Time-Delayed Feedback Control Method for Dynamical Symmetry
Breaking in a Periodically Driven Bistable System

Hiroki Tutu

Department of Applied Analysis and Complex Dynamical Systems, Graduate School
of Informatics, Kyoto University, Kyoto 606-8501, Japan

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With the aim of formulating a method to control dynamic phase transitions in periodically driven bistable systems with reversal symmetry, a time-delayed feedback control method to stabilize an unstable periodic orbit in the broken symmetric regime is studied. In order to overcome a limitation of the conventional time-delayed feedback method, another extended scheme is proposed, and its improved ability with respect to stabilization is proved. Through the linear stability analysis of model controlled systems driven by sinusoidal fields, basic differences between the conventional and proposed methods are extracted. It is clarified that a few characteristics around the bifurcation point from the pitchfork critical branch to the Hopf branch and the turning point of the Hopf critical branch classify essential features of the stability diagram and concern restrictions for stabilization. Within the linear stability treatment, this paper estimates a safe choice and an effective range of feedback gains in the proposed method.

§1. Introduction

Time-delayed feedback control (TDFC) is a powerful method to stabilize an unstable periodic orbit (UPO) embedded in a chaotic attractor and to suppress spatio-temporal chaos. There are several theoretical and experimental studies investigating the control of physical and biological systems.

Broken symmetry phenomena in periodically driven systems are also an active subject of study in nonequilibrium physics. They have been widely investigated in the context of dynamic phase transitions (DPT).

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Fig. 1. Schematic representations of hysteresis curves: (a) An SRO-cycle, and (b) two SBO-cycles (solid curves) and an unstable SRO-cycle (broken curve) in the broken symmetric phase. These are obtained from Eq. (2.1) with \( \Omega = 0.1 \), (a) \( h = 0.5 \), and (b) \( h = 0.4 \).

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* E-mail: tutu@acs.i.kyoto-u.ac.jp
A well-known example exhibiting a DPT is magnetization hysteresis phenomenon. While an alternating magnetic field $H(t)$ is applied to a magnetic material with period $T$ and symmetry $H(t) = -H(t + T/2)$ in time $t$, e.g., $H(t) \propto \cos(2\pi t/T)$, the magnetization cycle $M(t)$ will trace a closed curve in the $H$-$M$ plane, ignoring fluctuations of microscopic origin. Suppose that such a magnetic system possesses reversal symmetry with respect to the magnetization, which is characterized by a bistable potential. In this case, there would be two possible magnetization cycles with different kinds of dynamical symmetries, depending on the magnitude and period of the field. These two kinds of magnetization cycles are schematically represented in Fig. 1. One of the possible magnetization cycles restores the dynamical symmetry of the applied field, i.e., $M(t) = -M(t + T/2)$, and the other breaks this symmetry, i.e., $M(t) \neq -M(t + T/2)$. In our studies, the former is called symmetry restoring oscillation (SRO) and the latter symmetry breaking oscillation (SBO).

When the frequency of the applied field is sufficiently small and the amplitude of the field is sufficiently large, an SRO-type cycle [Fig. 1(a)] is observed, otherwise, leaving that state, as the frequency is raised beyond a critical value or as the amplitude is lowered below a critical value, symmetry breaking occurs and appears as an SBO-type cycle [Fig. 1(b)].

The aim of the present study is to theoretically develop a method for stabilizing the unstable SRO state in the broken symmetric phase. In order to stabilize the unstable SRO state, the TDFC is applied. This work is intended to be a preliminary study to construct the controlling scheme to stabilize an unstable SRO state in material systems with bistable symmetry, e.g., ferromagnetic and ferroelectric systems, under the symmetric periodic fields. If such a technique can be realized, some interesting phenomena that may lead to novel nonequilibrium states or useful applications would be expected. For example, such a technique may be applicable to a demagnetization technique.

The concept of the TDFC method is described in Fig. 2, which schematically represents the conventional TDFC scheme originally proposed. The “system” now corresponds to the magnetic material, which is under the control of the alternating magnetic field and the TDFC. Suppose that it is in the broken symmetric phase if the feedback is removed. It may be possible to generalize this system to systems which are described as bistable systems and in which the order parameter can be controlled.
by its conjugate field. The control device to stabilize the SRO state is surrounded by the square drawn by the broken line. This device works to synthesize a suitable feedback signal proportional to the quantity $M(t) + M(t - T/2)$. The technical details of the device are outside the scope of the present study. It is important that the feedback signal vanish while the SRO state is realized, and that the SRO state is maintained by a small control signal. This type of half-period delayed feedback has also been studied in a different context.\textsuperscript{25}

However, it is known that the conventional scheme cannot always stabilize UPOs.\textsuperscript{3,26} The limitation relevant to the present study is that it cannot stabilize UPOs of long period,\textsuperscript{3} which is now the unstable SRO state sustained by the low frequency forcing. To overcome this limitation is the main goal of the present study. This paper considers a simple dynamical system exhibiting the DPT of periodically driven bistable systems and proposes another feedback control method to stabilize the unstable SRO state even in the low frequency regime. Although we must consider the effect of noise in real applications, this paper does not treat it and leaves it for a future work.

This paper is organized as follows. Section 2 is mainly devoted to the argument regarding the conventional TDFC scheme and consists of three subsections. First, the model system to be controlled is briefly introduced. Next, fundamental properties of the conventional TDFC scheme are described. Based on a linear stability analysis, the stability criterion expressing the limitations of the method is elucidated. The last subsection presents numerical simulation results confirming the analytical results and interesting dynamical properties in the case that the system cannot be controlled. In §3, a method that overcomes the limitation of the conventional TDFC scheme is discussed, and its linear stability is demonstrated. Since the linear stability problem to be analyzed is somewhat intricate, it is treated with two different levels of approximation in two subsections. The first subsection analyzes the linear stability of a reduced single-component system, and the mechanism that leads to the improvement of the control capability is clarified. In addition, a safe choice of the feedback gain is proposed. The second subsection presents further analysis of the linear stability for the system reduced by the Fourier expansion approximation. Within this approximation, the limitation of the proposed method is studied. The final section gives a summary and remarks.

\section{Model}

The following is a simple model exhibiting dynamical symmetry breaking:

$$\dot{\psi} = -V'(\psi) + H(t), \quad (2.1)$$

where $V(\psi)$ is a bistable potential with fourth-order nonlinearity [explicitly, $V(\psi) = -\psi^2/2 + \psi^4/4$], the prime denotes differentiation with respect to the argument, and $H(t)$ is the sinusoidal force $H(t) = h \cos(\Omega t)$ ($T = 2\pi/\Omega$, $h > 0$, and $\Omega > 0$). Depending on the invariance of Eq. (2.1) under the transformations $t \to t + T/2$ and $\psi \to -\psi$, the system possesses either the symmetry restoring oscillation (SRO) state satisfying $\psi(t) = -\psi(t + T/2)$ or the symmetry breaking one (SBO state).
Both the average of \( \psi(t) \) over one period, \( \int_0^T \psi(t)dt/T \), and the quantity

\[
A(t) = \psi(t) + \psi(t - T/2)
\]

vanish in the SRO state but do not vanish in the SBO state. The order parameter \( A(t) \) plays an important role in this study.

The phase diagram for the SRO and SBO states is shown in Fig. 3, where the solid curve, \( h = h_{\text{SR-SB}}(\Omega) \), represents the critical boundary line between the two states, and the other curves are treated in another context. For a sufficiently large magnitude and low frequency of the applied field, the SRO state is stable. As the amplitude \( h \) is decreased from such a state with the frequency \( \Omega \) fixed, the SRO-SBO transition occurs and, the SRO state is unstable while the SBO is stable. The boundary between the two states, \( h = h_{\text{SR-SB}}(\Omega) \), is obtained as \( h_{\text{SR-SB}} \approx \sqrt{(1 + 4\Omega^2)/6} \) from the lowest-order truncated Fourier expansion approximation.\(^{13,14,19}\) The results for a higher-order approximation have been obtained in Ref. 20).

The basic idea of the TDFC of stabilizing the SRO state in the case that it is unstable is described below. First, let us consider the case in which the control of the system Eq. (2.1) is implemented by the conventional TDFC scheme:

\[
\dot{\psi} = -V'(\psi) + H(t) - K_0 \left[ \psi(t) + \psi(t - T/2) \right],
\]

where \( H(t) = h \cos(\Omega t) \), \( V(\psi) \) is the bistable potential as defined above, and we have assumed a linear relationship in the conversion from the state variable \( \psi \) of system to the output signal. The last term is the feedback input, which acts to suppress the deviation from the SRO state with feedback gain \( K_0 \). Once the SRO state is stabilized, the last term or the order parameter vanishes, and the SRO state, which is the same state as that without control, is maintained with a small feedback signal.

Fig. 3. (i) Phase diagram for the SRO and SBO states in the absence of control. The boundary between the two states, \( h = h_{\text{SR-SB}}(\Omega) \), is denoted by solid curve, which is numerically obtained from Eq. (2.1). (ii) Critical boundary lines \( h = h_c(\Omega; K_0) \) for three different values of \( K_0 \) with \( K_0^{(m)} + \delta K_0; \delta K_0 = 0.0 \) (circles), 0.05 (squares), and \(-0.05 \) (diamonds). For \( h < h_c(\Omega; K_0) \), the SRO state is unstable. The broken curve represents the critical boundary line \( h = h_c^{(3)}(\Omega; K_0^{(m)}) \) obtained from the Fourier expansion approximation.
The next subsection describes the linear stability analysis of Eq. (2.3), which is essentially the Floquet problem.

2.1. Linear stability analysis

Let \( \psi_p(t) \) and \( \phi(t) \) be the unstable SRO orbit in the uncontrolled system and the deviation from it, respectively. Substituting \( \psi(t) = \psi_p(t) + \phi(t) \) into Eq. (2.3), the linearized equation for the deviation \( \phi(t) \) is obtained as

\[
\dot{\phi} = \epsilon(t) - K_0 [\phi(t) + \phi(t - T/2)],
\]

where \( \epsilon(t) = -V''(\psi)\psi_p(t) \). From the periodicity, \( \epsilon(t - T/2) = \epsilon(t) \), Floquet’s theorem asserts that the particular solution \( \phi(t) \) can be written as \( \phi(t) = Q(t)e^{\Gamma t} \), with the Floquet exponent \( \Gamma \equiv \lambda + i\omega \) (where \( \lambda \) and \( \omega \) are real numbers, i.e., \( \lambda, \omega \in \mathbb{R} \)), where \( Q(t) \) is a periodic function satisfying \( Q(t) = Q(t - T/2) \). In the present case, there is no other choice of \( \epsilon \) and \( \phi \) such that it obeys the equation of motion \( \dot{Q} = [\epsilon(t) - \lambda]Q \), where \( \epsilon \) is the average of \( \epsilon(t) \) over one period, and that it equals the stability index of the SRO state in the absence of control. Substituting this into Eq. (2.4), the characteristic equation for \( \Gamma \) is found to be

\[
D(\Gamma; \epsilon) \equiv \epsilon - \Gamma - K_0(1 + e^{-\Gamma T/2}) = 0. \tag{2.5}
\]

This equation has an infinite number of solutions for \( \Gamma \). However, most of the valuable information may be that regarding its behavior around the marginal point of the instabilities, \( \lambda \approx 0 \), in Eq. (2.5).

