For these operators, it is easily verified that $U_{+,x}$ and $U_{-,x}$ are zero eigenfunctions; i.e. $\mathcal{L}_+ U_{+,x} = 0$ and $\mathcal{L}_- U_{-,x} = 0$.

The explicit form of $\mathcal{L}$ is $\mathcal{L} = \epsilon^2 \partial_x^2 + \frac{1}{2}(-3\bar{U}^2 + 1) + 2\beta(\partial_x \bar{U}) + 2\beta \bar{U} \partial_x$, and its adjoint is $\mathcal{L}^* = \epsilon^2 \partial_x^2 + \frac{1}{2}(-3U_{\pm}^2 + 1) - 2\beta \bar{U} \partial_x$. We can also obtain the approximate operators $\mathcal{L}_{\pm}^*$ near $x_1$ and $x_2$ as $\mathcal{L}_{\pm}^* = \epsilon^2 \partial_x^2 + \frac{1}{2}(-3U_{\pm}^2 + 1) - 2\beta U_{\pm} \partial_x$, respectively. It is easily verified that the zero eigenfunctions of $\mathcal{L}_{\pm}^*$ are $\left[\frac{1}{\cosh k_{\pm}(x-x_1)}\right]_{l_+} \equiv V_+$ and $\left[\frac{1}{\cosh k_{\pm}(x-x_2)}\right]_{l_-} \equiv V_-$, respectively, where $l_{\pm} = (\frac{1}{2\epsilon^2 k^2})$. We assume that $b_1(x, t)$ can be expanded in terms of the zero eigenfunctions of $\mathcal{L}_{\pm}^*$ as

$$b_1(x, t) = P_1^{(1)}(t) U_{+,x} + P_2^{(1)}(t) U_{-,x} + \bar{b}_1(x).$$

We consider separately the dynamics for each position of the kinks. Noting the condition $|x_2 - x_1| \gg |1/k_-$, when we consider $x \sim x_1$, the right-hand side of (3.1.5) is simplified as

$$\left[-f(U_+) - f(U_-) + f(U_+ + U_- - 1)\right] + \beta[\bar{U}^2 - U_+^2 - U_-^2],$$

$$= \frac{1}{2}(1 - 3U_+^2)(U_+ - 1) - \frac{3}{2}U_+(U_- - 1)^2 - \frac{1}{2}(U_- - 1)^3 - \frac{1}{2}U_-(1 - U_-)^2 + \beta \partial_x [2U_+(U_- - 1) - 2U_- + 1]$$

$$\sim \bar{U}(x) + \delta b_1(x,t),$$

where $B(x, t)$ corresponds to a correction and $\delta$ is a small parameter, with $O(\delta) \sim \exp[k_{-}(x_2 - x_1)]$. In the above, we have ignored the higher-order terms in order to focus on the first-order correction. Substituting the above form of $u(x, t)$ into (3.1.2) and expanding the right-hand side of (3.1.2) with respect to $\delta b_1$, we obtain

$$\epsilon^2 \partial_x^2 [U_+ + U_- - 1 + \delta b_1] + f(U_+ + U_- - 1 + \delta b_1) + \beta \partial_x [U_+ + U_- - 1 + \delta b_1]^2$$

$$\sim \left[\epsilon^2 \partial_x^2 + f'(\bar{U}) + 2\beta \bar{U} \partial_x + 2\beta(\partial_x \bar{U})\right] \delta b_1$$

$$+ \left[-f(U_+) - f(U_-) + f(\bar{U}) + \beta(\bar{U}^2),x - \beta(U_+^2),x - \beta(U_-^2),x\right]$$

$$+ \left[\frac{1}{2}f''(\bar{U}) + \beta \partial_x\right](\delta b_1)^2 + \cdots. (3.1.4)$$
\[ \sim \frac{1}{2}(1 - 3U_+^2)(U_+ - 1) - \frac{3}{2}U_+(U_+ - 1)^2 - \frac{1}{2}(U_- - 1)^3 - \frac{1}{2}U_-(1 - U_-^2) + 2\beta(U_+ - 1)\partial_x(U_+ - 1) \\
= -\frac{3}{2}(U_+ - 1)(U_+ - 1)(U_+ + U_-) + 2\beta(U_+ - 1)\partial_x(U_+ - 1) \\
\equiv g_+(U_+, U_-). \] (3.1.7)

Here, we remark that the value of \((U_- - 1)\) near \(x = x_1\) is \(O(\delta)\). Substituting (3.1.6) into (3.1.5) and considering \(O(\delta)\) terms, we obtain the temporal evolution equation for \(P_1^{(1)}\) as
\[
\delta(P_1^{(1)}U_{+,x} + \mathcal{L}_+\bar{b}_1) = g_+(U_+, U_-), \tag{3.1.8}
\]
where the suffix \(t\) denotes differentiation with respect to \(t\). Taking the inner product of (3.1.8) with the zero eigenfunction of \(\mathcal{L}_+\), we obtain the time dependence of \(P_1^{(1)}\) as
\[
P_{1,t}^{(1)} = \frac{\langle V_+, g_+(U_+, U_-) \rangle}{\delta \langle V_+, U_{+,x} \rangle}, \tag{3.1.9}
\]
where \(\langle \cdot \rangle\) is defined as follows:
\[
\langle F(x), G(x) \rangle = \int_{-\infty}^{+\infty} F(x)G(x)dx. \tag{3.1.10}
\]

By the similar procedure, the approximate form of the right-hand side of (3.1.5) near \(x = x_2\) is obtained as
\[
\left[-f(U_+) - f(U_-) + f(U_+ + U_- - 1)\right] + \beta[\bar{U}_-^2 - U_+^2 - U_-^2]_x \\
\sim \frac{1}{2}(1 - 3U_-^2)(U_- - 1) - \frac{3}{2}U_-(U_- - 1)^2 - \frac{1}{2}(U_- - 1)^3 - \frac{1}{2}U_+(1 - U_+^2) \\
+ \beta\partial_x\left[2U_-(U_- - 1) - 2U_+ + 1\right] \\
= -\frac{3}{2}(U_+ - 1)(U_- - 1)(U_+ + U_-) + 2\beta(U_+ - 1)\partial_x(U_- - 1) \\
\equiv g_-(U_+, U_-), \tag{3.1.11}
\]
where the value of \((U_+ - 1)\) near \(x = x_2\) is of order \(\exp[-k_+(x_2 - x_1)] \ll \delta\). Using the above approximate expression for the nonlinear terms, we obtain the time evolution equation of \(P_2^{(1)}\) as
\[
\delta(P_2^{(1)}U_{-,x} + \mathcal{L}_-\bar{b}_1) = g_-(U_+, U_-). \tag{3.1.12}
\]

Taking the inner product of (3.1.12) with the zero eigenfunction of \(\mathcal{L}_-\), we obtain the time-dependence of \(P_2^{(1)}\) as
\[
P_{2,t}^{(1)} = \frac{\langle V_-, g_-(U_+, U_-) \rangle}{\delta \langle V_-, U_{-,x} \rangle}. \tag{3.1.13}
\]
We note that the right-hand sides of (3.1.8) and (3.1.12) are independent of \( t \), and therefore \( P_1(1)(t) \) and \( P_2(1)(t) \) are polynomials of degree one. Thus we find that the first-order correction obtained from the naive perturbation expansion includes secular terms. Up to \( O(\delta) \), the solution representing two interacting kinks is

\[
\begin{align*}
\bar{u}(x, t) &= \bar{U}(x) + B(x, t) \\
&\sim \bar{U}(x) + \delta b_1(x, t) \\
&= U_+ + U_- - 1 \\
&\quad + \delta \left[ P_1(1)(t)U_{+,x} + P_2(1)(t)U_{-,x} + \bar{b}_1(x) \right].
\end{align*}
\]

(3.1.14)

It is easy to imagine from the expression (3.1.14) that the arbitrary constants \( x_1 \) and \( x_2 \) should be renormalized to time-dependent variables. For this purpose, we introduce an asymptotic renormalization transformation \( R \), and through this transformation, the arbitrary constants \( x_1 \) and \( x_2 \) are renormalized as \( \tilde{x}_1(t) = R(x_1, t) \) and \( \tilde{x}_2(t) = R(x_2, t) \). Although we treated up to \( O(\delta) \) terms, the formal expressions of the renormalized variables including higher-order terms are

\[
\begin{align*}
\tilde{x}_1(t) &= R(x_1, t) = x_1 \left[ 1 + \delta p_1^{(1)}(t) + \delta^2 p_1^{(2)}(t) + \cdots \right], \\
\tilde{x}_2(t) &= R(x_2, t) = x_2 \left[ 1 + \delta p_2^{(1)}(t) + \delta^2 p_2^{(2)}(t) + \cdots \right].
\end{align*}
\]

(3.1.15)

The above naive definition of the renormalization transformation is valid for small \( t \). We would like to obtain the Lie group from the above definition. Here we remark that the evolution equation (3.1.2) possesses translational symmetry, and hence a single kink solution possesses a translational symmetry with respect to spatial coordinate. However, interacting kinks no longer have the translational symmetry with respect to the distance between them, and for this reason the arbitrary constants change to time-dependent variables. At this point, we do not know whether the renormalization transformation expressed by \( \tilde{x}_1(t) = R(x_1, t) \) and \( \tilde{x}_2(t) = R(x_2, t) \) possesses translational symmetry with respect to time. In order to treat such cases, it is necessary to follow a general procedure, which we now describe. We first derive the relation for \( \tilde{x}_1(t) \), the relation for \( \tilde{x}_2(t) \) is easily obtained in a similar way. In order to rewrite the renormalization transformation (3.1.15) in the form of the translational group, we shift the time \( t \) by an arbitrary value \( \tau \). Then we have

\[
\begin{align*}
\tilde{x}_1(t + \tau) &= x_1 \left[ 1 + \delta p_1^{(1)}(t + \tau) + \delta^2 p_1^{(2)}(t + \tau) + \cdots \right] \\
&= x_1 \left[ 1 + \sum_{n=1}^{\infty} \delta^n p_1^{(n)}(t) \right] \\
&\quad + x_1 \left[ \tau \sum_{n=1}^{\infty} \delta^n p_{1,t}^{(n)}(t) + \frac{1}{2!} \tau^2 \sum_{n=1}^{\infty} \delta^n p_{1,tt}^{(n)}(t) \\
&\quad + \frac{1}{3!} \tau^3 \sum_{n=1}^{\infty} \delta^n p_{1,ttt}^{(n)}(t) + \cdots \right],
\end{align*}
\]

(3.1.16)
where $p^{(n)}_{1,kt}$ denotes the $k$-th derivative of $p^{(n)}_1$ with respect to $t$. As $x_1$ can be described by using $\tilde{x}_1(t)$ from (3.1.15), we can derive the asymptotic representation of the Lie group as

$$\tilde{x}_1(t + \tau) = \tilde{x}_1(t) + \left[ \tau \sum_{n=1}^{\infty} \delta^n p^{(n)}_{1,t}(t) + \frac{1}{2!} \tau^2 \sum_{n=1}^{\infty} \delta^n p^{(n)}_{1,2t}(t) + \frac{1}{3!} \tau^3 \sum_{n=1}^{\infty} \delta^n p^{(n)}_{1,3t}(t) + \cdots \right] \tilde{x}_1(t) \Big/ \left[ 1 + \sum_{n=1}^{\infty} \delta^n p^{(n)}_1(t) \right]. \quad (3.1.17)$$

When the renormalization transformation does not possess translational symmetry, the right-hand side of (3.1.17) depends explicitly on $t$. Then it is appropriate to formulate (3.1.17) in terms of an operator as

$$\tilde{x}_1(t + \tau) = G_\tau \left\{ \tilde{x}_1(t), T(t) \right\}, T(t) = t, \quad (3.1.18)$$

where $G_\tau$ is a translational group operator. The general derivation of (3.1.18) and proofs of the group operation $G_\tau$ are given in Ref. 16). According to the discussion given in this paper, we can derive an asymptotic expression for the generator of the Lie group, that is, an RG equation, as

$$\frac{d\tilde{x}_1(t)}{dt} = \partial_\tau \left[ G_\tau \left\{ \tilde{x}_1(t), T(t) \right\} \right] |_{\tau=0} = \left[ \frac{d}{dt} \left\{ \log \left( 1 + \sum_{n=1}^{\infty} \delta^n p^{(n)}_1(t) \right) \right\} \right] \tilde{x}_1(t). \quad (3.1.19)$$