Let us focus our attention on the critical situation for the SRO state with regard to its stability. Suppose that \( \epsilon \) is a relevant parameter causing an instability of the SRO state by changing it. Then, \( \lambda \) and \( \omega \) can be regarded as functions of \( \epsilon \). Furthermore, let \( \epsilon_c \) be the critical point of the SRO state, where \( \lambda = 0 \) and \( \omega = \omega_c \) (or \( \Gamma = i\omega_c \equiv \Gamma_c \)); hereafter \( \omega_c \) is referred to as the critical frequency. Then, near the critical points, \( D(\Gamma; \epsilon) \) can be expanded as \( D(\Gamma; \epsilon) = D(\Gamma_c; \epsilon_c) + \delta D(\Gamma_c; \epsilon_c) \), with

\[
\delta D(\Gamma_c; \epsilon_c) \equiv D'(\Gamma_c; \epsilon_c)\delta \Gamma + \frac{\partial}{\partial \epsilon} D(\Gamma_c; \epsilon) \bigg|_{\epsilon = \epsilon_c} \delta \epsilon + \cdots, \tag{2.6}
\]

where \( \delta \epsilon = \epsilon - \epsilon_c \), \( \delta \Gamma = \delta \lambda + i \delta \omega \), \( D'(X; \epsilon) \equiv \partial D(\Gamma; \epsilon)/\partial \Gamma \big|_{\Gamma = X} \). From the definition, we have \( D(\Gamma_c; \epsilon_c) = 0 \). Now, its real and imaginary parts are written

\[
\epsilon_c = K_0(1 + \cos w), \tag{2.7}
\]
\[
w = (K_0^2 T/2) \sin w, \tag{2.8}
\]

where \( w = \omega_c T/2 \) is the scaled critical frequency. Unless \( D'(\Gamma_c; \epsilon_c) = 0 \), for the change of \( \epsilon \) from the critical point, the Floquet exponent \( \Gamma \) varies as

\[
|D'(\Gamma_c; \epsilon_c)|^2 \delta \Gamma \approx \kappa(\Gamma_c; \epsilon_c) \delta \epsilon, \tag{2.9}
\]

with

\[
\kappa(\Gamma_c; \epsilon_c) \equiv -D'(\Gamma_c; \epsilon_c)^* \frac{\partial}{\partial \epsilon} D(\Gamma_c; \epsilon) \bigg|_{\epsilon = \epsilon_c}, \tag{2.10}
\]
where the superscript $^*$ represents the complex conjugate. The real part of Eq. (2.9) yields
\[
\delta \lambda \propto \text{Re} \, \kappa(\Gamma_c; \epsilon_c) \delta \epsilon. \tag{2.11}
\]
In the present case, denoting the real part of $\kappa(\Gamma_c; \epsilon_c)$ as $\text{Re} \, \kappa(\Gamma_c; \epsilon_c) \equiv \kappa_r(\epsilon)$, we have
\[
\kappa_r(\epsilon) = 1 - (K_0 T/2) \cos \omega
\]
from Eq. (2.5). The relation (2.11) suggests that the domain of $\epsilon$ for the stable SRO state is limited by $\epsilon \leq \epsilon_c$, corresponding to $\kappa_r(\epsilon) \geq 0$.

Note that Eq. (2.8) has only the solution $\epsilon = 0$ for $K_0 < K_0^m$ ($K_0^m \equiv 2/T$), and other solutions which emerge beyond $K_0^m$. Below, we limit the range of $\epsilon$ to be positive, without loss of generality. For a given $K_0$ for $K_0 < K_0^m$, Eq. (2.7) leads to $\epsilon_c = 2K_0 \equiv \epsilon_c^P(K_0)$ from $\epsilon = 0$. This is the so-called (supercritical) pitchfork bifurcation point, or the pitchfork branch when we refer to it as a function of $K_0$, $\epsilon = \epsilon_c^P(K_0)$. From $\kappa_r(0) > 0$, the SRO state is stable for $\epsilon < \epsilon_c^H(K_0)$. For a given $K_0$ satisfying $K_0 > K_0^m$, among the multiple solutions of $(\epsilon_c, \omega)$ in Eqs. (2.7) and (2.8), that for which $\epsilon$ is in the range $0 < \omega < \pi$ gives the onset point of instability, which is another type of critical point, the so-called Hopf bifurcation point, $\epsilon_c = \epsilon_c^H(K_0)$. Similarly, let us call the curve represented by $\epsilon = \epsilon_c^H(K_0)$ the Hopf bifurcation branch. Here we address the following fact. For the range $K_0 > K_0^m$, the solutions \{$(\epsilon_c = \epsilon_c^k(K_0), \omega^k)$\} ($k$ distinguishes the different solutions), except that with $\omega$ in the range $0 < \omega < \pi$, give larger values of $\epsilon_c$ than $\epsilon_c^H(K_0)$, i.e., $\epsilon_c^H(K_0) < \epsilon_c^k(K_0) \leq \epsilon_c^P(K_0)$. Therefore, these solutions cannot be regarded as critical points and critical frequencies (modes), and should be regarded as meaningless solutions within the linear stability argument. Consequently, the instability for $K_0 > K_0^m$ is described by the Hopf-type instability, with $\omega$ being in the range $0 < \omega < \pi$. The SRO state is therefore stable for $\epsilon < \epsilon_c^H(K_0)$ from $\kappa_r(\epsilon) > 0$, where the inequality can be understood through the graph of $\text{Im} \, D(\Gamma_c; \epsilon_c)$, noting the definition $\kappa_r(\epsilon) = -\partial(\text{Im} \, D(\Gamma_c; \epsilon_c))/\partial \omega_c$. The bifurcation point $K_0 = K_0^m$, at which the Hopf branch emerges from the pitchfork branch, i.e., $\epsilon_c^P(K_0^m) = \epsilon_c^H(K_0^m)$, gives the optimum value of the feedback gain $K_0$. This implies that with this value, the upper limit of the domain of $\epsilon$ in which the SRO state is stable is maximized within the linear stability argument, since $\epsilon_c^P(K_0^m) \leq \epsilon_c^P(K_0^m)$. Equivalently, the unstable SRO state with $\epsilon$ in the range
\[
\epsilon \geq \epsilon_c^P(K_0^m) = 4/T = 2\Omega/\pi \tag{2.13}
\]
is never stabilized with the present control scheme.

In the present system, Eq. (2.1), it is difficult to stabilize the unstable SRO state in low frequency regime. Specifically, if we decrease $h$ with $\Omega$ fixed in the lower frequency regime, the stability index $\epsilon$ of the SRO state increases, and it is easier to break the linear stability condition of the SRO state, $\epsilon \leq 2\Omega/\pi$, under optimum control. As we consider the applications mentioned in the previous section, in many cases, it may be preferable to use a low frequency and small amplitude external driving force in order to maintain the system in a reliable condition. This condition, however, conflicts with the stability criterion for the controllability. This is the main
reason for considering another control scheme to stabilize the SRO state even in such a regime.

2.2. Numerical results

Numerical simulations of Eq. (2.3), and also other differential equation in this paper, were carried out with the fourth-order Runge-Kutta method with a time step \( \Delta t = T/N_T \) (\( N_T = 2^{18} \)), where memory size of \( N_T \) elements is used to retain the trajectory during the past one period.

First, let us consider the test of the relation (2.13) via the stability diagram of the SRO state spanned by \( \Omega \) and \( h \). Figure 3 shows three samples of the critical boundary line \( h = h_c(\Omega; K_0) \) for three different values of \( K_0 \) with \( K_0^m + \delta K_0 \) (\( \delta K_0 = 0, \pm 0.05 \)). In this numerical experiment, the SRO state is identified by estimating the average of the order parameter Eq. (2.2) with a threshold value as \( |A| < 10^{-3} \). Recall that \( K_0^m \) gives the optimum feedback gain. Therefore, for \( K_0 \neq K_0^m \), the relationship \( h_c(\Omega; K_0^m) < h_c(\Omega; K_0) \) should be satisfied; i.e., the region of the unstable SRO state under the control is minimized with its boundary line \( h = h_c(\Omega; K_0^m) \) in the parameter space spanned by \( \Omega \) and \( h \). The figure evidently depicts this result.

The figure also shows an approximate result for the critical boundary line, \( h = h_c^{(3)}(\Omega; K_0^m) \), which is derived with an approximation using the lowest-order truncation of the Fourier expansion. Its details are given in Appendix A. That appendix also gives the results of a further higher-order approximation for the critical boundary lines as \( h = h_c^{(a)}(\Omega; K_0^m) \) (\( a \in \{5, 7\} \) indicates the order of the approximation).

It is also worthwhile to consider the dynamical behavior in the region in which the system is not under control. Since the system has an infinite number of degrees of freedom coming from the delayed feedback term, the behavior is not limited to forced oscillation of frequency \( \Omega \). Figure 4 shows the bifurcation diagrams in the dynamics governed by Eq. (2.3) for three samples of \( K_0 \): (a) \( K_0^m - 0.1 \), (b) \( K_0^m \), and (c) \( K_0^m + 0.1 \) with \( \Omega = 1.0 \). These indicate a stroboscopic time series of the order parameter \( A(nT) \) \( (n \in \mathbb{Z}) \) for every step of \( h \) in both the cases that the parameter \( h \) is increased and decreased. Those directions are indicated by arrows. The transient process \( (2000 \times T) \) before the system settles into attractors is omitted. The attractors should be symmetric with respect to the horizontal zero-axis. The bifurcation diagrams in Fig. 4 imply that there are several coexisting states; that is, there are hystereses for the settled states depending on path in which \( h \) changes. A typical example is displayed in Fig. 4(b). For the case of decreasing \( h \), the SRO state continuously bifurcates to the SBO state. Contrastingly, for the case of increasing \( h \), a period-three state settles down on the right-hand side of the former bifurcation point. This is the case that the SRO state is linearly stable with respect to infinitesimal disturbance but may be unstable with respect to finite ones. We should note that the critical boundary lines of the SRO state, \( h = h_c(\Omega; K_0) \), as shown in Fig. 3, have been obtained in the case of decreasing \( h \), in accordance with the linear stability argument. From Figs. 4(a) and (c), we also confirm that the SRO state is destabilized via the pitchfork bifurcation for (a) \( K_0 < K_0^m \) and the Hopf bifurcation for (c) \( K_0 > K_0^m \).