It is easily verified that the renormalization transformation (3.1.15) is recovered by formally integrating (3.1.19) with respect to $t$. For the truncated series of (3.1.19), we have

$$\tilde{x}_1(t) = \tilde{x}_1(0) \exp \left[ \delta p^{(1)}_1(t) + \delta^2 \left\{ p^{(2)}_1(t) - \left( \frac{1}{2} p^{(1)}_1(t) \right)^2 \right\} + \cdots \right]. \quad (3.1.20)$$

An expression similar to (3.1.20) can be obtained for $\tilde{x}_2(t)$. Then, we introduce the renormalized solution by replacing $x_{1,2}$ by $\tilde{x}_{1,2}(t)$ in the zero-th order solution. The renormalized solutions $u_{0,\pm}(x - \tilde{x}_{1,2})$ are expanded up to $O(\delta)$ as

$$\begin{align*}
  u_{0,+}(x - \tilde{x}_1) &= u_{0,+}(x - x_1) - \delta (\partial_x u_{0,+}) x_1 p^{(1)}_1, \\
  u_{0,-}(x - \tilde{x}_2) &= u_{0,-}(x - x_2) - \delta (\partial_x u_{0,-}) x_2 p^{(1)}_2.
\end{align*} \quad (3.1.21)$$

Thus, the renormalized interacting pulse solution up to $O(\delta)$ is

$$u(x, t) = u_{0,+}(x - \tilde{x}_1) + u_{0,-}(x - \tilde{x}_2) - 1 \sim U_+(x - x_1) + U_-(x - x_2) - 1 + \delta \left[ -\left( x_1 p^{(1)}_1(t) \right) U_+ - \left( x_2 p^{(1)}_2(t) \right) U_- \right]. \quad (3.1.22)$$
Comparing (3.1.14) with (3.1.22), we find the relations $-x_1P_1^{(1)}(t) = P_1^{(1)}(t)$ and $-x_2P_2^{(1)}(t) = P_2^{(1)}(t)$. Thus we can rewrite the renormalization transformations (3.1.15) up to $O(\delta)$ as

$$
\begin{align*}
\ddot{x}_1(t) &\sim x_1 - \delta P_1^{(1)}(t), \\
\ddot{x}_2(t) &\sim x_2 - \delta P_2^{(1)}(t).
\end{align*}
$$

(3.1.23)

Although we included the higher-order terms in the definition of the renormalization transformations (3.1.15), it is impossible to obtain their time dependences. When the renormalized interacting solution is expanded with respect to $\delta$, there appear higher-order derivatives of $U_\pm$ in the terms higher-order in $\delta$. However, we here assume that the correction $b_1(x, t)$ is expanded in terms of the zero eigenfunctions of $\mathcal{L}$. The reason for the assumption is that we would like to obtain the time-dependence of the coefficients by taking the inner product with the zero eigenfunction of $\mathcal{L}_t$. Comparing terms of equal order in $\delta$, we compare the interacting pulse solution obtained from the naive perturbation expansion with the Taylor expansion of the renormalized solution. At first order in $\delta$, as shown above, we can find the correspondence between the expansion coefficients of the renormalization transformation and the secular terms in $b_1$. For higher orders in $\delta^l$, the function $b_l(x, t)$ include only the zero eigenfunctions of $\mathcal{L}$, and for this reason, we cannot determine the correspondence for higher-order derivatives. (That is, although there appear higher-order derivatives in the Taylor expansion of the renormalized solution, there is no corresponding term in the interacting pulse solution obtained from the naive perturbation expansion.) Therefore we cannot obtain the time-dependences of the expansion coefficients of higher-order terms (that is, $O(\delta^l)$ for $l \geq 2$) in the renormalization transformation.

The functions $\tilde{x}_1(t + \tau)$ and $\tilde{x}_2(t + \tau)$ are expanded in terms of the generator $\tau \partial_t$ of the translational Lie group as

$$
\begin{align*}
\tilde{x}_1(t + \tau) &= \exp(\tau \partial_t) \tilde{x}_1(t) = \left(1 + \tau \partial_t + (\tau^2/2) \partial_t^2 + \cdots\right) \tilde{x}_1(t), \\
\tilde{x}_2(t + \tau) &= \exp(\tau \partial_t) \tilde{x}_2(t) = \left(1 + \tau \partial_t + (\tau^2/2) \partial_t^2 + \cdots\right) \tilde{x}_2(t).
\end{align*}
$$

(3.1.24)

By the definition of the renormalization transformation (3.1.18), we then obtain the asymptotic form of the generator of the translational Lie group, the RG equation, as

$$
\begin{align*}
\frac{d\tilde{x}_1}{dt} &= \partial_{\tau} \left[ G_{\tau}\left\{ \tilde{x}_1(t), T(t) \right\} \right] \bigg|_{\tau=0}, \\
\frac{d\tilde{x}_2}{dt} &= \partial_{\tau} \left[ G_{\tau}\left\{ \tilde{x}_2(t), T(t) \right\} \right] \bigg|_{\tau=0}.
\end{align*}
$$

(3.1.25)

Using (3.1.23), we obtain the dynamics of the position of pulse up to $O(\delta)$ as

$$
\begin{align*}
\frac{d\tilde{x}_1}{dt} &= -\delta \frac{dP_1^{(1)}(t)}{dt} = -\frac{\langle V_{+,g_+}(U_+U_-) \rangle}{\langle V_+, U_+, x \rangle}, \\
\frac{d\tilde{x}_2}{dt} &= -\delta \frac{dP_2^{(1)}(t)}{dt} = -\frac{\langle V_{-,g_-}(U_+U_-) \rangle}{\langle V_-, U_-, x \rangle}.
\end{align*}
$$

(3.1.26)
Now, let us derive the explicit form of pulse dynamics for the case \( |x_2 - x_1| k_- \gg 1\), which corresponds to the condition \( |x_2 - x_1| \gg \beta \). First, we focus on the temporal dependence of \( \tilde{x}_1(t) \). For later convenience, we make the definition
\[
\int_{-\infty}^{\infty} \left[ \frac{1}{\cosh x} \right]^p \exp(qx) dx \equiv E[p, q] > 0,
\]
where \( p \) and \( q \) are real, and we here treat pairs \((p, q)\) for which the integral converges.

Using the above definition, the denominator of (3.1.26) is
\[
\langle V_+, U_{+, x} \rangle = E[l_+ + 2, 0].
\]

Also, from (3.1.7), we see that the numerator of (3.1.26) is composed of two terms. The contribution from the first term becomes
\[
\langle V_+ - \frac{3}{2}(U_+ - 1)(U_- - 1)(U_+ + U_-) \rangle
\]
\[
= \left( -\frac{3}{2} \right) \int_{-\infty}^{\infty} \left[ \frac{1}{\cosh k_+ x} \right]^{l_+} \left[ \tanh k_- (x - L) - 1 \right]
\]
\[
\times \left[ \tanh k_+ x - 1 \right] \left[ \tanh k_+ x + \tanh k_- (x - L) \right] dx,
\]
where \((\tilde{x}_2 - \tilde{x}_1) = L\). We split the region of integral into two at \( x = L \). Now we note that \( \frac{1}{\cosh k_+ x} \) has a finite value near \( x = 0 \). However, it rapidly decreases with distance from there. Then we expand \( \tanh k_- (x - L) \) using the following formula for \( x > 0 \):
\[
\tanh x = 1 - 2 \sum_{n=1}^{\infty} (-1)^{n+1} e^{-2nx}.
\]

With this form, (3.1.29) becomes
\[
\sim \left( -\frac{3}{2} \right) \int_{-\infty}^{L} \left[ \frac{1}{\cosh k_+ x} \right]^{l_+} \left[ (-2) \sum_{n=1}^{\infty} (-1)^{n+1} e^{-2nk_- (x-L)} \right]
\]
\[
\times \left[ \tanh k_+ x - 1 \right] \left[ \tanh k_+ x + 1 \right] dx
\]
\[
- \left( \frac{3}{2} \right) \int_{L}^{\infty} \left[ \frac{1}{\cosh k_+ x} \right]^{l_+} \left[ (-2) \right] \left[ \tanh k_+ x - 1 \right] dx
\]
\[
\sim \left( -\frac{3}{2} \right) \int_{-\infty}^{\infty} (2) e^{2k_- L} \left[ \frac{1}{\cosh k_+ x} \right]^{l_+ + 2} e^{-2k_- x} dx
\]
\[
\equiv \left( -\frac{3}{k_+} \right) e^{2k_- L} E(l_+ + 2, -2 \left( \frac{k_-}{k_+} \right)),
\]

where the summation over \( n \) has been truncated at \( n = 1 \), and the upper limit of the integral has been extended to \( \infty \). The second term in the first line of (3.1.31) has been ignored, because the integrand is small in the region of integration.

In a similar way, the contribution from the second term of (3.1.7) becomes
\[
\langle V_+, 2\beta(U_- - 1) \partial_x (U_+ - 1) \rangle
\]
\[
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\]

\[
\int_{-\infty}^{\infty} \left[ \frac{1}{\cosh k_+ x} \right]^{l+2} 2\beta \left[ \tanh k_-(x-L) - 1 \right] \partial_x \left[ \tanh k_+ x - 1 \right] dx
\]

\[
\int_{-\infty}^{\infty} \left[ \frac{1}{\cosh k_+ x} \right]^{l+2} 2\beta k_+ \left[ \tanh k_-(x-L) - 1 \right] dx
\]

\[
\sim \int_{-\infty}^{\infty} \left[ \frac{1}{\cosh k_+ x} \right]^{l+2} 2\beta k_+ \left[ -2e^{2k_-} e^{-2k_+} \right] dx
\]

\[
\equiv (-4\beta)e^{2k_-} L E \left( l_+ + 2, -2 \left( \frac{k_-}{k_+} \right) \right). \tag{3.1.32}
\]

Finally, we obtain the time-dependence of \( \tilde{x}_1(t) \) as

\[
\frac{d\tilde{x}_1}{dt} = -\frac{\langle V_+, g_+(U_+, U_-) \rangle}{\langle V_+, U_+, x \rangle}
\]

\[
= \left( \frac{3}{k_+} + 4\beta \right) e^{2k_-} \frac{E \left( l_+ + 2, -2 \left( \frac{k_-}{k_+} \right) \right)}{E \left( l_+ + 2, 0 \right)}
\]

\[
= -2\beta + 6\sqrt{\beta^2 + \epsilon^2} e^{2k_- (\tilde{x}_2 - \tilde{x}_1)} \frac{E \left( l_+ + 2, -2 \left( \frac{k_-}{k_+} \right) \right)}{E \left( l_+ + 2, 0 \right)} > 0. \tag{3.1.33}
\]

In a similar way, the time dependence of \( \tilde{x}_2(t) \) is obtained as

\[
\frac{d\tilde{x}_2}{dt} = -\frac{\langle V_-, g_-(U_+, U_-) \rangle}{\langle V_-, U_-, x \rangle}
\]

\[
= \left( \frac{3}{k_-} + 4\beta \right) e^{-2k_+} \frac{E \left( l_- + 2, -2 \left( \frac{k_+}{k_-} \right) \right)}{E \left( l_- + 2, 0 \right)}
\]

\[
= -2\beta + 6\sqrt{\beta^2 + \epsilon^2} e^{-2k_+ (\tilde{x}_2 - \tilde{x}_1)} \frac{E \left( l_- + 2, -2 \left( \frac{k_+}{k_-} \right) \right)}{E \left( l_- + 2, 0 \right)} < 0. \tag{3.1.34}
\]

The above results suggest that there exists an attractive force between two asymmetric kinks. The symmetry breaking term \( \beta(\partial_x u^2) \) in (3.1-2) causes different time-dependences of the positions of the two kinks. In order to estimate (3.1-34), it is necessary to carry out numerical calculations. For the special case \( \beta = 0 \), (3.1-2) becomes the symmetric TDGL equation. In this case, the parameters in the above become \( k_\pm = \pm \left( \frac{1}{2\epsilon} \right), \ l_\pm = 2, \ E(4,0) = \frac{4}{3} \) and \( E(4,2) = \frac{8}{3} \). Using these values, we find that (3.1-33) and (3.1-34) become, respectively,

\[
\begin{cases}
\frac{d\tilde{x}_1}{dt} = 12\epsilon \exp \left[ -(\tilde{x}_2 - \tilde{x}_1)/\epsilon \right], \\
\frac{d\tilde{x}_2}{dt} = -12\epsilon \exp \left[ -(\tilde{x}_2 - \tilde{x}_1)/\epsilon \right].
\end{cases} \tag{3.1.35}
\]
This result is consistent with results obtained previously.\(^{18},^{20}\) Thus we have derived the dynamics of an asymmetric kink pair using the RG method based on the Lie group. Note that the solution in the symmetric kink case is recovered in the limit \(\beta \to 0\). We have also found that there appears an attractive force similar to that in the case of a symmetric kink pair.

### 3.2. Interaction of solitons in the KdV equation

In the previous subsection, we treated a kind of TDGL equation, which has semi-simple zero eigenvalues. In order to demonstrate the present method for a system that has a two-dimensional Jordan cell, we consider the pulse interaction in the KdV equation,

\[
\partial_t u + 6u\partial_x u + \partial_x^3 u = 0. \tag{3.2.1}
\]

This equation has a stationary pulse solution, a so-called soliton, taking the form of \(V(c, z) = \frac{\delta}{2} \text{sech}^2\left(\frac{\sqrt{\delta}}{2} z\right)\), where \(z\) is a moving coordinate that travels with speed \(c: z = x - ct\). It is important to note that \(c\) in the soliton solution is an arbitrary constant. In the moving frame represented by this coordinate, (3.2.1) becomes

\[
\partial_t u - c\partial_z u + 6u\partial_x u + \partial_x^3 u = 0. \tag{3.2.2}
\]

We consider the interaction between two pulses, whose positions are at \(z = z_1\) and \(z = z_2\), where \(z_1 < z_2\). The KdV equation possesses a translational symmetry, and thus the position of a single pulse is an arbitrary constant. We consider the case in which the distance between two pulses is sufficiently large compared with the width of a pulse, that is, the case \(|z_2 - z_1| \gg (1/\sqrt{\delta})\). In such a situation, the solution for two interacting pulses is written \(u(z, t) = V(c, z - z_1) + V(c, z, z - z_2) + B(z, t) \equiv V_1 + V_2 + B\). Here, \(B\) is a correction expanded as \(B(x, t) = \sum_{l=1}^{\infty} \delta^l b_l(z, t)\), where \(\delta\) is a small parameter, \(O(\delta) \sim \exp[-\sqrt{\delta}(z_2 - z_1)]\). We here treat the first-order correction, writing \(u(z, t)\) as

\[
u(z, t) = V_1 + V_2 + B = V_1 + V_2 + \sum_{l=1}^{\infty} \delta^l b_l(z, t) \sim V_1 + V_2 + \delta b_1(z, t). \tag{3.2.3}
\]

Substituting this form of \(u(z, t)\) into (3.2.2), we obtain the time evolution of \(b_1\) as

\[
\delta \partial_t b_1 = \delta M b_1 + g, \tag{3.2.4}
\]

where \(M = c\partial_z - 6[V_1\partial_z + V_2\partial_z + \partial_z(V_1 + V_2)] - \partial_x^3\) and \(g = -6\partial_z(V_1V_2)\). As we consider the situation that the pulses are well localized, we can simplify \(M\). First, we consider region near \(z = z_1\). Because \(V_2 = 0\) near \(z = z_1\), we have \(M \sim c\partial_z - 6V_1\partial_z - 6(\partial_z V_1) - \partial_x^3\). For this \(M\), it is easily verified that \(M(\partial_z V_1) = 0\). Contrastingly, we have \(M(\partial_c V_1) = - (\partial_z V_1)\) and \(M^2(\partial_c V_1) = 0\). These suggest that the system possesses a two-dimensional Jordan cell, and that \((\partial_z V_1)\) and \((\partial_c V_1)\) are zero eigenfunctions of \(M\). The situation is similar near \(z = z_2\). We here assume
that the correction term $b_1$ can be expanded in terms of these zero eigen functions as $b_1(z,t) = P_1^{(1)}(t)(\partial_z V_1) + P_2^{(1)}(t)(\partial_z V_2) + Q_1^{(1)}(t)(\partial_z V_1) + Q_2^{(1)}(t)(\partial_z V_2) + \bar{b}_1(z)$. Then, substituting $b_1(z,t)$ into (3.2.4), we obtain

$$
\delta \partial_t b = \delta \left[ P_{1,t}^{(1)}(\partial_z V_1) + P_{2,t}^{(1)}(\partial_z V_2) + Q_{1,t}^{(1)}(\partial_z V_1) + Q_{2,t}^{(1)}(\partial_z V_2) \right]
= \delta \left[ -Q_1^{(1)}(\partial_z V_1) - Q_2^{(1)}(\partial_z V_2) + M\bar{b}_1 \right] + g. \tag{3.2.5}
$$

In the situation we consider, the overlap of two pulses is small so that we can make use of the relations

$$
\int (\partial_z V_1)(\partial_z V_2) dz \sim 0,
\int (\partial_z V_2)(\partial_z V_1) dz \sim 0. \tag{3.2.6}
$$

The operator adjoint to $\mathcal{M}$ is defined by $\mathcal{M}^\dagger = -c\partial_z + 6[V_1\partial_z + V_2\partial_z] + \partial_z^3$. We note that we have the approximate relations $\mathcal{M}^\dagger V_1 \sim 0$ and $\mathcal{M}^\dagger V_2 \sim 0$ near $z = z_1$ and $z = z_2$, respectively. Taking the inner product of (3.2.5) with the zero eigenfunction of $\mathcal{M}^\dagger$ near $z = z_1$ and $z_2$, we obtain the time evolution of $Q_1$ and $Q_2$ as

$$
\begin{cases}
\delta Q_{1,t}^{(1)} \langle V_1, (\partial_z V_1) \rangle = \langle V_1, g \rangle, \\
\delta Q_{2,t}^{(1)} \langle V_2, (\partial_z V_2) \rangle = \langle V_2, g \rangle. \tag{3.2.7}
\end{cases}
$$

From the above, we see that $Q_{1,t}^{(1)}$ and $Q_{2,t}^{(1)}$ are constant, and hence $Q_1^{(1)}(t)$ and $Q_2^{(1)}(t)$ are polynomials of degree one. On the other hand, we note that (3.2.5) becomes

$$
\delta[P_{1,t}^{(1)} + Q_1^{(1)}](\partial_z V_1) + \delta[P_{2,t}^{(1)} + Q_2^{(1)}](\partial_z V_2) + \delta Q_{1,t}^{(1)}(\partial_z V_1) + \delta Q_{2,t}^{(1)}(\partial_z V_2)
\quad = g + \delta \mathcal{M}\bar{b}_1. \tag{3.2.8}
$$

Then, it is necessary that

$$
\begin{cases}
P_{1,t}^{(1)} + Q_1^{(1)} = \text{constant}, \\
P_{2,t}^{(1)} + Q_2^{(1)} = \text{constant}. \tag{3.2.9}
\end{cases}
$$

This implies that $P_1^{(1)}(t)$ and $P_2^{(1)}(t)$ are polynomials of degree two, and from (3.2.7) we find

$$
\begin{cases}
P_{1,t}^{(1)} = -Q_{1,t}^{(1)} = -\frac{\langle V_1, g \rangle}{\delta \langle V_1, (\partial_z V_1) \rangle}, \\
P_{2,t}^{(1)} = -Q_{2,t}^{(1)} = -\frac{\langle V_2, g \rangle}{\delta \langle V_2, (\partial_z V_2) \rangle}. \tag{3.2.10}
\end{cases}
$$

Thus we obtain the solution describing two interacting pulses from the naive perturbation expansion up to $O(\delta)$ as

$$
u(z,t) \sim V_1 + V_2 + \delta b_1(z,t)
\quad = V_1 + V_2
\quad + \delta \left[ P_{1}^{(1)}(t)(\partial_z V_1) + P_{2}^{(1)}(t)(\partial_z V_2) + Q_{1}^{(1)}(t)(\partial_z V_1) + Q_{2}^{(1)}(t)(\partial_z V_2) \right] + \delta \bar{b}_1(z). \tag{3.2.11}
$$
Because $b_1(z,t)$ includes secular terms, we renormalize arbitrary constants in order to avoid the divergence. The KdV equation (3.2.2) possesses translational symmetry with respect to spatial coordinate, and the traveling speed $c$ of the soliton solution is also an arbitrary constant. As in the case of the TDGL equation, we introduce the following renormalization transformation:

$$
\begin{align*}
\tilde{z}_1(t) &= R(z_1,t) = z_1 \left[ 1 + \delta p_1^{(1)}(t) + \delta^2 p_1^{(2)}(t) + \cdots \right], \\
\tilde{z}_2(t) &= R(z_2,t) = z_2 \left[ 1 + \delta p_2^{(1)}(t) + \delta^2 p_2^{(2)}(t) + \cdots \right], \\
\tilde{c}(t) &= R(c,t) = c \left[ 1 + \delta q^{(1)}(t) + \delta^2 q^{(2)}(t) + \cdots \right].
\end{align*}
$$

Replacing the arbitrary constants in the zero-th order solution by these renormalized solutions is also an arbitrary constant. As in the case of the TDGL equation, we find the relations

$$
\begin{align*}
\tilde{z}_1(t) &= z_1 - \delta P_1^{(1)}(t), \\
\tilde{z}_2(t) &= z_2 - \delta P_2^{(1)}(t), \\
\tilde{c}(t) &= c + \delta Q_1^{(1)}(t).
\end{align*}
$$

Thus we have obtained asymptotic forms of the generators of the Lie group as

$$
\begin{align*}
\frac{d^2 \tilde{z}_1}{dt^2} &= \partial_{\tau}^2 G_\tau \left\{ \tilde{z}_1(t), T(t) \right\} \bigg|_{\tau=0} = -\delta \frac{d^2 P_1^{(1)}}{dt^2} = \left\langle V_1, g \right\rangle \left\langle \frac{\partial V_1}{\partial c} \right\rangle, \\
\frac{d^2 \tilde{z}_2}{dt^2} &= \partial_{\tau}^2 G_\tau \left\{ \tilde{z}_2(t), T(t) \right\} \bigg|_{\tau=0} = -\delta \frac{d^2 P_2^{(1)}}{dt^2} = \left\langle V_2, g \right\rangle \left\langle \frac{\partial V_2}{\partial c} \right\rangle, \\
\frac{dc}{dt} &= \partial_{\tau} G_\tau \left\{ \tilde{c}(t), T(t) \right\} \bigg|_{\tau=0} = \delta \frac{dQ_1^{(1)}}{dt} = \left\langle V_1, g \right\rangle \left\langle \frac{\partial V_1}{\partial c} \right\rangle = \delta \frac{dQ_2^{(1)}}{dt} = \left\langle V_2, g \right\rangle \left\langle \frac{\partial V_2}{\partial c} \right\rangle.
\end{align*}
$$

Comparing (3.2.11) with (3.2.14), we find the relations $P_1^{(1)} = -z_1 p_1^{(1)}$, $P_2^{(1)} = -z_2 p_2^{(1)}$ and $Q_1^{(1)} = Q_2^{(1)} = c q^{(1)}$. Thus the renormalization transformation (3.2.12) can be rewritten up to $O(\delta)$ as

$$
\begin{align*}
\tilde{z}_1(t) &\sim z_1 - \delta P_1^{(1)}(t), \\
\tilde{z}_2(t) &\sim z_2 - \delta P_2^{(1)}(t), \\
\tilde{c}(t) &\sim c + \delta Q_1^{(1)}(t) = c + \delta Q_2^{(1)}(t).
\end{align*}
$$

Thus we have obtained asymptotic forms of the generators of the Lie group as

$$
\begin{align*}
\frac{d^2 \tilde{z}_1}{dt^2} &= \partial_{\tau}^2 G_\tau \left\{ \tilde{z}_1(t), T(t) \right\} \bigg|_{\tau=0} = -\delta \frac{d^2 P_1^{(1)}}{dt^2} = \left\langle V_1, g \right\rangle \left\langle \frac{\partial V_1}{\partial c} \right\rangle, \\
\frac{d^2 \tilde{z}_2}{dt^2} &= \partial_{\tau}^2 G_\tau \left\{ \tilde{z}_2(t), T(t) \right\} \bigg|_{\tau=0} = -\delta \frac{d^2 P_2^{(1)}}{dt^2} = \left\langle V_2, g \right\rangle \left\langle \frac{\partial V_2}{\partial c} \right\rangle, \\
\frac{dc}{dt} &= \partial_{\tau} G_\tau \left\{ \tilde{c}(t), T(t) \right\} \bigg|_{\tau=0} = \delta \frac{dQ_1^{(1)}}{dt} = \left\langle V_1, g \right\rangle \left\langle \frac{\partial V_1}{\partial c} \right\rangle = \delta \frac{dQ_2^{(1)}}{dt} = \left\langle V_2, g \right\rangle \left\langle \frac{\partial V_2}{\partial c} \right\rangle.
\end{align*}
$$
Following Ref. 18), we can evaluate the above formulas and verify that there exists a repulsive force between two solitons. We here have demonstrated the application of the present method to a system that has doubly degenerate zero eigenfunctions.