Figure 5 displays an chaotic behavior observed at \( h = 0.245 \) (\( \Omega = 1.0 \), \( K_0 = 0 \).
Fig. 4. Bifurcation diagrams for the following parameter values: (a) $\delta K_0 = -0.1$, (b) $\delta K_0 = 0.0$, and (c) $\delta K_0 = 0.1$ with $\Omega = 1.0$. These plot the stroboscopic time series $A(nT)$ ($n \in \mathbb{Z}$) obtained from Eq. (2.3) by changing of $h$ in small steps. $h$ is changed in both increasing and decreasing directions, where the former dynamical state is retained in each step of $h$, and the directions are indicated by the arrows. The outer lines indicate the bifurcation diagram without control, i.e., that of Eq. (2.1), namely, the pitchfork bifurcation.

$K_0^m$), whose location in Fig. 4(b) is indicated by the thick arrow. The stroboscopically plotted points in the $A$-$\psi$ plane exhibit a nested structure, which often appears as a characteristic of strange attractors in dissipative dynamical systems. In fact, the Lyapunov exponent of the attractor has a small positive value (0.075); this implies a chaotic state.\(^{27}\)
\[ \Omega=1.0, \, h=0.245, \, \delta K_0=0.0 \]

Fig. 5. Strange attractor observed in the Poincaré plot in the plane spanned by \( \psi(nT) \) and \( \mathcal{A}(nT) \) \((n \in \mathbb{Z})\). The plot is generated by Eq. (2.3) with \( h = 0.245, \delta K_0 = 0.0, \) and \( \Omega = 1.0 \). This parameter set is included in Fig. 4(b), and the corresponding location is indicated by the thick arrow there. The square window is the magnification of the small square in the lower left of the attractor.

**§3. Proposed method**

How can we overcome the limitation of the TDFC implemented by Eq. (2.3)? Some extended methods to improve control have been proposed.\(^{2,28-32}\) The present paper, however, proposes another extended scheme as

\[
\begin{align*}
\dot{\psi} &= -V'(\psi) + H(t) - K_0 [\psi(t) + \psi(t - T/2)] - v, \\
\dot{v} &= \gamma_1 \left[ K_1 \left\{ \psi(t) + \psi(t - T/2) \right\} - v \right],
\end{align*}
\]

where the parameters \( K_0, \, K_1 \) and \( \gamma_1 \) are positive constants. In addition to Eq. (2.3), the new variable \( v(t) \), which plays the role of suppressing the time derivative of the deviation from the SRO state, is incorporated. With the variable \( v(t) \), Eq. (3.1) takes into account the memory effect of the control device under consideration, whose time scale is characterized by \( \gamma_1^{-1} \). Note that both \( v(t) \) and \( \mathcal{A}(t) \) vanish when the SRO state is realized.

### 3.1. Numerical results

First, we present the numerical evidence that the proposed method, Eq. (3.1), stabilizes the SRO state, especially in the parameter region where the previous method Eq. (2.3) fails. Figure 6 displays the bifurcation diagrams obtained from Eq. (3.1) for three values of \( K_0 \). Each of them shows that the SRO state is destabilized into the SBO states via the pitchfork bifurcation, and its bifurcation point on the \( h \)-axis decreases as \( K_0 \) is increased. In particular, at \( K_0 = 0.55 \), the SRO state is stabilized in the whole domain of \( h \). It is shown in the theoretical treatment given
Fig. 6. Bifurcation diagrams obtained from Eq. (3.1) with \( \Omega = 0.3, K_1 = 4.0, \gamma_1 = 2.0, \) and \( K_0 = 0.35, 0.45, 0.55. \) For each value of \( K_0 \) with different runs, the stroboscopic time series \( \mathcal{A}(nT) \ (n \in \mathbb{Z}) \) for a sufficiently large number \( n \) is plotted for every step of \( h. \) There is no hysteresis when \( h \) is increased or decreased.

Fig. 7. The trajectory of an SRO cycle (broken curve) and two trajectories of SBO cycles (solid curves) for the parameter values \( \Omega = 0.3 \) and \( h = 0.1. \) The former is unstable without control, and also unstable under control with the conventional scheme. This figure describes the control realized using Eq. (3.1) with \( K_0 = 0.55, K_1 = 4.0, \gamma_1 = 2.0. \)

below, when we choose sufficiently large values for \( \gamma_1 \) and \( K_1, \) the SRO state can be stabilized in the whole domain of \( h \) with a suitable choice of \( K_0. \)

Figure 7 plots the trajectories of the stable SBO cycles and the unstable SRO cycle at the parameter values \( \Omega = 0.3 \) and \( h = 0.1 \) in the absence of control. Although this SRO state cannot be stabilized by the conventional TDFC method with any choice of the control gain, the proposed method Eq. (3.1) reproduces it. Whether an SRO state numerically obtained with control coincides with the intrinsic unstable SRO state in the absence of the control can be checked by determining whether the time-reversed trajectory obtained from Eq. (2.1), constructed by reversing its time steps, follows the obtained trajectory backward starting from the last state of the former trajectory, because the unstable SRO state of Eq. (2.1) is stable under
Fig. 8. Transient process on the way to the SRO state. The solid and broken curves represent $A(t)$ and $v(t)$, respectively. The parameter values are the same as in Fig. 7.

time-reversed flow. Figure 8 plots the transient process on the way to reaching the SRO state. Both the variables $A(t)$ and $v(t)$ exhibit damped oscillation with a phase difference $\approx \pi/2$. This implies that these variables compensate for each other.

3.2. Linear stability analysis based on the adiabatic approximation

In the first treatment of the linear stability using Eq. (3.1), we consider the case in the absence of the memory effect, i.e., $\gamma_1 \to \infty$. Hereafter, this treatment is referred as the adiabatic approximation, for convenience. By taking the limit $\gamma_1 \to \infty$, Eq. (3.1) is reduced to

$$
\dot{\psi} \approx -V'(\psi) + H(t) - K_0 [\psi(t) + \psi(t - T/2)] - K_1 \{\dot{\psi}(t) + \dot{\psi}(t - T/2)\}. 
$$

(3.2)

This is equivalent to

$$(1 + K_1)\dot{\psi} = -V'(\psi) + H(t) - K_0 [\psi(t) + \psi(t - T/2)] - K_1 \dot{\psi}(t - T/2).$$

(3.3)

Here, it should be noted that taking the limit $\gamma_1 \to \infty$ is difficult, or the expression (3.3) is impossible, from the physical point of view, because any device will have some kind of memory or latency time. The finite $\gamma_1$ case is treated in the next subsection. There, we see a significant influence of the memory effect on the stability. However, this simplification helps us see how the limitation in the conventional TDFC method is overcome. Thus, this subsection treats the linear stability analysis of Eq. (3.3) before proceeding to that of Eq. (3.1).

Let $\psi_p(t)$ be a periodic orbit of the SRO state and $\phi(t)$ be a deviation from it. Substituting $\psi(t) = \psi_p(t) + \phi(t)$ into Eq. (3.3), linearized equation for $\phi(t)$ is obtained as

$$(1 + K_1)\dot{\phi} = \epsilon(t)\phi(t) - K_0 [\phi(t) + \phi(t - T/2)] - K_1 \dot{\phi}(t - T/2),$$

(3.4)

where $\epsilon(t) = -V''(\psi)_{\psi=\psi_p(t)}$. From the periodicity $\epsilon(t) = \epsilon(t + T/2)$, $\phi(t)$ can be written as $\phi(t) = Q(t)e^{\Gamma t}$ with the periodic function $Q(t) = Q(t + T/2)$ and
the Floquet exponent $\Gamma \equiv \lambda + i\omega$ ($\lambda, \omega \in \mathbb{R}$) (Floquet’s theorem). Given that the periodic function $Q(t)$ obeys \( \{1 + x_1(\Gamma)\} \dot{Q} = \{\epsilon(t) - \epsilon\} Q \), we obtain the characteristic equation for $\Gamma$ as

$$D(\Gamma; \epsilon) \equiv \epsilon - \Gamma - \chi_0(\Gamma) - \Gamma \chi_1(\Gamma) = 0,$$  \tag{3.5}

where $\chi_j(\Gamma) \equiv K_j(1 + e^{-\Gamma T/2})$ ($j = 0, 1$).

In the same way as in the previous section, we focus our attention here on the critical situation of the SRO state. Then let $\epsilon$ and $\epsilon_c$ be the parameters governing the stability of the SRO state and its critical value, respectively. From the definition, at the critical point, we have $\lambda = 0$ and $D(\Gamma_c; \epsilon_c) = 0$, where $\Gamma_c = i\omega_c$ and $\omega_c$ is the critical frequency. The real and imaginary parts of the latter equation are respectively written as

$$\epsilon_c = K_0(1 + \cos w) + \frac{2K_1}{T} w \sin w,$$  \tag{3.6}

$$[1 + K_1(1 + \cos w)] w = \frac{K_0 T}{2} \sin w,$$  \tag{3.7}

where $w = \omega_c T/2$ is the (scaled) critical frequency. Combining these, we also get

$$\epsilon_c = K_0 \frac{(1 + 2K_1)(1 + \cos w)}{1 + K_1(1 + \cos w)},$$  \tag{3.8}

where $w = w(K_0)$ is a function of $K_0$ with Eq. (3.7). This shows that $\epsilon_c$ can be regarded as a function of $K_0$ through $w$. This expression is sometimes used in the later calculations.

As given by Eq. (2.9) in the previous section, with $\delta \epsilon = \epsilon - \epsilon_c$, the Floquet exponent $\Gamma$ satisfies $\delta \Gamma \propto \kappa(\Gamma_c; \epsilon_c) \delta \epsilon$ if $D'(\Gamma_c; \epsilon_c) \neq 0$, where $\delta \Gamma = \delta \lambda + i\delta \omega$, and $\kappa(\Gamma_c; \epsilon_c)$ is defined in Eq. (2.10). This relationship leads to some useful, definitive results. Its imaginary part is rewritten as $\partial \omega/\partial \epsilon|_{\epsilon = \epsilon_c} \propto \text{Im} \kappa(\Gamma_c; \epsilon_c)$; in particular, for the pitchfork bifurcation point $\epsilon = \epsilon_c^P(K_0)$ ($\Gamma_c = 0$), this generally vanishes, i.e., $\partial \omega/\partial \epsilon|_{\epsilon = \epsilon_c^P(K_0)} = 0$, from the parities of $D(\Gamma_c; \epsilon_c)$ for the reversal of $\omega_c$. As mentioned in the previous section, its real part leads to the following: If $\epsilon_c$ is the critical point, the domain of the stable SRO state is limited by $\epsilon \leq \epsilon_c$, corresponding to $\text{Re} \kappa(\Gamma_c; \epsilon_c) \gtrless 0$. This statement is revisited in the next paragraph. In the present case, the real part of Eq. (2.10) yields

$$\text{Re} \kappa(\Gamma_c; \epsilon_c) = -\text{Re} D'(\Gamma_c; \epsilon_c)^* = -\frac{\partial \text{Im} D(\omega_c; \epsilon_c)}{\partial \omega_c}.$$  \tag{3.9}

Using the notation $\text{Re} \kappa(\Gamma_c; \epsilon_c) \equiv \kappa_\tau(w)$, $\kappa_\tau(w)$ is written as

$$\kappa_\tau(w) = 1 + K_1(1 + \cos w - w \sin w) - (K_0 T/2) \cos w.$$  \tag{3.10}

Now let us consider the solutions of $(\epsilon_c, w)$ in Eqs. (3.6) and (3.7) for a given $K_0$, with the other parameters fixed. Depending on the magnitude of $K_0$, the number of solutions for $w$ in Eq. (3.7) varies. Correspondingly, one or more solutions for
$\epsilon_c$ are allowed. In the latter case, as there are multiple Hopf branches, among the multiple solutions, one of the solutions relevant to the linear stability argument is that which belongs to the primitive Hopf critical branch $\epsilon = \epsilon_c^H(K_0)$ ($0 < \omega < \pi$). Let $\{[\epsilon_c^k(K_0), w^k(K_0)]|k = 0, 1, \ldots \}$ be a set of solutions identified as the critical points belonging to the critical branches, i.e., the primitive pitchfork and Hopf branches, for a given $K_0$. Then the domain(s) of $\epsilon$ for the stable SRO state is (are) bounded as $\{\epsilon \leq \epsilon_c^k(K_0)\}$, corresponding to $\kappa_{r}[i\omega_c^k(K_0); \epsilon_c^k(K_0)] \geq 0$ ($w^k = \omega_c^k T/2$), where $k$ distinguishes different critical points for a given $K_0$.