3.3. Interaction of solitons in the KdV equation with a perturbation

We here consider the interaction of pulses in the KdV equation with a perturbation. When the dissipative terms in the form of second- and forth-order derivatives are regarded as the perturbation, the equation is called the Benney equation. It is known that when there is a pulse solution in the Benney equation, the asymptotic traveling speed \( c^* \) is uniquely selected by the perturbation. We first derive the pulse dynamics in the case of a general type of perturbation term. Let us consider the equation

\[
\partial_t u + 6u\partial_x u + \partial_x^3 u = \epsilon g(u), \tag{3.3.1}
\]

where \( g(u) \) is the perturbation term, and \( \epsilon \) is a positive small constant. We now treat the case in which there is a single pulse in the system. We introduce the moving coordinate \( z = x - ct \), and the position of the pulse is \( z_1 \). The solution can be expanded with respect to \( \epsilon \) as

\[
u(z,t) = u_0(c,z - z_1) + \sum_{l=1}^{\infty} \epsilon^l u_l(z,t). \tag{3.3.2}
\]

Substituting the above form of \( u(z,t) \) into (3.3.1), and taking \( O(\epsilon) \) terms, we obtain the time evolution equation for \( u_1 \) as

\[
\partial_t u_1 + M u_1 = g(u_0), \tag{3.3.3}
\]

where \( M = -c\partial_z + 6[u_0\partial_z + (\partial_z u_0)] + \partial_x^3 \). It is easy to verify that \( M(\partial_z u_0) = 0 \), \( M(\partial_c u_0) = (\partial_z u_0) \) and \( M^2(\partial_c u_0) = 0 \). Next, we expand \( u_1 \) in terms of these zero eigenfunctions as

\[
u_1(z,t) = P^{(1)}_1(t)(\partial_z u_0) + Q^{(1)}_1(t)(\partial_c u_0) + \tilde{u}_1(z). \tag{3.3.4}
\]

In the above, \( [P^{(1)}_1(t) + Q^{(1)}_1(t)] \) and \( Q^{(1)}_1(t) \) should be time-independent. The operator adjoint to \( M \) is \( M^\dagger = c\partial_z - 6u_0\partial_z - \partial_x^3 \), and it is easily verified that \( M^\dagger u_0 = 0 \). Then, taking the inner product of (3.3.4) with the zero eigenfunction of \( M^\dagger \), we obtain

\[
\begin{cases}
P^{(1)}_{1,t} + Q^{(1)}_1(t) = \frac{\langle u_0,g(u_0) \rangle}{\langle u_0,\partial_c u_0 \rangle}, \\
P^{(1)}_{1,tt} + Q^{(1)}_1(t) = 0. \tag{3.3.5}
\end{cases}
\]

From the above, we see that \( P^{(1)}_1(t) \) and \( Q^{(1)}_1(t) \) are polynomials of degree two and one, respectively. Thus, the solution obtained through the naive perturbation
variables, we obtain the renormalized solution. The renormalized solution replacing the arbitrary constants in the zero-th order solution by the renormalized

Comparing (3.3.8) with (3.3.6), we find the relations

Repeating the arbitrary constants in the zero-th order solution by the renormalized variables, we obtain the renormalized solution. The renormalized solution $u(\tilde{c}(t), z - \tilde{z}_1(t))$ is expanded up to $O(\epsilon)$ as

Comparing (3.3.8) with (3.3.6), we find the relations

Thus, the renormalization transformations (3.3.7) are rewritten up to $O(\epsilon)$ as

The time evolution of the traveling speed is given by

This implies that the traveling speed changes gradually in $O(\epsilon)$. For the Benney equation, in which $g(u) = -(\partial_z^2 + \partial_z^4)u$, the stationary traveling speed $c^*$ is determined by the condition $\langle u_0, g(u_0) \rangle = \langle u_0, (\partial_z^2 + \partial_z^4)u_0 \rangle = 0$. Thus the stationary traveling speed is determined uniquely by the dissipative perturbation.\(^{18}\)

Now, we consider the interaction between two pulses in the Benney equation,

For the stationary state of a single pulse, the traveling speed is uniquely determined as $c^*$. However, for the two pulse state, the traveling speed is modified by interaction.
We consider the situation in which two pulses have the same traveling speed, $c$, which converges to $c^*$ in the limit $\epsilon \to 0$. The positions of the pulses are arbitrary constants and assumed to be $z = z_1$ and $z_2$, where $z_1 < z_2$.

We write the solution $u(z, t)$ representing two interacting pulses as $u(z, t) = V(z - z_1) + V(z - z_2) + B(z, t) \equiv V_1 + V_2 + B \equiv V(z) + B$, where $B$ is a correction and $V(z)$ is a stationary solution of the Benney equation satisfying

$$-c\partial_z V + 6V \partial_z V + \partial_z^3 V = -\epsilon(\partial_z^2 + \partial_z^4)V.$$  \hspace{1cm} (3.3.13)

We expand $B(z, t)$ as $B(z, t) = \sum_{i=1}^{\infty} \delta^i b_i(z, t)$, where $\delta$ is a small parameter, $O(\delta) \sim \exp[-\sqrt{c}(z_2 - z_1)]$. Then, substituting $u(z, t)$ into (3.3.1) and considering $O(\delta)$ terms, we obtain the evolution equation for $b_1$ as

$$\delta \partial_t b_1 + \delta \mathcal{L} b_1 = H,$$  \hspace{1cm} (3.3.14)

where $\mathcal{L} = -c\partial_z + 6[\partial_z V + \bar{V} \partial_z] + \partial_z^3 + \epsilon(\partial_z^2 + \partial_z^4)$ and $H = [-6\partial_z(V_1 V_2)]$. As we assume that the distance between two pulses is large compared with the width of the pulse, we can assume that $\mathcal{L} \sim \mathcal{L}_i = -c\partial_z + 6[\partial_z V_i + V_i \partial_z] + \partial_z^3 + \epsilon(\partial_z^2 + \partial_z^4)$ near $z = z_i$, where $i = 1$ and 2. For these approximate linear operators, we find zero eigenfunctions satisfying $\mathcal{L}_i(\partial_z V_i) = 0$. The operator adjoint to $\mathcal{L}$ is $\mathcal{L}_i^\dagger = c\partial_z - 6\bar{V}\partial_z - \partial_z^3 + \epsilon(\partial_z^2 + \partial_z^4)$. Using analysis similar to that given above, we obtain the analogous operators approximating $\mathcal{L}$ near $z = z_i$ to be $\mathcal{L}_i^\dagger = c\partial_z - 6V_i \partial_z - \partial_z^3 + \epsilon(\partial_z^2 + \partial_z^4)$. We write their zero eigenfunctions as $W_i$, which satisfy $\mathcal{L}_i^\dagger W_i = 0$.

Using these zero eigenfunctions of $\mathcal{L}$, we expand the correction $b_1$ as

$$b_1(z, t) = P_1^{(1)}(t)(\partial_z V_1) + P_2^{(1)}(t)(\partial_z V_2) + \bar{b}_1(z).$$  \hspace{1cm} (3.3.15)

Substituting (3.3.15) into (3.3.14), we obtain

$$\delta \left[ P_1^{(1)}(t)(\partial_z V_1) + P_2^{(1)}(t)(\partial_z V_2) \right] + \delta \mathcal{L} \bar{b} = H.$$  \hspace{1cm} (3.3.16)

Here, it is necessary that $P_1^{(1)}$ and $P_2^{(1)}$ be constant. Then, $P_i^{(1)}(t)$ is a polynomial of degree one, and the correction $u_1(z, t)$ includes secular terms. Taking the inner product of (3.3.16) with the zero eigenfunction of $\mathcal{L}_i^\dagger$, we obtain

$$\begin{cases}
    P_1^{(1)} \frac{\langle W_1, H \rangle}{\delta\langle W_1, \partial_z V_1 \rangle}, \\
    P_2^{(1)} \frac{\langle W_2, H \rangle}{\delta\langle W_2, \partial_z V_2 \rangle}.
\end{cases} \hspace{1cm} (3.3.17)
$$

The solution representing two interacting pulses obtained from the naive perturbation expansion up to $O(\delta)$ is

$$u(z, t) \sim V(z - z_1) + V(z - z_2) + \delta b_1(z, t)$$ $$= V_1 + V_2$$ $$+ \delta \left[ P_1^{(1)}(t)(\partial_z V_1) + P_2^{(1)}(t)(\partial_z V_2) + \bar{b}_1(z) \right].$$  \hspace{1cm} (3.3.18)
We remark that the traveling speed $c$ is not an arbitrary constant in this system. In order to avoid divergence, the arbitrary constants $z_1$ and $z_2$ are renormalized. We introduce the renormalization transformation as

$$\begin{cases}
\tilde{z}_1(t) = z_1 \left[ 1 + \delta_{1}^{(1)} + \delta^2 \delta_{1}^{(2)} + \cdots \right], \\
\tilde{z}_2(t) = z_2 \left[ 1 + \delta_{2}^{(1)} + \delta^2 \delta_{2}^{(2)} + \cdots \right].
\end{cases} \quad (3.3.19)$$

Then, replacing $z_{1,2}$ by $\tilde{z}_{1,2}(t)$ in the zero-th order solution, we obtain the renormalized solution. The renormalized interacting solution is expanded up to $O(\delta)$ as

$$u(z, t) = V \left( z - \tilde{z}_1(t) \right) + V \left( z - \tilde{z}_2(t) \right) \sim V_1 + V_2 + \delta \left[ -z_1 p_{1}^{(1)} (\partial_z V_1) - z_2 p_{2}^{(1)} (\partial_z V_2) \right]. \quad (3.3.20)$$

Comparing (3.3.18) with (3.3.20), we find the relations $z_1 p_{1}^{(1)} = -P_{1}^{(1)}$ and $z_2 p_{2}^{(1)} = -P_{2}^{(1)}$. Thus the renormalization transformation (3.3.19) is rewritten up to $O(\delta)$ as

$$\begin{cases}
\tilde{z}_1(t) = z_1 - \delta P_{1}^{(1)}(t), \\
\tilde{z}_2(t) = z_2 - \delta P_{2}^{(1)}(t).
\end{cases} \quad (3.3.21)$$

Using analysis similar to that given in the previous subsection, we obtain the dynamics of pulse positions from the asymptotic forms of the generators of the Lie group as

$$\begin{align*}
\frac{d\tilde{z}_1}{dt} = \partial_\tau \left[ G_\tau \left\{ \tilde{z}_1(t), T(t) \right\} \right]_{\tau = 0} = -\delta \frac{dP_1}{dt} = -\frac{\langle W_1, H \rangle}{\langle W_1, \partial_\tau V_1 \rangle}, \\
\frac{d\tilde{z}_2}{dt} = \partial_\tau \left[ G_\tau \left\{ \tilde{z}_2(t), T(t) \right\} \right]_{\tau = 0} = -\delta \frac{dP_2}{dt} = -\frac{\langle W_2, H \rangle}{\langle W_2, \partial_\tau V_2 \rangle}. \quad (3.3.22)
\end{align*}$$

Generally, we cannot obtain the explicit forms of the zero eigenfunctions of $\mathcal{L}$ and $\mathcal{L}^\dagger$. However, when $\epsilon = 0$, (3.3.1) becomes the KdV equation, and therefore we know the explicit forms of these eigenfunctions, and $c$ is $c^*$. In such a limiting case, we define $\mathcal{M} = -c^* \partial_z + 6(\partial_z \tilde{V}(0)) + 6\tilde{V}(0) \partial_z + \partial^3_z$ and its adjoint operator $\mathcal{M}^\dagger = c^* \partial_z - 6\tilde{V}(0) \partial_z - \partial^3_z$, where $\tilde{V}(0) = (V_1^{(0)} + V_2^{(0)})$. Here, we choose $V_i^{(0)} = \left[ \frac{c^*}{2} \tanh^2(\sqrt{c^*}(z - z_i)) \right]$ for $i = 1$ and 2. These satisfy the relations $[-c^* \partial_z + 6\tilde{V}_i(0) \partial_z + \partial^3_z] V_i^{(0)} = 0$. Employing arguments similar to those used in the case of $\mathcal{L}$ and $\mathcal{L}^\dagger$, we can demonstrate the validity of the approximation $\mathcal{M} \sim \mathcal{M}_i \equiv -c^* \partial_z + 6(\partial_z \hat{V}_i(0)) + 6\hat{V}_i(0) \partial_z + \partial^3_z$, and similarly for its adjoint operator $\mathcal{M}^\dagger \sim \mathcal{M}_i^\dagger$ near $z = z_i$ for $i = 1$ and 2. For these operators, we can verify that $\mathcal{M}_i(\partial_z \hat{V}_i(0)) \sim 0$, $\mathcal{M}_i(\partial_z \hat{V}_i(0)) \sim (\partial_z \hat{V}_i(0))$ and $\mathcal{M}_i^\dagger \hat{V}_i(0) \sim 0$ near $z = z_i$. For later convenience, we write these zero eigenfunctions of $\mathcal{M}_i^\dagger$ as $W_i^{(0)}$.

In order to approximate the denominator and numerator of (3.3.22), we expand
\[V_1, V_2, W_1, W_2 \text{ and } c \text{ about } \epsilon = 0 \text{ up to } O(\epsilon) \text{ as}
\]
\[
\begin{align*}
V_1 &= V(z - \tilde{z}_1) = V_1^{(0)} + \epsilon V_1^{(1)}, \\
V_2 &= V(z - \tilde{z}_2) = V_2^{(0)} + \epsilon V_2^{(1)}, \\
W_1 &= W(z - \tilde{z}_1) = W_1^{(0)} + \epsilon W_1^{(1)}, \\
W_2 &= W(z - \tilde{z}_2) = W_2^{(0)} + \epsilon W_2^{(1)}, \\
c &= c^* + \epsilon c^{(1)}. \\
\end{align*}
\]

We first derive the time-dependence of \(\tilde{z}_1\). Substituting the above into the numerator of (3.3.22), we obtain, up to \(O(1)\),
\[
\begin{align*}
\langle W_1, H \rangle &= \left\langle (W_1^{(0)}, -6\partial_z(V_1^{(0)}V_2^{(0)})) \rightangle \\
&= \left\langle (V_1^{(0)}, -6\partial_z(V_1^{(0)}V_2^{(0)})) \rightangle \\
&\sim -6\partial_z V_2^{(0)}|_{z=\tilde{z}_1} \int_{-\infty}^{\infty}(V_1^{(0)})^2 dz \\
&\sim -C \exp \left[{-\sqrt{c^*(\tilde{z}_2 - \tilde{z}_1)}}\right],
\end{align*}
\]
where \(C\) is a positive constant. In addition, we approximate the denominator of (3.3.22) up to \(O(\epsilon)\) as
\[
\begin{align*}
\langle W_1, \partial_z V_1 \rangle &= \int_{-\infty}^{\infty} \left[ W_1^{(0)} + \epsilon W_1^{(1)} \right] \left[ \partial_z(V_1^{(0)} + \epsilon V_1^{(1)}) \right] dz \\
&\sim \epsilon \int_{-\infty}^{\infty} \left[ V_1^{(0)}(\partial_z V_1^{(1)}) + W_1^{(1)}(\partial_z V_1^{(0)}) \right] dz.
\end{align*}
\]
In order to carry out the integral in (3.3.25), we use the following relations:
\[
\begin{align*}
\langle \partial_z V_1^{(0)}, W_1^{(1)} \rangle &= \langle M\partial_c V_1^{(0)}, W_1^{(1)} \rangle = \langle \partial_c V_1^{(0)}, M^\dagger W_1^{(1)} \rangle, \\
M^\dagger W_1^{(1)} &= M^\dagger V_1^{(1)} - 2(\partial_z^2 + \partial_z^4)V_1^{(0)}, \\
\langle \partial_c V_1^{(0)}, M^\dagger V_1^{(1)} \rangle &= \langle M\partial_c V_1^{(0)}, V_1^{(1)} \rangle \\
&= \langle \partial_z V_1^{(0)}, V_1^{(1)} \rangle = -\langle V_1^{(0)}, \partial_z V_1^{(1)} \rangle.
\end{align*}
\]
Here, we can directly verify the relation \(\partial_c V_1^{(0)} = \frac{V_1^{(0)}}{c} + \left(\frac{\dot{c}}{2c}\right)\partial_z V_1^{(0)}\) for \(V_1^{(0)}(z) = \frac{c}{2}\sech^2(\sqrt{\frac{c}{2}}z)\). Then, using (3.3.26) and the above, we obtain, up to \(O(\epsilon)\),
\[
\begin{align*}
\langle W_1, \partial_z V_1 \rangle &\sim (-2\epsilon) \int_{-\infty}^{\infty} \left( \partial_c V_1^{(0)} \right) \left[ (\partial_z^2 + \partial_z^4)V_1^{(0)} \right] dz \\
&= \left(\frac{-\epsilon}{c^*}\right) \int_{-\infty}^{\infty} (\partial_z V_1^{(0)})^2 dz < 0.
\end{align*}
\]
Thus, we obtain the time-dependence of $\tilde{z}_1$ to be

$$\frac{d\tilde{z}_1}{dt} = -\frac{\langle W_1, H \rangle}{\langle W_1, \partial_z V_1 \rangle} \sim -\frac{C}{\epsilon} \exp\left[-\sqrt{c^*}(\tilde{z}_2 - \tilde{z}_1)\right]. \quad (3.3.28)$$

The evolution equation for $\tilde{z}_2$ is obtained in a similar way, the dynamics of $\tilde{z}_2$ is minus sign of (3.3.28). Thus, the interaction between two pulses in the Benney equation is repulsive.$^{18}$ Here, we have derived this result using the RG method based on the Lie group.

3.4. Interaction of traveling pulses in a reaction diffusion system

We here treat the interaction of pulses in a two-component reaction diffusion system. We denote an activator and an inhibitor by $u$ and $v$, respectively. The reaction terms for $u$ and $v$ are denoted by $f$ and $g$, respectively. For the sake of mathematical tractability, we choose a piecewise linear function for $f(u, v)$. Then, the time evolution equations are

$$\begin{cases}
\epsilon \sigma \frac{\partial u}{\partial t} = \epsilon^2 \frac{\partial^2 u}{\partial x^2} + f(u, v), \\
\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} + g(u, v),
\end{cases} \quad (3.4.1)$$

where $f(u, v) = H(u - a) - u - v$ and $g(u, v) = \mu u - v$. Here, $a$ is a parameter in the range $0 < a < 1$, and $\mu$, $\sigma$, $\epsilon$ and $D$ are positive constants. $H(x)$ is the Heaviside step function satisfying $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. In the case $\epsilon^2 \gg D$, the above system has a traveling pulse solution, whereas in the opposite limit, $\epsilon^2 \ll D$, a motionless pulse is stable for large $\sigma$. When $\sigma$ is decreased, this pulse becomes destabilized through an oscillatory bifurcation, and the interface begins to oscillate with a breathing-like motion. However, under the special condition $\int (u + v) dx = \text{constant}$, the motionless pulse bifurcates to a traveling pulse solution as $\sigma$ decreases.$^{21,22}$ In such a situation, we consider the pulse interaction.

We first derive the profile of a single traveling pulse solution. In the limit $\epsilon \to 0$, we assume that there are two interfaces, at $z_-$ and $z_+$, where $z_- < z_+$. In this limit, we have the relation

$$u + v = H(u - a) = \begin{cases} 1, & (z_- < z < z_+) \\ 0, & \text{(otherwise)} \end{cases} \quad (3.4.2)$$

This implies that $H(u - a)$ can be replaced by $H_+(-z + z_+)H_-(z - z_-) \equiv H_+H_-$. In the moving frame represented by the coordinate $z = x - ct$, the time evolution for $v$ is governed by

$$\frac{\partial v}{\partial t} = c \frac{\partial v}{\partial z} + D \frac{\partial^2 v}{\partial z^2} - (\mu + 1)v + \mu H(u - a). \quad (3.4.3)$$
We can obtain the profile of the stationary traveling pulse solution of (3.4.3) as

\[ v(z) = \begin{cases} 
C_1e^{\lambda_+-z}, & (-\infty < z < z_-) \\
C_2e^{\lambda_+} + C_3e^{\lambda_-} + \left(\frac{\mu}{\mu+1}\right), & (z_- < z < z_+) \\
C_4e^{\lambda_--z}, & (z_+ < z < \infty) 
\end{cases} \tag{3.4.4} \]

where the coefficients \( C_i \ (i = 1, \cdots, 4) \) are given by

\[
\begin{align*}
C_1 &= \left(\frac{\lambda_-}{\lambda_+ - \lambda_-}\right) \left(\frac{\gamma}{\gamma+1}\right) (e^{-\lambda_+ z_+} - e^{-\lambda_- z_-}), \\
C_2 &= \left(\frac{\lambda_+}{\lambda_+ - \lambda_-}\right) \left(\frac{\gamma}{\gamma+1}\right) e^{-\lambda_+ z_+}, \\
C_3 &= \left(\frac{-\lambda_+}{\lambda_+ - \lambda_-}\right) \left(\frac{\gamma}{\gamma+1}\right) e^{-\lambda_- z_-}, \\
C_4 &= \left(\frac{\lambda_+}{\lambda_+ - \lambda_-}\right) \left(\frac{\gamma}{\gamma+1}\right) (e^{-\lambda_- z_+} - e^{-\lambda_- z_-}),
\end{align*} \tag{3.4.5} \]

and \( \lambda_{\pm} \) are the eigenvalues of the linear operator \( L = D\partial_z^2 + c\partial_z - (\mu + 1) \), explicitly \( \lambda_{\pm} = \frac{1}{2D}[-c \pm \sqrt{c^2 + 4D(\mu + 1)}] \). The traveling speed of the leading front is determined by the value of the inhibitor at the interface, denoted by \( v_I \):

\[ c = c(v_I) = \frac{2 \left(-\frac{1}{2} + a + v_I\right)}{\sigma[(a + v_I)(1 - a - v_I)]^{1/2}}. \tag{3.4.6} \]

The velocity of the rear front is the opposite of the above. In order for a traveling pulse to be maintained, it is necessary that \( c(v_{I,-}) = -c(v_{I,+}) \), where \( v_{I, \pm} \) are the values of the inhibitor at \( z_{\pm} \), respectively. These conditions give the relation between the traveling speed \( c \) and the width of the pulse \( d = (z_+ - z_-) \) for a given value of \( a \). We here assume that the above conditions are satisfied and that there are stationary traveling pulses.

Now, we consider interactions among the pulses in a pulse train. We denote the positions of the interfaces of the \( i \)-th pulse by \( z_{i,\pm} \). We assume that the interval of each pulse is sufficiently large compared with the pulse width, \( |z_{i,+} - z_{i,-}| \equiv d_i \). There are refractory zones behind the fronts of these traveling pulses, and these maintain the intervals between pulses. This assumption allows us to employ the nearest-neighbor approximation for the interactions among pulses.

In the limit \( \epsilon \to 0 \), we write the wave train solution of (3.4.1) as \( v(z,t) = \bar{V}(z) + B(z,t) = \sum_k v_k + \sum_{l=1}^{\infty} \delta^l b_l(z,t) \). Here, \( \delta \) is a small parameter, with \( O(\delta) \sim \exp[-\lambda_+ L] \), and \( L \) is the characteristic distance between consecutive pulses. Thus, \( b_l(z,t) \) is a correction of \( O(\delta^l) \) to the zero-th order solution consisting of a superposition of non-interacting pulses. The \( k \)-th pulse, \( v_k \), satisfies

\[ cv_{k,z} + Dv_{k,zz} - (\mu + 1)v_k + \mu H(u_k - a) = 0, \tag{3.4.7} \]

where \( u_k \) is given by \( u_k = H(u_k - a) - v_k \), and the suffix \( z \) denotes differentiation with respect to \( z \). The above \( v \) is substituted into (3.4.3), and we thereby obtain the
time evolution equation for $\delta b_1$ as
\[
\partial_t(\delta b_1) = \mathcal{L}(\delta b_1) + \mathcal{L} \sum_k v_k + \mu H(u - a) \\
= \mathcal{L}(\delta b_1) - \sum_k \mu H(u_k - a) + \mu H(u - a), \tag{3.4.8}
\]

where $u = \sum_k u_k$. When we focus on the dynamics of the $j$-th pulse, the contribution from the $k$-th pulse ($k \neq j$) is small in the region near the $j$-th pulse. For this reason, we can use the approximation $H(u - a) = H(u_j - a + \sum_{k \neq j} u_k) \approx H(u_j - a) + H'(u_j - a) \sum_{k \neq j} u_k$.