For a sufficiently small $K_0$ in Eq. (3.7), only the solution $w = 0$ exists, but other solutions emerge as $K_0$ is increased beyond a certain value. There are two possible ways that they can emerge. Figures 9(a) and (b) depict them. There, the functions on both sides of Eq. (3.7) are represented, and their intersections correspond to the solutions $w$. These show that, depending on $K_1$, there is a (a) continuous or (b) discontinuous bifurcation of the critical frequency $w \neq 0$ as $K_0$ is increased beyond the critical value. Here, let us estimate the critical value of $K_1$ separating these two ways in which $w \neq 0$ solutions emerge. Consider the expansion of the imaginary part of Eq. (3.5) around $w = 0$, i.e., $\kappa_{r}(0)w + \kappa''_{r}(0)w^3/6 \approx 0$ from Eq. (3.9). Then, in the neighborhood of the marginal point $K_0 = K_0^m \equiv 2(1 + 2K_1)/T$, given by $\kappa_{r}(0) = 0$, we have (a) a supercritical bifurcation for $K_1 < 1$ from $\kappa''_{r}(0) = K_0 T/2 - 3K_1 > 0$, and a (b) subcritical one for $K_1 > 1$. The upper panels of both Figs. 10(a) and (b) display typical bifurcation structures of the critical frequency for the cases (a)
$K_1 < 1$ and (b) $K_1 > 1$.

In the case (a), for a given $K_0$ satisfying $K_0 \leq K_0^m$, since $w = 0$, we have $\epsilon_c = 2K_0 \equiv \epsilon_c^P(K_0)$ from Eqs. (3.6) and (3.7), and $\kappa_\nu(0) \geq 0$ from Eq. (3.10). The SRO state is therefore stable for $\epsilon < \epsilon_c^P(K_0)$. For a given $K_0$ satisfying $K_0 > K_0^m$, letting $\epsilon_c^H(K_0)$ and $w$ be a solution of Eqs. (3.8) and (3.7) with the restriction $0 < w < \pi$, we have $\kappa_\nu(w) > 0$. Therefore, the SRO state is stable for $\epsilon < \epsilon_c^H(K_0)$. As mentioned above, the solutions $(\epsilon_c, w) = (2K_0, 0)$ and others with $w > \pi$ are not regarded as critical points in $K_0 > K_0^m$. The lower panel of Fig. 10(a) shows both critical boundary lines $\epsilon = \epsilon_c^P(K_0)$ as functions of $K_0$. It is found that the upper limit of $\epsilon$, by which the controllable zone for the SRO state is bounded, is maximized at $K_0 = K_0^m$. This gives the optimum feedback gain in the case (a).

The lower panel of Fig. 10(b) represents the critical boundary line as a function of $K_0$, which consists of the pitchfork and Hopf bifurcation branches: $\epsilon = \epsilon_c^P(K_0)$ ($K_0 < K_0^m$) and $\epsilon = \epsilon_c^H(K_0)$ ($K_0 > K_0^m$). There, the latter branch is further separated into two segments by its turning point, and those are denoted by different signs, as shown in the panel. Those segments also correspond to the two segments $w^\pm$ ($0 < w^- < w^+$) on the curve of the critical frequency in the upper panel. Let the position of the turning point in $K_0$ be $K_0 = K_0^\nu$. Using this, $\epsilon_c^H(K_0)$ is defined in the range $K_0^n < K_0 < K_0^m$, where $\kappa_\nu(w^-) < 0$, and $\epsilon_c^H(K_0)$ is defined for $K_0 > K_0^m$, where $\kappa_\nu(w^+) > 0$. At the point $K_0 = K_0^n$, the conditions $\kappa_\nu(w) = 0$ and Eq. (3.7)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig10.png}
\caption{Solutions $w$ (upper panels) and $\epsilon_c$ (lower panels) of Eqs. (3.6) and (3.7) as functions of $K_0$, which were obtained numerically from those equations for (a) $K_1 = 0.9$ and (b) $K_1 = 2.0$ with $\Omega = 0.3$. The horizontal axes, $K_0$, have the same scale in both upper and lower panels. The upper panels exhibit typical bifurcation structures, (a) supercritical ($K_1 < 1$) and (b) subcritical $K_1 > 1$ ones, of the critical frequency $w$ ($> 0$) as $K_0$ is changed around the marginal point $K_0 = K_0^m$. The lower panels exhibit typical critical boundary lines $\epsilon = \epsilon_c^H(K_0)$ as functions of $K_0$ for the cases (a) $K_1 < 1$ and (b) $K_1 > 1$. The critical boundary line consists of the pitchfork and Hopf bifurcation branches, $\epsilon = \epsilon_c^P(K_0)$ and $\epsilon = \epsilon_c^H(K_0)$. In the case (b), the Hopf bifurcation branch folds up over the domain $K_0^P \leq K_0 \leq K_0^m$, and it is further separated into two segments, $\epsilon = \epsilon_c^H(K_0)$, with the turning point $[K_0^n, \epsilon_c^H(K_0^n)]$. This turning point also corresponds to the marginal point of a pair of nontrivial critical frequencies $w^\pm$, which suddenly appear as $K_0$ is increased beyond $K_0^m$. The broken curve in the lower panel of the graph (b) represents the trace of the turning point as $K_1$ is increased.}
\end{figure}
are simultaneously satisfied for \(0 < w < \pi\). The point \(K_0^n\) must be in the range \(8K_1(1 + 2K_1)/[T(1 + K_1)^2] \leq K_0^n \leq K_0^m\). Consequently, the SRO state is stable in the ranges \(\epsilon < \epsilon_c^P(K_0)\) for \(K_0 < K_0^n\), \(\epsilon_c^H-(K_0) < \epsilon < \epsilon_c^P(K_0)\) and \(\epsilon < \epsilon_c^H+(K_0)\) for \(K_0^n \leq K_0 < K_0^m\), or \(\epsilon < \epsilon_c^H+(K_0)\) for \(K_0^m < K_0\).

In the case (a), we have determined the optimum feedback gain to \(K_0^m\) in the sense that the upper limit of the domain of \(\epsilon\) for the stable SRO state is maximized under the control with that choice. In the case (b), however, this sense is not suitable to define the optimum gain, but instead another criterion is necessary. In a practical sense, it might be safe to start the stabilization process from an unstable SRO state with a possibly small value of \(\dot{h}\) and to continue it by changing \(h\) while decreasing \(\dot{h}\) slowly up to a certain target state with static control gains. That is, we should take into account the pathway of the stabilization process. Below, we restrict ourselves to the pathway mentioned here. Then, from the lower panel of Fig. 10(b), in the stabilization process under the control with a given \(K_0\) in the range \(K_0^n < K_0 < K_0^m\), the SRO state that may be safely treated should be regarded to be limited by the critical state given as \(\epsilon = \epsilon_c^H+(K_0)\). It is also found that if \(K_0\) is set slightly below \(K_0^n\), the SRO state can be treated by changing \(h\) under the control with the static gains up to the upper limit \(\epsilon_c^P(K_0^n)\) in a single range of \(\epsilon\). Hence, we may choose the optimum feedback gain of \(K_0\) to be slightly below \(K_0^n\).

Before ending this subsection, we give a reason why the proposed method improves the stabilization, and show that there is no limitation on the stabilization in the proposed method within the adiabatic approximation. Recall that in the conventional method, the critical point \(\epsilon_c\) takes the maximum value \(\epsilon_c^P(K_0^m) = 2\Omega/\pi\) at the optimum feedback gain \(K_0^m = \Omega/\pi\). The restriction of that method is that we cannot move the maximum value of \(\epsilon_c\) without changing \(\Omega\). Contrastingly, in the proposed method, for a given \(K_1\) satisfying \(K_1 > 1\), if we set the feedback gain \(K_0\) slightly below \(K_0^n\), denoting it as \(K_0^n-\), the destabilization point is given by \(\epsilon_c^P(K_0^n-)\), and it can be further lifted up by increasing \(K_1\). This difference can also

![Fig. 11.](https://academic.oup.com/jpapa/article-abstract/11/4/987/5680)
be read from Fig. 9(b), in which the two types of curves correspond to the two sides of Eq. (3.7), and the appearance of the intersections, other than that of \( w = 0 \), implies the emergence of the Hopf instability. By increasing \( K_1 \), the curve corresponding to the left-hand side of Eq. (3.7) is further lifted up. This makes it possible to avoid the emergence of the contact point corresponding to the Hopf instability. Thus, a large value of \( K_0 \) up to \( K_0^n \), at which two curves contact, is allowed for stabilization. With the conventional method corresponding to the case \( K_1 = 0 \) there is no way to avoid the instability. This is a qualitative explanation of the improvement obtained with the proposed method.

To give a proof of the second statement in the previous paragraph, let us show how the turning point \([K_0^n, \epsilon^c_-(K_0^n)]\) behaves in the limit of large \( K_1 \). Figure 11 plots \( K_0^n \) and \( \epsilon^c_-(K_0^n) \equiv \epsilon^n_c \) as functions of \( K_1 \) for \( K_1 > 1 \). It is found that both quantities are monotonically increasing functions of \( K_1 \), and this means that any unstable SRO state can be stabilized under the control with a sufficiently large value of \( K_1 \). However, we should add the technical restriction that a larger value of \( K_1 \) requires a greater cost for the management of the time resolution; e.g., in the computation, a sufficiently smaller step size of the time update is required. The asymptotic forms of \( K_0^n \) and \( \epsilon^c_+ \) for \( K_1 \gg 1 \) are estimated as follows:

\[
K_0^n \approx \Omega \left( \sqrt{1 + 2K_1} - 2\pi^{-1} \right),
\]

\[
\epsilon^c_+ \approx \Omega \sqrt{1 + 2K_1}.
\]

These certainly show that \( \epsilon^n_c \) and also \( \epsilon^c_p(K_0^n) \) are monotonically increasing functions of \( K_1 \). These results are derived in Appendix B.

### 3.3. Linear stability analysis

Although the above treatment does not include the memory effect of Eq. (3.1), it elucidates how the control in the proposed method improves. However, it misses an important property which comes from the memory effect. We see below that this eventually imposes a limitation on the capability of this method. This subsection deals with Eq. (3.1) under simplification that is intended to extract the essential property coming from the memory effect.

Let \([\phi_1(t), \phi_2(t)]\) be an infinitesimal deviation from the SRO state \([\psi(t), v(t)] = [\psi_p(t), 0]\) in Eq. (3.1). A set of linearized equations for \([\dot{\phi}_1(t), \dot{\phi}_2(t)]\) is given by

\[
\begin{align*}
\dot{\phi}_1 &= \epsilon(t)\phi_1 - K_0 [\phi_1(t) + \phi_1(t - T/2)] - \phi_2, \\
\dot{\phi}_2 &= \gamma_1 \left[ K_1 \left\{ \dot{\phi}_1(t) + \dot{\phi}_1(t - T/2) \right\} - \phi_2 \right],
\end{align*}
\]

where \( \epsilon(t) = -V''(\psi)|_{\psi=\psi_p(t)} \). From the periodicity \( \epsilon(t + T/2) = \epsilon(t) \), the particular solution form for \([\phi_1(t), \phi_2(t)]\) may be written \([q_1(t), q_2(t)]e^{it} \) with the periodic functions \( q_k(t + T/2) = q_k(t) \) \((k = 1, 2)\) and \( \Gamma \equiv \lambda + i\omega \), where \( \lambda \) and \( \omega \) are the expansion rate and frequency. Substituting this into Eq. (??), we find

\[
\dot{M}\dot{Q} = \dot{R}(t)Q,
\]
with $Q \equiv [q_1(t), q_2(t)]^T$ and
\[
\dot{M} \equiv \begin{bmatrix} 1 & 0 \\ -\gamma_1 \chi_1(\Gamma) & 1 \end{bmatrix}, \quad \dot{R}(t) \equiv \begin{bmatrix} \epsilon(t) - \Gamma - \chi_0(\Gamma) & -1 \\ \gamma_1 \Gamma \chi_1(\Gamma) & -\gamma_1 - \Gamma \end{bmatrix},
\]
and
\[
\chi_k(\Gamma) \equiv K_k [1 + e^{-\Gamma T/2}] (k = 0, 1).
\]

Equation (3.14) represents a usual Floquet problem. The most promising method for solving this problem may be the perturbation approximation based on the Fourier expansion. From the $(T/2)$-periodicity of $\epsilon(t)$, $\dot{R}(t)$, and $Q$, they are expanded as follows:
\[
\epsilon(t) = \sum_{n \in \mathbb{Z}} \tilde{\epsilon}_n e^{2i\Omega t}, \quad \dot{R}(t) = \sum_{n \in \mathbb{Z}} \tilde{R}_n e^{2i\Omega t}, \quad Q(t) = \sum_{n \in \mathbb{Z}} \tilde{Q}_n e^{2i\Omega t}.
\]

Now let us approximate Eq. (3.14) with the three Fourier coefficients, $\{\tilde{R}_n\}_{n=0, \pm 1}$ and $\{\tilde{Q}_n\}_{n=0, \pm 1}$. Solving the set of equations relating $\{\tilde{Q}_n\}_{n=0, \pm 1}$ to $\tilde{Q}_0$, we find $\dot{D}\tilde{Q}_0 \approx 0$, with
\[
\dot{D} \equiv \dot{R}_0 - \dot{R}_1 - \frac{\tilde{\epsilon}}{\tilde{R}_0 - 2i\Omega \tilde{M}} \dot{R}_1 - \frac{\tilde{\epsilon}}{\tilde{R}_0 + 2i\Omega \tilde{M}} \dot{R}_0,
\]
where
\[
\dot{R}_0 \equiv \begin{bmatrix} \epsilon - \Gamma - \chi_0(\Gamma) & -1 \\ \gamma_1 \Gamma \chi_1(\Gamma) & -\gamma_1 - \Gamma \end{bmatrix}, \quad \epsilon \equiv \overline{\epsilon(t)}
\]
and $\dot{R}_{\pm 1}$ respectively have only one non-vanishing component ($\dot{R}_{\pm 1})_{1,1} = \tilde{\epsilon}_{\pm 1}$. From the last fact, the one non-vanishing component of the second term on the right-hand side of Eq. (3.17) is found to be
\[
\left( \frac{\tilde{\epsilon}}{\tilde{R}_0 - 2i\Omega \tilde{M}} \dot{R}_1 \right)_{1,1} = -\frac{\gamma_1 + \Gamma + 2i\Omega}{|\tilde{R}_0 - 2i\Omega \tilde{M}|} \sigma(\epsilon),
\]
where $\sigma(\epsilon) \equiv |\tilde{\epsilon}_1|^2$. Here, we have assumed that $|\tilde{\epsilon}_1|^2$ can be written as a function of $\epsilon$. The third term is also obtained with the reversal of sign $\Omega \to -\Omega$ in Eq. (3.19). Substituting these results into Eq. (3.17), we get
\[
\dot{D} = \begin{bmatrix} (\tilde{R}_0)_{1,1} + \beta(\Gamma; \epsilon) & -1 \\ \gamma_1 \Gamma \chi_1(\Gamma) & -\gamma_1 - \Gamma \end{bmatrix},
\]
where
\[
\beta(\Gamma; \epsilon) = \frac{2(\gamma_1 + \Gamma)[(\tilde{R}_0) - 4\Omega^2] - 8\Omega^2[\text{Tr} \tilde{R}_0 - \gamma_1 \chi_1(\Gamma)]}{(|(\tilde{R}_0) - 4\Omega^2|^2 + 4\Omega^2[\text{Tr} \tilde{R}_0 - \gamma_1 \chi_1(\Gamma)]^2)} \sigma(\epsilon).
\]

The condition that $\tilde{Q}_0$ has a nontrivial solution is $|\dot{D}| = 0$. Furthermore, this condition is expanded as
\[
D_0(\Gamma; \epsilon) \equiv |\tilde{R}_0| - (\gamma_1 + \Gamma) \beta(\Gamma; \epsilon) = 0.
\]
From the point of view of the present approximation, which is regarded as a series expansion in powers of $|\epsilon|^2$, we should regard $|\hat{R}_0| = 0$ in Eq. (3.22) as the zeroth order approximation and the second term as the first correction to it. Hence, substituting $|\hat{R}_0| = 0$ into Eq. (3.21), we have

$$\beta(\Gamma; \epsilon) \approx \frac{2\gamma_1^2 \chi_1(\Gamma)(\gamma_1 + \Gamma)}{4\Omega^2(\gamma_1 + \Gamma)^2 + [(\gamma_1 + \Gamma)^2 + \gamma_1^2 \chi_1(\Gamma)]^2} \sigma(\epsilon), \quad (3.23)$$

where the term $(\hat{R}_0)_{1,1}$ in Eq. (3.21) is replaced with $\gamma_1 \Gamma \chi_1(\Gamma)/(\gamma_1 + \Gamma)^{-1}$. Hereafter, we use this expression for $\beta(\Gamma; \epsilon)$.

Before proceeding to the analysis of Eq. (3.22), let us elucidate how the present treatment differs from the previous treatment in the adiabatic approximation with regard to their characteristic equations (3.5) and (3.22). First, in the limit $\Gamma \to \infty$, it is found that Eq. (3.22) agrees with Eq. (3.5), i.e., Eq. (3.5) is included in $|\hat{R}_0| = 0$. Second, in the previous treatments, although only the average of the time-dependent coefficient $\epsilon(t)$ contributes to the characteristic equation, in the present case, with the correction term (3.23), Eq. (3.22) includes the modulation effect of $\epsilon(t)$, i.e., $\sigma(\epsilon)$, in the form of coupling with the other feedback gain $K_1$. This suggests that the feedback gain $K_1$ enhances the influence of the detailed structures of the periodic orbit in the presence of the memory effect.

There are two ways to know the functional form of $\sigma(\epsilon)$, from the numerical data fitting and from the Fourier expansion approximation given by Eq. (A.6) in Appendix A. Both methods suggest that $\sigma(\epsilon)$ can be approximately written as

$$\sigma(\epsilon) \approx \sigma_0(\epsilon - \epsilon_m)^2, \quad (3.24)$$

with the two constants $\sigma_0 \approx 0.1 \sim 0.3$ and $\epsilon_m \approx 1.0$. The easiest method for the former approach may be the following. By solving Eq. (3.1) directly, we can obtain the pitchfork bifurcation points $\epsilon$ as functions of $K_0$, i.e., $\epsilon = \epsilon_p(K_0)$ for some runs with different values of $K_1$. Then, comparing these with the analytic expression $D_0(0; \epsilon) = 0$, i.e.,

$$\epsilon - 2K_0 + \frac{4\gamma_1 K_1 \sigma(\epsilon)}{4\Omega^2 + \gamma_1^2 (1 + 2K_1)^2} = 0, \quad (3.25)$$

we may determine the parameter values of Eq. (3.24) needed to fit them into a single functional form. However, we should note that Eq. (3.25) is limited to the case that $\gamma_1$ is sufficiently large. For the case of smaller $\gamma_1$, it is expected that an approximation incorporating higher order harmonics of $\epsilon(t)$ is required. In comparison with the previous result for the pitchfork bifurcation point $\epsilon_p(K_0) = 2K_0$, the value of the critical point $\epsilon$ given by Eq. (3.25) is smaller, and the stability of the SRO state is weaker. This implies that the memory effect contributes to lower the pitchfork bifurcation point.