Let us consider the dynamics of the $i$-th pulse, which is regarded as interacting only with the $(i \pm 1)$-th pulses in the nearest-neighbor approximation. Then, (3.4.8) is approximated as
\[
\partial_t(\delta b_1) = \mathcal{L}(\delta b_1) - \mu \sum_{k \neq i} H(u_k - a) + \mu H'(u_i - a) \sum_{k \neq i} u_k \\
\sim \mathcal{L}(\delta b_1) - \mu H(u_{i-1} - a) - \mu H(u_{i+1} - a) + \mu H'(u_i - a)[u_{i-1} + u_{i+1}] \\
\equiv \mathcal{L}(\delta b_1) + g_i. \tag{3.4.9}
\]

In order to find the zero eigenfunction of $\mathcal{L}$, we differentiate (3.4.7) with respect to $z$. Then we obtain
\[
\left[ c\partial_z + D\partial_z^2 - (\mu + 1) \right] v_{k,z} + \mu \left[ \delta(z - z_{k,-}) - \delta(z - z_{k,+}) \right] \\
= \mathcal{L} v_{k,z} + \mu \left[ \delta(z - z_{k,-}) - \delta(z - z_{k,+}) \right] = 0. \tag{3.4.10}
\]

This implies that, in the limit $z_{k,-} \to z_{k,+}$, we can regard $v_{k,z}$ as a zero eigenfunction of $\mathcal{L}$. Similarly, a stationary traveling pulse with velocity $-c$, denoted $w_k(z)$, satisfies the equation
\[
-cw_{k,z} + Dw_{k,zz} - (\mu + 1) w_k + \mu H(u_k - a) \\
\equiv \mathcal{L}^\dagger w_k + \mu H(u_k - a) = 0. \tag{3.4.11}
\]

The eigenvalues of $\mathcal{L}^\dagger$ are $\lambda_{\pm} = \frac{1}{2D}[c \pm \sqrt{c^2 + 4D(\mu + 1)}]$. Thus, the explicit form of $w(z)$ is obtained by replacing $c$ by $-c$ in (3.4.4) and (3.4.5). With arguments similar to those used with regard to $\mathcal{L}$, it can be shown that $w_{k,z}$ can be regarded as the zero eigenfunction of $\mathcal{L}^\dagger$ in the limit $z_{k,-} \to z_{k,+}$.

If the pulse width $d_i$ is assumed to be sufficiently smaller than distances between pulses, the pulse dynamics can be described in terms of the central positions of the pulses. In this situation, we denote the position of the $i$-th pulse as $z_i$. Although $b_1(z,t)$ includes corrections of $O(\delta)$ to all pulses, we now focus on the correction to the $i$-th pulse. For this purpose, we assume that $b_1$ can be expanded near $z = z_i$ in terms of the zero eigenfunction of $\mathcal{L}$ as $b_1(z,t) = P_i^{(1)}(t) v_{i,z} + \tilde{b}_1(z)$. Substituting $b_1(z,t)$ into (3.4.9), we obtain
\[
\delta P_i^{(1)} v_{i,z} = \delta \mathcal{L} \tilde{b}_1 + g_i. \tag{3.4.12}
\]
Taking the inner product of (3.4.12) with the zero eigenfunction of \( \mathcal{L}^\dagger \), the time-dependence of \( P_i^{(1)} \) is obtained as

\[
P_{i,t}^{(1)} = \frac{\langle w_i,z,g_i \rangle}{\delta \langle w_i,z,v_i,z \rangle} = \text{constant.} \tag{3.4.13}
\]

From the above, we find that \( P_i^{(1)}(t) \) is a polynomial of degree one. Although we have treated the correction to the \( i \)-th pulse, similar arguments hold for every pulse. Then, summing up all contributions from every pulse, we obtain the solution describing a train of interacting pulses up to \( O(\delta) \) as

\[
v(z, t) \sim \sum_i v_i + \delta b_1(z, t) = \sum_i v_i + \delta \left[ \sum_i P_i^{(1)}(t)(\partial_z v_i) + b_1(z) \right]. \tag{3.4.14}
\]

Thus, the solution obtained from the naive perturbation expansion includes secular terms. The reaction diffusion system considered here possesses translational symmetry, and therefore the position of a single pulse is an arbitrary constant. However, interactions among pulses in a train result in a time dependence of the position of each pulse. In order to avoid the divergence in the correction \( b_1(z, t) \), the arbitrary constants \( z_i \) are renormalized formally as

\[
\tilde{z}_i(t) = R(z_i, t) = z_i \left[ 1 + \delta p_i^{(1)}(t) + \delta^2 p_i^{(2)}(t) + \cdots \right]. \tag{3.4.15}
\]

Then, replacing \( z_i \) in the zero-th order solution by \( \tilde{z}_i(t) \), we obtain the renormalized solution. In this way, the renormalized interacting pulse train becomes, up to \( O(\delta) \),

\[
v(z, t) = \sum_i v \left( z - \tilde{z}_i(t) \right)
\begin{align*}
&\sim \sum_i v(z - z_i) + \delta \left[ \sum_i (-z_i p_i^{(1)}(t))(\partial_z v) \right] \\
&= \sum_i v_i + \delta \left[ \sum_i (-z_i p_i^{(1)}(t))(\partial_z v_i) \right]. \tag{3.4.16}
\end{align*}
\]

Comparing (3.4.14) with (3.4.16), we find the relation \( -z_i p_i^{(1)}(t) = P_i^{(1)}(t) \). Then, the renormalization transformation (3.4.15) is rewritten up to \( O(\delta) \) as

\[
\tilde{z}_i(t) \sim z_i - \delta P_i^{(1)}(t). \tag{3.4.17}
\]

Thus, we obtain the time evolution equation for \( \tilde{z}_i(t) \) as the asymptotic form of the generator of the Lie group:

\[
\frac{d\tilde{z}_i(t)}{dt} = \partial_\tau \left[ G_\tau \left\{ \tilde{z}_i(t), T(t) \right\} \right] \bigg|_{\tau=0} = -\delta \frac{dP_i^{(1)}}{dt} = -\frac{\langle w_i,z,g_i \rangle}{\langle w_i,z,v_i,z \rangle}. \tag{3.4.18}
\]
Now, let us approximate the integral of (3.4.18) by using the explicit forms of \( v_i \) and \( w_i \). The main assumptions are the following. i) The nearest-neighbor interactions are taken into consideration; that is, the \( i \)-th pulse is regarded as interacting with the \((i \pm 1)\)-th pulses. ii) The width of each pulse, \( d_i \), is negligibly small compared with the characteristic distance between pulses, and for this reason, the position of a pulse is regarded as its central position. In this situation, the pulse dynamics are determined using asymptotic profiles. The asymptotic profiles of the \( i \)-th pulse, \( v_i \) and \( w_i \), are given as

\[
v_i = \begin{cases} 
C'_1 e^{\lambda_+ (z - \tilde{z}_i)}, & (-\infty < z < \tilde{z}_i) \\
C'_1 e^{\lambda_- (z - \tilde{z}_i)}, & (\tilde{z}_i < z < +\infty)
\end{cases}
\]  

and

\[
w_i = \begin{cases} 
D'_1 e^{\lambda_+ (z - \tilde{z}_i)}, & (-\infty < z < \tilde{z}_i) \\
D'_1 e^{\lambda_- (z - \tilde{z}_i)}, & (\tilde{z}_i < z < +\infty)
\end{cases}
\]

where \( C'_1 \) and \( D'_1 \) are positive constants. Using these expressions, the numerator of (3.4.18) is approximated as

\[
\langle w_{i,z}, g_i \rangle \sim \int_{-\infty}^{+\infty} \left[ -\mu H(u_{i-1} - a) - \mu H(u_{i+1} - a) + \mu \left\{ \delta(z - z_{i-1}) - \delta(z - z_{i+1}) \right\} \left\{ -v_{i-1}(z) - v_{i+1}(z) \right\} \right] w_{i,z} dz
\]

\[
\sim \int_{-\infty}^{+\infty} (-\mu) \left[ \delta(z - \tilde{z}_{i-1}) + \delta(z - \tilde{z}_{i+1}) \right] 
\times \left[ \lambda_+^{\dagger} D'_1 \exp[\lambda_+^{\dagger}(z - \tilde{z}_i)] H(-z + \tilde{z}_i) + D'_1 \lambda_+^{\dagger} \exp[\lambda_+^{\dagger}(z - \tilde{z}_i)] H(z - \tilde{z}_i) \right] dz
\]

\[
\sim -a \exp[-\lambda_+^{\dagger}(\tilde{z}_i - \tilde{z}_{i-1})] + b \exp[-|\lambda_+^{\dagger}|(\tilde{z}_{i+1} - \tilde{z}_i)],
\]  

(3.4.21)

where \( a = D'_1 \lambda_+^{\dagger} \mu > 0 \) and \( b = |D'_1 \lambda_+^{\dagger} \mu| > 0 \). Contrastingly, the denominator of (3.4.18) is approximated as

\[
\langle w_{i,z}, v_{i,z} \rangle \sim \int_{-\infty}^{\tilde{z}_i} C'_1 D'_1 \lambda_+ \lambda_+^{\dagger} \exp[\lambda_+^{\dagger}(z - \tilde{z}_i)] \exp[\lambda_+(z - \tilde{z}_i)] dz
\]

\[
+ \int_{\tilde{z}_i}^{+\infty} C'_1 D'_1 \lambda_- \lambda_+^{\dagger} \exp[\lambda_+^{\dagger}(z - \tilde{z}_i)] \exp[\lambda_-(z - \tilde{z}_i)] dz
\]

\[
= C'_1 D'_1 \frac{\lambda_+ \lambda_+^{\dagger}}{(\lambda_+ + \lambda_+^{\dagger})} - C'_1 D'_1 \frac{\lambda_- \lambda_+^{\dagger}}{(\lambda_- + \lambda_+^{\dagger})}
\]

\[
= C'_1 D'_1 \frac{2(\mu + 1)}{\sqrt{c^2 + 4D(\mu + 1)}} > 0.
\]  

(3.4.22)
Using (3.4.21) and (3.4.22), we obtain an asymptotic expression for the evolution equation of $\tilde{z}_i(t)$,

$$
\frac{d\tilde{z}_i(t)}{dt} = -\frac{\langle w_{i,z}, g_i \rangle}{\langle w_{i,z}, v_{i,z} \rangle} \sim a' \exp\left[-\lambda_+^i(\tilde{z}_i - \tilde{z}_{i-1})\right] - b' \exp\left[-|\lambda_-^i|(\tilde{z}_{i+1} - \tilde{z}_i)\right],
$$

where $a' = a/\langle w_{i,z}, v_{i,z} \rangle$ and $b' = b/\langle w_{i,z}, v_{i,z} \rangle$. Thus, the time dependence of the position of the $i$-th pulse is described in terms of the distances between the nearest-neighbor pulses. Although a similar result was previously reported by Elphick et al.,\textsuperscript{23} we have derived it using the RG method based on the Lie group. In the stationary state, all pulses are separated by equal distances due to the repulsive interaction. The mean distance between pulses $\bar{L}$ is approximated by setting (3.4.23) to zero. We then obtain $\bar{L} \sim \frac{1}{(\lambda_+^i - |\lambda_-^i|)} \log\left(\frac{\lambda_+^i}{|\lambda_-^i|}\right) = \frac{D}{c} \log\left(\frac{(c+\sqrt{c^2+4D(\mu+1)})^2}{4D(\mu+1)^2}\right)$.

3.5. Interaction of traveling pulses in a three-component reaction diffusion system

Recently, three-component (one activator and two inhibitors) reaction diffusion systems have been actively studied by many researchers, and several noteworthy phenomena, such as the elastic collision of fast traveling pulses and self-duplicating spots, have been reported.\textsuperscript{24,25} For this reason, it is meaningful to study pulse dynamics in three-component reaction diffusion systems.

We here study the following three-component reaction diffusion system:

$$
\begin{align*}
\epsilon\sigma \frac{\partial u}{\partial t} &= \epsilon^2 \frac{\partial^2 u}{\partial x^2} + H\left(u - a(w)\right) - u - v, \\
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + \mu u - v, \\
\tau \frac{\partial w}{\partial t} &= d \frac{\partial^2 w}{\partial x^2} + u + v - w - s_0.
\end{align*}
$$

(3.5.1)

Here, $u$ is an activator, and $v$ and $w$ are inhibitors. $H(x)$ is the Heaviside step function satisfying $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. The function $a(w)$ is given by $a(w) = \frac{1}{2}[1 + \tanh(\alpha w + \hat{a}_0)]$, where $\alpha$ and $\hat{a}_0$ are positive constants. In addition, $\sigma, \mu, \tau, d$ and $s_0$ are positive constants, and $\epsilon$ is a small positive constant. We here consider the situation that the diffusion constant $d$ of the second inhibitor, $w$, is sufficiently larger than that of the first one, and the specific time constant $\tau$ is very small. In other words, we consider the situation in which $d \gg 1$ and $\tau \ll 1$. In this situation, it has been shown using singular perturbation analysis that the above system has a traveling pulse solution.\textsuperscript{26} We here consider interactions among the pulses in a train.