Again, let us regard $\epsilon$ as a control parameter to handle the stability of the SRO state. The Floquet exponent $\Gamma$ is parameterized by $\epsilon$, and, at the critical point $\epsilon_c$, we have $\Gamma = \Gamma_c = i\omega_c$, where $\omega_c$ is the critical frequency. The critical point is given by

$$D(\Gamma_c; \epsilon_c) \equiv \epsilon_c + \beta(\Gamma_c; \epsilon_c) - \Gamma_c - \chi_0(\Gamma_c) - \frac{\gamma_1 \Gamma_c}{\gamma_1 + \Gamma_c} \chi_1(\Gamma_c) = 0, \quad (3.26)$$
where we have rewritten Eq. (3.22). Using the definitions \( \text{Re} D(T; \epsilon_c) \equiv D_{\tau}(T) \) and \( -(T/2) \text{Im} D(T; \epsilon_c) \equiv D_{\iota}(T) \), the real and imaginary parts of \( D(T; \epsilon_c) \) are written as

\[
D_{\tau}(T) = \epsilon_c + \beta_{\tau}(w; \epsilon_c) - \left\{ K_0 + \frac{2}{T} \frac{\gamma_1}{\gamma_1^2 + w^2} \right\} (1 + \cos w) - \frac{2}{T} \frac{\gamma_1^2 K_1}{\gamma_1^2 + w^2} w \sin w, \tag{3.27}
\]

\[
D_{\iota}(T) = -\frac{T}{2} \beta_{\iota}(w; \epsilon_c) + w \left[ 1 + \frac{\gamma_1^2 K_1}{\gamma_1^2 + w^2} (1 + \cos w) \right] - \left\{ \frac{K_0 T}{2} + \frac{\gamma_1^2 K_1 w^2}{\gamma_1^2 + w^2} \right\} \sin w, \tag{3.28}
\]

where \( w \equiv \omega_c T/2 \) (scaled critical frequency), \( \gamma_1 \equiv \gamma_1 T/2 \), \( \text{Re} \beta(T; \epsilon_c) \equiv \beta_{\tau}(w; \epsilon_c) \), and \( \text{Im} \beta(T; \epsilon_c) \equiv \beta_{\iota}(w; \epsilon_c) \). The quantities \( D_{\tau}(T) \) and \( D_{\iota}(T) \) have even and odd parity with respect to sign reversal, \( w \to -w \), respectively. For values of \( \epsilon \) near the critical point, writing \( \delta \epsilon = \epsilon - \epsilon_c \), the stability index \( \lambda \) behaves as

\[
\delta \lambda \propto \text{Re} \kappa(T; \epsilon_c) \delta \epsilon \tag{3.29}
\]

near its vanishing point, where \( \kappa(T; \epsilon_c) \) is defined by Eq. (2.10). The real part of \( \kappa(T; \epsilon_c) \), \( \text{Re} \kappa(T; \epsilon_c) \equiv \kappa_{\tau}(w) \), is rewritten as

\[
\kappa_{\tau}(w) = \left[ \frac{\partial D_{\tau}(T)}{\partial \epsilon_c} \frac{\partial D_{\iota}(T)}{\partial w} - \frac{\partial D_{\iota}(T)}{\partial w} \frac{\partial D_{\tau}(T)}{\partial \epsilon_c} \right]. \tag{3.30}
\]

As noted in the previous subsection, from Eq. (3.29), using the set of critical point(s) \( \{ \epsilon_c^k, w^k \} \) for a given \( K_0 \), where the superscript \( k \) distinguishes different critical points that belong to one critical branch, the domain(s) of \( \epsilon \) in which the SRO state is stable is (are) bounded by the inequality(ies) \( \{ \epsilon \equiv \epsilon_c^k \} \) corresponding to \( \kappa_{\tau}(i \omega_c^k; \epsilon_c^k) \gtrless 0 \).

Now, let us study how the root \( w \) (or \( \omega_c \)) of the equation \( D_{\iota}(T) = 0 \) bifurcates from the trivial one \( w = 0 \) as \( K_0 \) is increased. It should be noted that \( D_{\iota}(T) \) includes \( \epsilon_c \) via \( \beta_{\iota}(w; \epsilon_c) \), which implicitly depends on \( w \) through the relationship \( D_{\tau}(T) = 0 \). Therefore \( \epsilon_c \) must be regarded as a function of \( w \), i.e., \( \epsilon_c(w) \), under the constraint \( D_{\tau}(T) \equiv D_{\tau}(T; \epsilon_c) = 0 \). Differentiation of a given function \( G(T; \epsilon_c) \equiv G \), which depends on both \( w \) and \( \epsilon_c \), with respect to \( w \) is, therefore, defined as

\[
D_w G \equiv \frac{\partial G}{\partial w} + \frac{\partial \epsilon_c(w)}{\partial w} \frac{\partial G}{\partial \epsilon_c}.
\]

\[
= \left[ \frac{\partial D_{\tau}(T)}{\partial \epsilon_c} \right]^{-1} \left[ \frac{\partial D_{\iota}(T)}{\partial \epsilon_c} \frac{\partial \epsilon_c}{\partial w} - \frac{\partial D_{\iota}(T)}{\partial w} \frac{\partial \epsilon_c}{\partial \epsilon_c} \right] G. \tag{3.31}
\]

Note that we have \( D_w D_{\iota}(T) = 0 \), and, from Eq. (3.30),

\[
D_w D_{\iota}(T) = \left[ \frac{\partial D_{\tau}(T)}{\partial \epsilon_c} \right]^{-1} \kappa_{\tau}(w). \tag{3.32}
\]

Using these relations, we obtain the expansion of \( D_{\iota}(T) \) around \( w = 0 \) as

\[
\bar{\kappa}_{\tau}(0) w + \bar{\kappa}_{\tau}''(0) w^3 / 6 \approx 0, \tag{3.33}
\]
with

$$
\tilde{\kappa}_r(0) = \left[ \frac{\partial D_r(0)}{\partial \epsilon_c} \right]^{-1} \kappa_r(0)
$$

$$
= 1 + 2K_1 - \frac{K_0 T}{2} - \frac{T}{2} \frac{\partial \beta_1(w; \epsilon^P)}{\partial w} \bigg|_{w=0},
$$

(3.34)

$$
\tilde{\kappa}_r''(0) \equiv D_w^2 \left[ \frac{\partial D_r(\Gamma_c)}{\partial \epsilon_c} \right]^{-1} \kappa_r(w) \bigg|_{w=0} = \kappa_r(0) D_w^2 \left[ \frac{\partial D_r(\Gamma_c)}{\partial \epsilon_c} \right]^{-1} \kappa_r''(0),
$$

(3.35)

$$
\kappa_r''(0) \equiv D_w^2 \kappa_r(w) \bigg|_{w=0},
$$

(3.36)

and

$$
(\epsilon^P = \epsilon^P(K_0)) \text{ is the pitchfork bifurcation point defined by Eq. (3.25). When } \beta(\Gamma_c; \epsilon_c) \ll 2K_0, \epsilon^P(K_0) \text{ is approximately given by}
$$

$$
\epsilon^P(K_0) \approx 2K_0 - \frac{4\gamma_1 K_1 \sigma(2K_0)}{4\Omega^2 + \gamma_1^2(1 + 2K_1)^2}.
$$

(3.37)

The marginal point of the Hopf critical branch, \( K_0 = K^m_0 \), is given by the condition \( \tilde{\kappa}_r(0) = 0 \). From Eq. (3.35), we get

$$
\frac{K^m_0 T}{2} \approx 1 + 2K_1 - \frac{T}{2} \frac{\partial \beta_1[w; 4(1 + 2K_1)/T]}{\partial w} \bigg|_{w=0},
$$

(3.38)

where the last term depends on \( K_1 \) in a complicated manner. This shows that, in contrast to the adiabatic treatment, the memory effect causes the marginal point \( K^m_0 \) to shift slightly by an amount that depends on \( K_1 \), but that dependence is not simple. In the following treatment, the term \( \beta(\Gamma_c; \epsilon_c) \) is omitted, because the results obtained by incorporating it are not expected to add any distinctive ingredients. At the marginal point \( K_0 = K^m_0 \), we find two types of bifurcation, (a) a supercritical bifurcation and (b) a subcritical one, corresponding to the cases \( \kappa_r''(0) > 0 \) (\( K_1 < K^c_1 \)) and \( \kappa_r''(0) < 0 \) (\( K_1 > K^c_1 \)), respectively, where the critical value \( K^c_1 \) is defined with \( \kappa_r''(0) = 0 \) as

$$
K^c_1 \approx \left[ 1 + 12/\gamma_1 T + 48/\gamma_1 T^2 \right]^{-1}.
$$

(3.39)

In this calculation, the term \( \beta(\Gamma_c; \epsilon_c) \) has been ignored for simplicity. This is a good approximation for the large \( \gamma_1 \) regime.

Figure 12 displays typical types of behavior of the critical frequency \( w \) and critical boundary line \( \epsilon_c \) as functions of \( K_0 \) in the regime of relatively large \( \gamma_1 \). The upper and lower panels represent \( w \) and \( \epsilon_c \) for three typical values of \( K_1 \), namely 0.15 (\( < K^c_1 \)), 0.76 (\( \approx K^c_1 \)) and 1.75 (\( > K^c_1 \)).

In the case (a) (\( K_1 < K^c_1 \)), for a given \( K_0 \) satisfying \( K_0 \leq K^m_0 \), one stability condition of the SRO state is \( \epsilon < \epsilon^P_c(K_0) = 2K_0 \). For a given \( K_0 \) satisfying \( K_0 > K^m_0 \), letting \( \epsilon_c = \epsilon^H_c(K_0) \) be the Hopf bifurcation branch satisfying the condition \( D(\Gamma_c; \epsilon^H_c) = 0 \) (\( 0 < w < \pi \)), another is given by \( \epsilon < \epsilon^H_c(K_0) \). This is exactly the same as the previous arguments.
branch. These two segments also correspond to those of the critical frequencies $w$ and $\Omega = 0.3, \gamma_1 = 2.0$ and $\Omega = 0.3$ fixed. The solid curves are the numerical results obtained with $D(I_c; \sigma) = 0$ for these values of $K_1$, where the parameters in Eq. (3.24) are determined as $\sigma_0 = 0.3$ and $\epsilon_m = 1.0$, so that the hypothetical pitchfork bifurcation branch defined by Eq. (3.25) meets the results from Eq. (3.1). The broken curve represents the trace of the point satisfying $\kappa_c(w) = 0$ on the Hopf bifurcation branch, i.e., the turning point defined by $[K_0^H, \epsilon_c^{H-}(K_0^H)]$, as $K_1$ changes from $K_1^c$.

In the case (b) ($K_1 > K_1^c$), the domain of $\epsilon$ for the stable SRO state is bounded by the critical boundary lines consisting of the pitchfork bifurcation branch ($w = 0$), $\epsilon = \epsilon_c^P(K_0)$ [$K_0 \leq K_0^n$], and the Hopf bifurcation branch ($0 < w < \pi$). As shown in the lower panel in Fig. 12, the latter consists of two segments: $\epsilon = \epsilon_c^H(K_0)$, defined for $K_0^n < K_0 < K_0^w$ in which $\kappa_c(w) < 0$, and $\epsilon_c^{H+}(K_0)$, defined for $K_0 > K_0^n$ in which $\kappa_c(w) > 0$. The turning point $[K_0^n, \epsilon_c^{H-}(K_0^n)]$ is defined at $\kappa_c(w) = 0$ on the branch. These two segments also correspond to those of the critical frequencies $w^\pm$ ($0 < w^- < w^+ < \pi$), as shown in the upper panel of Fig. 12.