We first derive the profile of a single traveling pulse solution. In the moving frame represented by the coordinate $z = x - ct$, (3.5.1) becomes
where $z_\pm$ are the positions of the right and left interfaces, respectively. Then the time evolution of $v(z,t)$ and $w(z,t)$ is governed by

$$
\frac{\partial}{\partial t} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \frac{\partial^2}{\partial z^2} + c \frac{\partial}{\partial z} - (\mu + 1) & 0 \\ 0 & \frac{\partial^2}{\partial z^2} + c \frac{\partial}{\partial z} - \frac{1}{\tau} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} \mu H(u - a(w)) \\ -s_0 + H(u - a(w)) \end{pmatrix} + \frac{\mu H(u - a(w))}{s_0 + H(u - a(w))}.
$$

The profile of the stationary traveling pulse solution is obtained as

$$
v(z) = \begin{cases} 
C_1 e^{\lambda_{v,+} z}, & (-\infty < z < z_-) \\
C_2 e^{\lambda_{v,-} z} + C_3 e^{\lambda_{v,+} z} + \frac{\mu}{\mu + 1}, & (z_- < z < z_+) \\
C_4 e^{\lambda_{v,-} z} & (z_+ < z < \infty)
\end{cases}
$$

and

$$
w(z) = \begin{cases} 
D_1 e^{\lambda_{w,+} z} - s_0, & (-\infty < z < z_-) \\
D_2 e^{\lambda_{w,-} z} + D_3 e^{\lambda_{w,+} z} + 1 - s_0, & (z_- < z < z_+) \\
D_4 e^{\lambda_{w,-} z} - s_0. & (z_+ < z < \infty)
\end{cases}
$$

Here, $\lambda_{v,\pm}$ and $\lambda_{w,\pm}$ are respectively

$$
\begin{align*}
\lambda_{v,\pm} &= \frac{1}{2d} \left[ -c \pm \sqrt{c^2 + 4(\mu + 1)} \right], \\
\lambda_{w,\pm} &= \frac{1}{2d} \left[ -\tau c \pm \sqrt{(\tau c)^2 + 4d} \right],
\end{align*}
$$

and the coefficients $C_i$ and $D_i$ $(i = 1, 2, 3, 4)$ are as follows:

$$
\begin{align*}
C_1 &= \left( \frac{\mu}{\mu + 1} \right) \left( \frac{\lambda_{v,-}}{\lambda_{v,+} - \lambda_{v,-}} \right) (e^{-\lambda_{v,+} z} - e^{-\lambda_{v,-} z}), \\
C_2 &= -\left( \frac{\mu}{\mu + 1} \right) \left( \frac{\lambda_{v,+}}{\lambda_{v,+} - \lambda_{v,-}} \right) e^{-\lambda_{v,-} z}, \\
C_3 &= \left( \frac{\mu}{\mu + 1} \right) \left( \frac{\lambda_{v,-}}{\lambda_{v,+} - \lambda_{v,-}} \right) e^{-\lambda_{v,+} z}, \\
C_4 &= \left( \frac{\mu}{\mu + 1} \right) \left( \frac{\lambda_{v,+}}{\lambda_{v,+} - \lambda_{v,-}} \right) (e^{-\lambda_{v,-} z} - e^{-\lambda_{v,+} z}).
\end{align*}
$$
and

\[
\begin{aligned}
D_1 &= \left( \frac{\lambda w, -}{\lambda w, + - \lambda w, -} \right) (e^{-\lambda w, + z_+} - e^{-\lambda w, + z_-}), \\
D_2 &= -\left( \frac{\lambda w, +}{\lambda w, + - \lambda w, -} \right) e^{-\lambda w, - z_-}, \\
D_3 &= \left( \frac{\lambda w, -}{\lambda w, + - \lambda w, -} \right) e^{-\lambda w, + z_+}, \\
D_4 &= \left( \frac{\lambda w, +}{\lambda w, + - \lambda w, -} \right) (e^{-\lambda w, - z_+} - e^{-\lambda w, - z_-}).
\end{aligned}
\tag{3.5.9}
\]

In the present system, the traveling speed \(c\) is determined by the values of \(v\) and \(w\) at the interface, denoted by \(v_I\) and \(w_I\), respectively:

\[
C(v_I; a(w_I)) = \frac{2 \left( \frac{1}{2} - v_I - a(w_I) \right)}{\sigma \left( v_I + a(w_I) \right) \left( 1 - v_I - a(w_I) \right)^{\frac{1}{2}}},
\tag{3.5.10}
\]

In order for a stationary traveling pulse to be maintained, the set \((\sigma, c, z_+ - z_-)\) must take values such that the condition

\[
-C(v_{I,-}; a(w_{I,-})) = C(v_{I,+}; a(w_{I,+})) = c
\tag{3.5.11}
\]

holds, where \(v_{I,\pm}\) and \(w_{I,\pm}\) are the values of the inhibitors at \(z = z_{\pm}\), respectively. We here assume that there is a stationary traveling pulse solution, for which the above condition is satisfied.

We consider interactions among pulses in a train in order to elucidate the effects of the second inhibitor. The positions of interfaces of the \(i\)-th pulse are denoted by \(z_i\). The width of the pulse is \(d_i = |z_{i, +} - z_{i, -}|\), which is assumed to be sufficiently smaller than the characteristic distance between pulses. As in the previous subsection, we would like to obtain the dynamics of pulses described by the central positions of the pulses. We begin by writing the solution \(\vec{S}(z, t)\) describing a train of interacting pulses as

\[
\vec{S}(z, t) = \left( \begin{array}{c} v(z, t) \\ w(z, t) \end{array} \right) = \sum_k \left( \begin{array}{c} v_k \\ w_k \end{array} \right) + \vec{B}(z, t),
\tag{3.5.12}
\]

where the \(k\)-th pulse \(t(v_k, w_k)\) satisfies

\[
\mathcal{L} \left( \begin{array}{c} v_k \\ w_k \end{array} \right) + \left( \begin{array}{c} \mu H(u_k - a(w_k)) \\ -s_0 + H(u_k - a(w_k)) \end{array} \right) = 0,
\tag{3.5.13}
\]

and \(\vec{B}(z, t)\) is a correction term, which is expanded as

\[
\vec{B}(z, t) = \sum \delta^l \vec{B}_l(z, t).
\tag{3.5.14}
\]

Here, \(\delta\) is a small parameter, with \(O(\delta) \sim \exp[-\lambda w,+ L]\), and \(L\) is a characteristic distance between consecutive pulses. Substituting \(\vec{S}\) into (3.5.4), we obtain the time
evolution equation of $\delta \tilde{B}_1$ as

$$
\begin{align*}
\delta \partial_t \tilde{B}_1 &= \delta \mathcal{L} \tilde{B}_1 + \mathcal{L} \sum_k \begin{pmatrix} v_k \\ w_k \end{pmatrix} \bigg| \begin{pmatrix} \frac{\mu H(u-a(w))}{-s_0+H\left(u-a(w)\right)} \end{pmatrix} \\
&= \delta \mathcal{L} \tilde{B}_1 - \sum_k \begin{pmatrix} \frac{\mu H(u-a(w_k))}{-s_0+H\left(u-a(w_k)\right)} \end{pmatrix} + \begin{pmatrix} \frac{\mu H(u-a(w))}{-s_0+H\left(u-a(w)\right)} \end{pmatrix},
\end{align*}
$$

(3.5-15)

where $u = \sum_k u_k$ and $w = \sum_k w_k$. As in the previous subsection, we first focus on the correction to the $i$-th pulse. Only interactions between nearest-neighbor pulses are considered, and the Heaviside step function is expanded near the $i$-th pulse. Then we obtain an approximate time evolution equation for $\tilde{B}_1$ as

$$
\begin{align*}
\delta \partial_t \tilde{B}_1 &= \delta \mathcal{L} \tilde{B}_1 \\
&+ \left( \mu \left[ -H(i-1) - H(i+1) + H'(i) \left( u_{i-1} + u_{i+1} - a'(w_i)(w_{i-1} + w_{i+1}) \right) \right] \right) \\
&\equiv \delta \mathcal{L} \tilde{B}_1 + \tilde{g}_i,
\end{align*}
$$

(3.5-16)

where $H(i) \equiv H(u_i-a(w_i))$ and $H'(i) \equiv \frac{dH(x)}{dx} \big|_{x=u_i-a(w_i)}$. We consider the situation that the width of a pulse, $d_i$, is negligibly small compared with the characteristic distance between pulses. In this situation, we can describe the pulse dynamics in terms of the central positions of the pulses, $z_i$. Then, using a procedure similar to that used in the previous subsection, we can easily verify that the approximate zero eigenfunction of $\mathcal{L}$ is $t(v_{i,z}, w_{i,z}) \equiv t \tilde{E}_{i,z}$, where the suffix $z$ denotes differentiation with respect to $z$, and $\tilde{E}_i$ is defined by

$$
\tilde{E}_i = \begin{pmatrix} v_i \\ w_i \end{pmatrix} = \begin{cases} \\
C_i^+ e^{\lambda_{v,+}(z-z_i)}, & (-\infty < z < z_i) \\
D_i^+ e^{\lambda_{w,+}(z-z_i)}, & (z_i < z < +\infty)
\end{cases},
$$

(3.5-17)

where $C_i^+$ and $D_i^+$ are positive constants.

The adjoint operator $\mathcal{L}^\dagger$ is obtained by replacing $c$ by $-c$ in the definition of $\mathcal{L}$. The eigenvalues of $\mathcal{L}^\dagger$ are $\lambda_{v,\pm}^i = \frac{1}{2} \left[ c \pm \sqrt{c^2 + 4(\mu + 1)} \right]$ and $\lambda_{w,\pm}^i = \frac{1}{2d} \left[ \tau c \pm \sqrt{(\tau c)^2 + 4d} \right]$. We denote the zero eigenfunction of $\mathcal{L}^\dagger$ by $\tilde{F}_{i,z}$, where $\tilde{F}_i$ is defined by

$$
\tilde{F}_i = \begin{pmatrix} \end{pmatrix} = \begin{cases} \\
C_{a,1} e^{\lambda_{v,+}(z-z_i)}, & (-\infty < z < z_i) \\
D_{a,1} e^{\lambda_{w,+}(z-z_i)}, & (z_i < z < +\infty)
\end{cases},
$$

(3.5-18)
where $C'_{a,1}$ and $D'_{a,1}$ are positive constants.