As discussed near the end of the previous subsection, choosing the feedback gain $K_0$ slightly below $K_0^n$ can be regarded as a safe choice for stabilizing the SRO state if we want to carry out the stabilization process by changing $\epsilon$ up to a target state while maintaining the system in the SRO state under control with a fixed feedback gain. In this case, the allowable target SRO state under control extends up to the state for which its value of $\epsilon$ is below $\epsilon_c^P(K_0^m)$. Carrying out this process further and proceeding through the narrow region $\epsilon_c^{H-}(K_0) < \epsilon < \epsilon_c^P(K_0)$ toward the region of largest $\epsilon$ around the corner near $[K_0^m, \epsilon_c^P(K_0^m)]$ by adjusting both $\epsilon$ and $K_0$, increase the possibility that the system will suffer a catastrophe. We now give an explanation for this. Although the boundary line for $K_1 = 1.75$ in the lower panel of Fig. 12 certainly indicates that the SRO state is linearly stable in the region $\epsilon_c^{H-}(K_0) < \epsilon < \epsilon_c^P(K_0)$ $(K_0^n < K_0 < K_0^w)$, this does not rule out the possibility that there are other coexisting states. Figure 13 presents a typical bifurcation diagram...
in the case that the feedback gain $K_0$ is slightly beyond $K_0^n \approx 0.380$, i.e., $K_0 = 0.39$ ($K_1 = 1.75, \gamma_1 = 2.0$). As predicted from the stability diagram in Fig. 12, the SRO state, which corresponds to the state in which the order parameter $A(nT)$ ($n \in \mathbb{Z}$) vanishes, appears twice in two separate domains of $h$, where $h$ and $\epsilon$ are regarded to have a one-to-one correspondence. In the bifurcation diagram, near the right edge of the middle window, however, we find that the system exhibits a discontinuous transition to the SRO state from another one as $h$ is decreased, despite the fact that the linear stability analysis predicts a continuous bifurcation. Such a discontinuity is regarded as dangerous from a technological point of view.

In the adiabatic treatment, it is found that the arc around the turning point $[K_0^n, \epsilon_c^K(K_0^n)]$ in the Hopf bifurcation branch moves in the upper right direction in the $K_0-\epsilon$ space as $K_1$ increases [see Fig. 10(b)]. In the present case, however, the memory effect characterized by $\gamma_1$ tends to obstruct such motion of the turning point. Figure 14 plots critical boundary lines as functions of $K_0$ in a case of relatively small $\gamma_1$ case, $\gamma_1 = 0.5$, for three different values of $K_1$, with $\Omega = 0.3$ fixed. The solid curves and different kinds of symbols correspond respectively to the approximation results (solid curves) obtained from Eq. (3.26) and the numerical results (symbols) of Eq. (3.1) with $K_1 = 0.387$, [corresponding to $\approx K_1^c$ in the approximation result Eq. (3.40)], $2.5 (> K_1^{c2})$, where $K_1^{c2}$ is defined in the next paragraph, and $3.8$. Reflecting a departure of $\gamma_1$ from the range of validity of the approximation, the approximation results (solid curves) significantly differ from the numerical results, but the former still provide a hint for explaining the latter. For the constants in Eq. (3.24), $\sigma_0 = 0.1$ and $\epsilon_m = 1.0$ are used for convenience. The numerical results for the stability diagram show that the maximum value of $\epsilon_c$ for the optimum or safe value of $K_0$ begins to decrease as $K_1$ passes beyond a certain value, c.f., the
squares ($K_1 = 2.5$) and triangles ($K_1 = 3.8$) in the graph. The corresponding approximation results provide a qualitative explanation for this retreat behavior regarding stability. As $K_1$ rises beyond a certain value, the upper part of the Hopf bifurcation branch $\epsilon = \epsilon_c^{H-}(K_0)$ passes through the pitchfork branch, as seen from its behavior in the sequence of solid curves in the graph. As a result, the motion of the turning point $[K_0^n, \epsilon_c^H(K_0^n)]$ changes to being in the upper-left direction from the upper-right direction in the $K_0-\epsilon$ space; the broken curve represents the trace of the turning point as $K_1$ is increased. This eventually suppresses the expansion of the domain of $\epsilon$ in which the SRO state is stable.

Because the retreat is followed by the appearance of a region in which $\epsilon_c^P(K_0) < \epsilon_c^{H-}(K_0)$ on $K_0$ below $K_0^m$, we can characterize the beginning of the retreat based on the behavior of the quantity $[\epsilon_c^{H-}(K_0) - \epsilon_c^P(K_0)]$ near $K_0 = K_0^m$. Below, we consider $K_1 = K_1^{c2}$ to be the critical point of $K_1$ at which $\partial[\epsilon_c^{H-}(K_0) - \epsilon_c^P(K_0)]/\partial K_0|_{K_0=K_0^m}$ vanishes. Near the marginal point of the Hopf bifurcation branch, for $\delta K_0 = K_0 - K_0^m < 0$, we have

$$w^2 \approx -\frac{6\tilde{\kappa}_r(0)}{\tilde{\kappa}_r''(0)} \approx \frac{3T}{\tilde{\kappa}_r''(0)} \delta K_0$$

(3.41)

from Eq. (3.33), where $\tilde{\kappa}_r''(0) < 0$ ($K_1 > K_1^c$). Using this and the expansion of Eq. (3.27) up to order $w^2$, we get

$$\epsilon_c^{H-}(K_0) - \epsilon_c^P(K_0) = -3\frac{1-K_1/K_1^{c2}}{\tilde{\kappa}_r''(0)} \delta K_0,$$

(3.42)

with $K_1^{c2} = \gamma_1 T/8$, where the term $\beta(\Gamma_c; \epsilon_c)$ has been ignored for simplicity. Thus,
for the range $K_1^c < K_1^{c2} < K_1$, we have a domain of $K_0 (< K_0^m)$ in which $\epsilon_c^P(K_0) < \epsilon_c^{H-}(K_0)$ holds.

Recall that the domain of $\epsilon$ in which the SRO state is stable under control with relatively large $\gamma_1$ is bounded by the conditions $\epsilon < \epsilon_c^P(K_0)$, $\epsilon_c^{H-}(K_0) < \epsilon < \epsilon_c^P(K_0)$ and $\epsilon < \epsilon_c^{H+}(K_0)$ for a given $K_0$. However, the second set of inequalities are not effective when $K_1$ is much larger than $K_1^{c2}$. The domain of $\epsilon$ for the SRO state is, therefore, limited by the first and third conditions, and the optimum feedback gain is given by the point of $K_0$ (which we denote as $K_0^o$) at which the two critical lines, $\epsilon = \epsilon_c^P(K_0)$ and $\epsilon = \epsilon_c^{H+}(K_0)$, intersect. It is expected that the point $K_0 = K_0^o$ decreases as the turning point $[K_0^m, \epsilon_c^{H-}(K_0^m)]$ goes backward. Thus, we can say that the capability for stabilizing the SRO state declines or is suppressed for values of $K_1$ above $K_1^{c2}$, and the effective range of $K_1$ is thus limited below $K_1^{c2}$.

§4. Summary

In this paper we have discussed a method for stabilizing the unstable SRO state in a periodically driven bistable system with a potential with fourth-order nonlinearity. First, a linear stability analysis of the system with the conventional TDFC method was presented. Then its limitation regarding control in the low frequency and low amplitude regime of the external force was shown. In the main part of the paper, a new method designed to overcome this limitation was proposed, and its properties with regard to the linear stability were elucidated. In the proposed method, in addition to the original time-delayed feedback input term, the time derivative of that term is also employed to stabilize the UPO. The linear stability analysis of the proposed method was carried out in two steps.

The first analysis treats the model simplified by the adiabatic approximation. The remarkable features of the proposed method are as follows. (i) With the additional feedback term characterized by the parameter $K_1$, by increasing $K_1$, it is possible to rise the marginal point of the Hopf bifurcation branch, which is denoted by $K_0^m$. (ii) There are two cases in which the pitchfork and the Hopf bifurcation branches are connected: supercritical and subcritical types. The latter type occurs in the case that $K_1$ is larger than a critical value. Hence, it is expected that we necessarily encounter the latter case when the Floquet exponent of the UPO, $\epsilon$, in the absence of control is greater. In the latter case, another characteristic value of the feedback gain $K_0$, which is denoted by $K_0^n$, and which corresponds to the turning point of the Hopf bifurcation branch, plays an important role. The results of this paper suggest that a safe choice of the parameter $K_0$ is one for which its magnitude is slightly smaller than $K_0^n$. Since $K_0^n$ increases as $K_1$ increases, such a choice ensures that the domain of $\epsilon$ in which the SRO state is stable expands.

The second analysis treats the memory effect of the model within the truncated Fourier expansion approximation. It was shown that as the characteristic time scale of the memory effect increases, both the pitchfork and Hopf bifurcation branches are significantly distorted in comparison with the adiabatic limit. This can be interpreted as the memory effect has a slippage contribution for the feedback inputs. One of the remarkable points is that the pitchfork bifurcation branch is no longer given by
a simple linear relation between the stability index $\epsilon$ and the feedback gain $K_0$, but depends on the details of the periodic orbit in a complex way. Another remarkable point is that, due to the overhanging of the Hopf bifurcation branch as the control parameter $K_1$ is increased beyond a certain value, the turning point of the branch is eventually drawn back, and the capability for stabilizing the SRO state decreases. We have estimated the critical point of $K_1$, $K_{c2}^{0}$, above which this decrease begins, from the geometrical features around the connection point between the pitchfork and Hopf bifurcation branches, and we identified the effective range of $K_1$ as that satisfying $K_1 < K_{c2}^{0}$.

Finally, we mention some remaining problems. Consideration of the robustness with respect to noisy disturbances, which is obviously quite important, remains for a future study. Investigations for systems with many degrees of freedom also remain.

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Appendix A

--- Fourier Expansion Approximation ---

One purpose of this section is to derive the expression for the critical boundary line, $h = h_c(\Omega; K^{m}_0)$, as shown in Fig. 3, corresponding to the case of equality in the condition (2.13), which describes the parameter domain where the SRO state is unstable under the control given by the conventional TDFC method (2.3) with the optimum feedback gain $K_0 = K^{m}_0$. The second purpose is to confirm the validity of the relation (3.24) in §3.3.

Both questions are treated using an approximation method based on the Fourier expansion. First, we need information concerning the orbit of the SRO state. The SRO state $\psi_p(t)$ can be represented by a Fourier expansion series with the fundamental frequency $\Omega$ of the external force as

$$\psi_p(t) = \sum_{n \in \mathbb{Z}_o} \tilde{\psi}_n e^{i n \Omega t}, \quad (A.1)$$

where $\mathbb{Z}_o$ represents the set of odd integers, and $\tilde{\psi}_n$ is the Fourier coefficient of the $n$-th order harmonic. From the definition of the SRO state, we have $\tilde{\psi}_n = 0$ for even $n$. Also, because it is real, we have $\tilde{\psi}_n = \tilde{\psi}_n^*$. From Eq. (2.1), the Fourier coefficients satisfy

$$-in\Omega\tilde{\psi}_n + \tilde{\psi}_n - \sum_{n_1, n_2, n_3 \in \mathbb{Z}_o} \delta_{n, n_1 + n_2 + n_3} \tilde{\psi}_{n_1} \tilde{\psi}_{n_2} \tilde{\psi}_{n_3}$$
\[+ \frac{\hbar}{2} (\delta_{n,1} + \delta_{n,-1}) = 0, \quad (A.2)\]

where \(\delta_{i,j}\) is the Kronecker delta: \(\delta_{i,j} = 1\) if \(i = j\) and \(\delta_{i,j} = 0\) if \(i \neq j\).