We assume that the correction $\vec{B}_1(z,t)$ can be expanded in terms of the zero eigenfunction of $\mathcal{L}$ as

$$\vec{B}_1(z,t) = P_i^{(1)}(t)\vec{E}_{i,z} + \vec{F}_1(z).$$

(3.5.19)

Substituting $\vec{B}_1(z,t)$ into (3.5.16), and taking the inner product with the zero eigenfunction of $\mathcal{L}^\dagger$, we obtain the time dependence of $P_i^{(1)}$ as

$$P_i^{(1)} = \frac{\langle t \vec{F}_{i,z}, \vec{g}_i \rangle}{\delta \langle t \vec{F}_{i,z}, \vec{E}_{i,z} \rangle} = \text{constant}. \quad (3.5.20)$$

From the above, we find that $P_i^{(1)}(t)$ is a polynomial of degree one. Then, summing up all the contributions from each pulse, we obtain a solution representing a train of interacting pulses, up to $O(\delta)$, as

$$\vec{S}(z,t) \sim \sum_i \vec{E}_i + \delta \vec{B}_1(z,t)$$

$$= \sum_i \left( \begin{array}{c} v_i \\ w_i \end{array} \right) + \delta \sum_i P_i^{(1)}(t) \left( \begin{array}{c} (\partial_z v_i) \\ (\partial_z w_i) \end{array} \right). \quad (3.5.21)$$

Thus, the solution obtained from the naive perturbation expansion includes secular terms. In order to avoid the divergence of $\vec{B}_1$, the arbitrary constant $z_i$ is renormalized as

$$\tilde{z}_i(t) = R(z_i, t) = z_i \left[ 1 + \delta p_i^{(1)}(t) + \delta^2 p_i^{(2)}(t) + \cdots \right]. \quad (3.5.22)$$

Next, replacing $z_i$ by $\tilde{z}_i(t)$ in the zero-th order solution, we obtain a renormalized solution. The renormalized solution of the pulse train up to $O(\delta)$ is

$$\tilde{S}(z,t) = \sum_i \left( \begin{array}{c} v_i \\ w_i \end{array} \right)$$

$$\sim \sum_i \left( \begin{array}{c} v(z - z_i) \\ w(z - z_i) \end{array} \right) + \delta \sum_i \left( -z_i p_i^{(1)}(t) \right) \left( \begin{array}{c} (\partial_z v) \\ (\partial_z w) \end{array} \right)$$

$$= \sum_i \left( \begin{array}{c} v_i \\ w_i \end{array} \right) + \delta \sum_i \left( -z_i p_i^{(1)}(t) \right) \left( \begin{array}{c} (\partial_z v)_i \\ (\partial_z w)_i \end{array} \right). \quad (3.5.23)$$

Comparing (3.5.21) with (3.5.23), we find the relation $-z_i p_i^{(1)}(t) = P_i^{(1)}(t)$. Then, the renormalization transformation (3.5.22) is rewritten up to $O(\delta)$ as

$$\tilde{z}_i(t) \sim z_i - \delta P_i^{(1)}(t). \quad (3.5.24)$$

Thus, we have obtained the time evolution equation for $\tilde{z}_i(t)$ as an asymptotic form of the generator of the Lie group:

$$\frac{d\tilde{z}_i(t)}{dt} = \partial_\tau \left[ G_\tau \{ \tilde{z}_i(t), T(t) \} \right] \bigg|_{\tau=0} = -\delta \frac{dP_i^{(1)}}{dt} = -\frac{\langle t \vec{F}_{i,z}, \vec{g}_i \rangle}{\langle t \vec{F}_{i,z}, \vec{E}_{i,z} \rangle}. \quad (3.5.25)$$
The actual procedure for approximating (3.5-25) is similar to that employed in the previous subsection. Below, we give the final expressions.

The numerator of (3.5-25) is

\[
\langle t \vec{F}_{i,z}, \bar{g}_i \rangle \sim [(-\mu)C'_{a,1} \{ \lambda_{v,+} \bar{c} - \lambda_{v,-} |(\bar{z}_{i+1} - \bar{z}_i) \}] 
+ \left( \frac{1}{\tau} \right) D_{a,1} \{ \lambda_{w,+} \bar{c} - \lambda_{w,-} |(\bar{z}_{i+1} - \bar{z}_i) \}
\]

where \(a_v = \mu C'_a \lambda_{v,+} \), \(b_v = |\mu C'_a | \lambda_{v,-} \), \(a_w = \frac{1}{\tau} D'_{a,1} \lambda_{w,+} \) and \(b_w = |\frac{1}{\tau} D'_{a,1}| \lambda_{w,-} \).

Contrastingly, the denominator of (3.5-25) is approximated as

\[
\langle t \vec{F}_{i,z}, \bar{E}_{i,z} \rangle \sim C'_a \frac{2(\mu + 1)}{\sqrt{c^2 + (\mu + 1)^2}} + D'_a \frac{2}{\sqrt{(\tau c)^2 + 4d}} > 0. \tag{3.5-27}
\]

Using (3.5-26) and (3.5-27), we obtain the time-dependence of \(\bar{z}_i\) as

\[
\frac{d\bar{z}_i(t)}{dt} = -\langle t \vec{F}_{i,z}, \bar{g}_i \rangle \sim \left[ a'_v \exp \left[ -\lambda_{v,+} |(\bar{z}_i - \bar{z}_{i-1}) \right] - b'_v \exp \left[ -\lambda_{v,-} |(\bar{z}_{i+1} - \bar{z}_i) \right] \right] 
+ \left[ a'_w \exp \left[ -\lambda_{w,+} |(\bar{z}_i - \bar{z}_{i-1}) \right] - b'_w \exp \left[ -\lambda_{w,-} |(\bar{z}_{i+1} - \bar{z}_i) \right] \right]. \tag{3.5-28}
\]

where \(a'_v = a_v / \langle t \vec{F}_{i,z}, \bar{E}_{i,z} \rangle \), \(b'_v = b_v / \langle t \vec{F}_{i,z}, \bar{E}_{i,z} \rangle \), \(a'_w = a_w / \langle t \vec{F}_{i,z}, \bar{E}_{i,z} \rangle \) and \(b'_w = b_w / \langle t \vec{F}_{i,z}, \bar{E}_{i,z} \rangle \).

Now, let us approximate the mean distance between pulses \(\bar{L}\) by setting (3.5-28) equal zero. For \(\tau \sim 1\) and \(d \sim 1\), the two terms in (3.5-28) are comparable. However, when \(\tau \gg 1\) and \(d \ll 1\), the first term in (3.5-28) is dominant. In this case, we obtain \(\bar{L} \sim \bar{L}_v = \frac{1}{(\lambda_{v,+} - |\lambda_{v,-}|)} \log \left( \frac{\lambda_{v,+}}{|\lambda_{v,-}|} \right) = \frac{1}{c} \log \left( \frac{(\tau c + \sqrt{(\tau c)^2 + 4d})^2}{4(\mu + 1)} \right) \). This is essentially the same as the result for the two-component reaction diffusion system considered in the previous subsection. By contrast, for \(\tau \ll 1\) and \(d \gg 1\), the second term in (3.5-28) is dominant, and we obtain \(\bar{L} \sim \bar{L}_w = \frac{1}{(\lambda_{w,+} - |\lambda_{w,-}|)} \log \left( \frac{\lambda_{w,+}}{|\lambda_{w,-}|} \right) = \frac{d}{\tau c} \log \left( \frac{(\tau c + \sqrt{(\tau c)^2 + 4d})^2}{4d} \right) \approx \sqrt{d} \left[ 1 - \frac{1}{24} \left( \frac{\tau c}{d} \right)^2 \right] \). This means that when the second inhibitor has a short relaxation time and long diffusion length, the mean interval of pulses is approximately \(\sqrt{d}\). Also, we note that \(\bar{L}_w \gg \bar{L}_v\). This can be understood intuitively by noting that the second inhibitor causes neighboring pulses to strongly repell one another, so that the mean distance between pulses becomes large.

When fast traveling pulses in two-component reaction diffusion systems (such as the FitzHugh-Nagumo system) meet, they annihilate each other on collision. This is
because one pulse cannot go beyond the refractory zone of the other pulse. However, pulses with very slow speeds can collide elastically. The mechanism in the latter case has been theoretically explained by Ohta.\textsuperscript{27} It is applicable to traveling pulses with arbitrarily slow speeds in the vicinity of a translational bifurcation point. Contrastingly, fast traveling pulses in the three-component reaction diffusion system can collide elastically under the condition that the second inhibitor has a short relaxation time and long diffusion length.\textsuperscript{26} This is because the second inhibitor has a long tail in front of the pulse, and this tail reduces the traveling speed before collision, and through this effect, the propagation direction of each pulse is reversed. Although perturbation theories are not applicable to phenomena far from the bifurcation point, the present result indirectly supports this elastic collision of fast traveling pulses in three-component reaction diffusion systems.

§4. Conclusion

The RG method has been reformulated on the basis of a naive renormalization transformation and the Lie group.\textsuperscript{16} This reformulated RG method has been applied to chaotic discrete systems, symplectic maps and general reaction diffusion systems.\textsuperscript{28}–\textsuperscript{30} In the present work, we studied pulse dynamics as an application of this reformulated perturbative RG method based on the Lie group. As examples, we treated asymmetric kink solutions of the TDGL equation, a modified KdV equation and reaction diffusion systems.

When there exists a single pulse, the position of the pulse is an arbitrary constant, due to the translational symmetry of the system. However, when there are two or more pulses in the system, interactions among pulses cause the positions of the pulses to become time dependent variables. In order to describe interacting pulses, we used the single pulse solution in the system as the zero-th order solution. It is necessary to make corrections to the superposition of these zero-th order solutions. As we consider the situation that the interaction is very small, we treat it as a perturbation. We were able to find a small parameter representing the size of the interaction, and corrections were expanded in terms of it. We linearized the nonlinear terms with respect to this small parameter, and the time evolution equation for the correction was thereby obtained. In this process, there exists a linear operator $L$ in terms of which we express the correction. The key point in the present method is to assume that the corrections can be expanded in terms of the zero eigenfunctions of $L$, and their coefficients are generally time-dependent variables. We easily obtained the time-dependence of the coefficients by taking the inner product of them with the zero eigenfunction of the adjoint operator $L^\dagger$. Then we noted that the corrections obtained from the naive perturbation expansion includes secular terms. Therefore, in order to avoid divergence, the arbitrary constants in the zero-th order solution, such as the position of the pulse, were renormalized to time-dependent variables.

The expression for the interacting pulses solution obtained using the above scheme can be interpreted as the Taylor expansion of the renormalized solution, which is obtained by replacing the arbitrary constants in the zero-th order solution by the renormalized variables, with respect to the small parameter. Comparing the
interacting pulse solution obtained from the naive perturbation expansion with the Taylor expansion of the renormalized solution, we find the correspondence between the renormalized parameters and the secular terms. Using these, the renormalization transformation is constructed from the secular terms. We can interpret the renormalization transformation as the translational group operator. The generator of the Lie group yields the time-dependence of the renormalized variables, and with this, we obtain the asymptotic time evolution equation for the position of pulse.

We here remark that the present method is a first-order approximation, it is impossible to obtain higher-order corrections with this method. We assumed that the corrections to the superposition of the zero-th order solution can be expanded in terms of the zero eigenfunction of $L$. The reason for this assumption is that we would like to obtain the time dependence by taking the inner product with the zero eigenfunction of $L^\dagger$. However, when the renormalized solution is expanded with respect to the small parameter, there appear higher-order derivatives of the zero-th order solution. In order to find the correspondence between the expansion coefficients of the renormalization transformation and the secular terms, we compare the interacting pulse solution obtained from the naive perturbation expansion with the Taylor expansion of the renormalized solution by comparing terms of equal orders in the small parameter. At first-order in this small parameter, the correspondence is found. However, for higher-orders, there is no correspondence of the coefficients in terms of the higher-order derivatives. Thus, in the present method, only the first-order correction to the superposition of the zero-th order solution is taken into consideration in the renormalization transformation. The truncated renormalization transformation reproduces the results obtained with other methods.

We briefly compare the present method with other methods. In the scheme proposed by Ei and Ohta,\textsuperscript{18) the positions of pulses are treated as time-dependent variables a priori. The time evolution equation of the position of a pulse is obtained from the solvability condition. This requires that the inhomogenous terms should be orthogonal to the zero eigenfunction of the operator adjoint to the linear operator. Contrastingly, in the present RG method based on the Lie group, the time-dependence of the positions of the pulses is introduced a posteriori in order to avoid the divergence of the correction term. The arbitrary constants are renormalized to time-dependent variables, and the generator of the Lie group yields the time-dependence of the positions of the pulses. We here remark that all of the secular terms are assumed to be renormalized to arbitrary constants, even if the validity of this assumption is not clear a priori. By contrast, in the theory of Ei, Fujii and Kunihiro,\textsuperscript{17) the correction to the unperturbed solution is obtained with the RG method under some initial conditions. The initial value is also expanded perturbatively, and it is chosen such that the secular terms in the perturbed solution are canceled out. In their RG prescription, the step of setting the initial time $t_0 = t$ in the RG equation is natural, and the time-dependence of the positions of the pulses is obtained from the RG equation. In terms of the concept of invariant manifolds, the RG method constructs invariant manifolds perturbatively, order by order. The higher order-solutions determine the deformation of the invariant manifolds. In their procedure, these deformations are obtained systematically by performing a formal
integral with the help of the projection operators. Contrastingly, in the present method, we cannot obtain the exact higher-order corrections because it is assumed that solutions are obtained only from an expansion in the zero eigenfunctions of $L$. This is a limitation of the present method, because it implies that the method is valid only up to first order in the deformation of the invariant manifolds. However, when the higher-order (no less than second order) corrections are ignored, there is no difference between two methods, as we have found.

We have thus found that the concepts and limitations of these three methods are quite different. However, in each method, it is necessary to know the zero eigenfunctions of the linear operator $L$ and its adjoint $L^\dagger$ for actual calculations of the explicit time-dependence.

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**References**

12) D. V. Shirkov, hep-th/9602024.
30) Y. Masutomi and K. Nozaki, Physica D 151 (2001), 44.