From the definition \(\epsilon(t) = -V''(\psi)|_{\psi=p(t)} = 1 - 3\psi_p(t)^2\), the Fourier coefficient \(\tilde{\epsilon}_n\) defined in Eq. (3.16) is written
\[\tilde{\epsilon}_n = \delta_{n,0} - 3 \sum_{n_1,n_2 \in \mathbb{Z}_0} \delta_{n_1+n_2,2n} \tilde{\psi}_{n_1} \tilde{\psi}_{n_2}. \quad (A.3)\]

This yields
\[\epsilon = \tilde{\epsilon}_0 = 1 - 6 \sum_{k=1}^{\infty} |\tilde{\psi}_{2k-1}|^2, \quad (A.4)\]
\[|\tilde{\epsilon}_1|^2 = 9 \sum_{n_1,n_2 \in \mathbb{Z}_0} \tilde{\psi}_{n_1} \tilde{\psi}_{-n_2} \tilde{\psi}_{2-n_1} \tilde{\psi}_{-2+n_2}. \quad (A.5)\]

If we retain only the coefficients \(\tilde{\psi}_{\pm 1}\) in the above expansions, we have
\[\sigma(\epsilon) \equiv |\tilde{\epsilon}_1|^2 \approx (\epsilon - 1)^2/4. \quad (A.6)\]

We find that this expression does not contradict Eq. (3.24), and this answers the second question mentioned above.

In §2.1, from Eq. (2.13), we found that in the domain of \(\epsilon\) satisfying \(\epsilon \geq 2\Omega/\pi\), the SRO state is never stabilized in the conventional TDFC, even with any choice of the feedback gain. Noting that \(\epsilon < 1\) from Eq. (A.4), for a given \(\Omega\) satisfying \(\Omega \geq \pi/2\), that relation is never satisfied. Therefore for such values of \(\Omega\), the system can be controlled. Contrastingly, for \(\Omega < \pi/2\), there is a domain of \(\epsilon\) in which the system cannot be controlled. The critical boundary representing the limit of control is thus rewritten as
\[\sum_{k=1}^{\infty} |\tilde{\psi}_{2k-1}|^2 = \frac{1}{6} \left(1 - \frac{2\Omega}{\pi}\right). \quad (\Omega < \pi/2) \quad (A.7)\]

This can be read as a relationship between \(h\) and \(\Omega\), i.e., \(h = h_c(\Omega; K^m_0)\). From Eq. (A.2), it is found that the Fourier coefficient \(\tilde{\psi}_n\) is a function of \(\Omega\) and \(h\).

Below, Eqs. (A.2) and (A.7) are approximately solved with their truncated expansions. The following describes each step of the truncated expansions at \(O(|\tilde{\psi}_1|^\alpha)\) \((\alpha = 3, 5, 7, 9)\) for Eqs. (A.2) and (A.7).

1. \(O(|\tilde{\psi}_1|^3)\) truncated equations: The equation (A.7) is reduced to
\[|\tilde{\psi}_1|^2 = \frac{1}{6} \left(1 - \frac{2\Omega}{\pi}\right). \quad (A.8)\]

Then, from Eq. (A.2), the equation for \(\tilde{\psi}_1\) is reduced to
\[\left[-i\Omega + 1 - 3|\tilde{\psi}_1|^2\right] \tilde{\psi}_1 + \frac{h}{2} = 0. \quad (A.9)\]
2. $O(|\tilde{\psi}_1|^5)$ truncated equations: The equation (A.7) is reduced to Eq. (A.8). Then, from Eq. (A.2), the equations for $\tilde{\psi}_n$ $(n = 1, 3)$ are reduced to

$$
\begin{align*}
- i\Omega + 1 - 3|\tilde{\psi}_1|^2 - 6|\tilde{\psi}_3|^2 \tilde{\psi}_1 - 3\tilde{\psi}_1^2 \tilde{\psi}_3 + \frac{h}{2} &= 0, \\
-3i\Omega + 1 - 6|\tilde{\psi}_1|^2 \tilde{\psi}_3 - 3\tilde{\psi}_1 \tilde{\psi}_5 &= 0.
\end{align*}
$$

(A.10) (A.11)

3. $O(|\tilde{\psi}_1|^7)$ truncated equations: The equation (A.7) is reduced to

$$
|\tilde{\psi}_1|^2 + |\tilde{\psi}_3|^2 = \frac{1}{6} \left( 1 - \frac{2\Omega}{\pi} \right).
$$

(A.12)

Then, from Eq. (A.2), the equations for $\tilde{\psi}_n$ $(n = 1, 3, 5)$ are reduced to

$$
\begin{align*}
- i\Omega + 1 - 3|\tilde{\psi}_1|^2 - 6|\tilde{\psi}_3|^2 \tilde{\psi}_1 - 3\tilde{\psi}_1^2 \tilde{\psi}_3 + \frac{h}{2} &= 0, \\
-3i\Omega + 1 - 6|\tilde{\psi}_1|^2 \tilde{\psi}_3 - 3\tilde{\psi}_1 \tilde{\psi}_5 &= 0, \\
-5i\Omega + 1 - 6|\tilde{\psi}_1|^2 \tilde{\psi}_5 - 3\tilde{\psi}_1^2 \tilde{\psi}_3 &= 0.
\end{align*}
$$

(A.13) (A.14) (A.15)

The actual expansion of Eq. (A.2) has been carried out with the help of an algebraic manipulation program.

Let $h = h^{(3)}_c (\Omega; K_0^m)$ be the critical boundary line in $\Omega-h$ space in the $O(|\tilde{\psi}_1|^3)$ approximation. The equation (A.9) and the condition (A.8) lead to

$$
h^{(3)}_c = \sqrt{\frac{1}{6} \left( 1 - \frac{2\Omega}{\pi} \right) \sqrt{1 + \frac{4}{\pi} \Omega + \left( 4 + \frac{4}{\pi^2} \right) \Omega^2}}.
$$

(A.16)

with $\Omega < \pi/2$. Similarly, let $h = h^{(5)}_c (\Omega; K_0^m)$ be the critical boundary line in the $O(|\tilde{\psi}_1|^5)$ approximation. The equation (A.10) gives $\tilde{\psi}_3 = (-3i + 2/\pi)^{-1} \tilde{\psi}_1^3/\Omega$, and Eq. (A.11) leads to

$$
h^{(5)}_c = \sqrt{\frac{1}{6} \left( 1 - \frac{2\Omega}{\pi} \right) \left| -2i\Omega + 1 + \frac{2\Omega}{\pi} - \frac{(1 - 2\Omega/\pi)^2}{6(-3i + 2/\pi)\Omega} \right|}.
$$

(A.17)

with $\Omega < \pi/2$.

In the approximation of the order $O(|\tilde{\psi}_1|^7)$, we apply the reduced condition (A.8) for $|\tilde{\psi}_1|^2$ in the first term of the left-hand side of Eqs. (A.14) and (A.15) in order to keep the approximation at $O(|\tilde{\psi}_1|^7)$. Then, solving Eqs. (A.15) and (A.14), we get $\tilde{\psi}_5 = 3\tilde{\psi}_1^2 \tilde{\psi}_3/L_5$ with $L_5 = (-5i + 2/\pi)\Omega$ and $\tilde{\psi}_3 = \tilde{\psi}_1^3/L_3$ with

$$
L_3 = (-3i + 2/\pi)\Omega - \frac{(1 - 2\Omega/\pi)^2}{4L_5}.
$$

(A.18)

Combining these results with Eqs. (A.12) and (A.13), we obtain the transition curve $h = h^{(7)}_c (\Omega; K_0^m)$ in the $O(|\tilde{\psi}_1|^7)$ approximation as

$$
h^{(7)}_c = 2|\tilde{\psi}_1| \left( -i + 2/\pi \right)\Omega + 3|\tilde{\psi}_1|^2 - \frac{3|\tilde{\psi}_1|^4}{L_3}.
$$

(A.19)
where $|\tilde{\psi}_1|$ is determined by solving

$$
|\tilde{\psi}_1|^2 + \frac{|\tilde{\psi}_1|^6}{|L_3|^2} = \frac{1}{6} \left( 1 - \frac{2\Omega}{\pi} \right),
$$

(A.20)

Appendix B

Asymptotic Forms of $K_0^n$ and $c^n_c$ in the Case $K_1 \gg 1$

In the following, we present an algebraic calculation for the asymptotic form of $K_0^n$ and the corresponding value of $c_c^n$, in the case $K_1 \gg 1$, i.e., the derivation of the results Eqs. (3.11) and (3.12) in §3.2. At $K_0 = K_0^n$, Eqs. (3.7) and $\kappa_r(w) = 0$ are simultaneously satisfied, where $\kappa_r(w)$ is defined at Eq. (3.10). Then, Eqs. (3.7) and $\kappa_r(w) = 0$ yield

$$
1 + \cos w = \frac{b - \sqrt{b^2 - c}}{2K_1^2},
$$

(B.1)

with $b = K_0^n T(1 + K_1)/2 - 2K_1$ and $c = 4K_1^2(1 + K_0^n T/2)$. Here we have assumed the asymptotic form of $K_0^n$ as $K_0^n T/2 \to O(K_1^z)$ ($z > 0$) in the limit $K_1 \to \infty$, and therefore we have $b \to \infty$ and $w \to \pi$ as $K_1 \to \infty$. The validity of this assumption is verified below. Substituting Eq. (B.1) into Eq. (3.8), we get

$$
c_c^n = \frac{K_0^n (1 + 2K_1)}{K_1} \frac{b - \sqrt{b^2 - c}}{2K_1 + b - \sqrt{b^2 - c}},
$$

(B.2)

The factor $\sqrt{b^2 - c}$ is approximated as follows:

$$
\sqrt{b^2 - c} = \sqrt{\alpha K_1^2 + 2\beta K_1 + \gamma},
$$

where $\alpha \equiv (K_0^n T/2)^2 - 4K_0^n T$, $\beta \equiv (K_0^n T/2)^2 - K_0^n T$, and $\gamma \equiv (K_0^n T/2)^2$, is approximated as follows:

$$
\sqrt{b^2 - c} \approx \sqrt{\alpha (K_1 + \beta/\alpha)^2 + \gamma - \beta^2/\alpha}
$$

$$
\approx \sqrt{\alpha} K_1 + \beta/\sqrt{\alpha} - \frac{2}{1 + K_1^2} \alpha K_1 + \beta/\alpha + O \left( (K_0^n K_1)^{-1} \right)
$$

$$
\approx \left( \frac{K_0^n T}{2} - 4 \right) K_1 + \frac{K_0^n T}{2} \frac{2K_1}{1 + K_1} + O \left[ (K_0^n)^{-1} K_1 \right].
$$

(B.3)

From this and Eq. (B.1), we get

$$
1 + \cos w \approx \frac{1}{1 + K_1} + O \left( (K_0^n K_1)^{-1} \right),
$$

(B.4)

$$
\sin w \approx \frac{\sqrt{1 + 2K_1}}{1 + K_1},
$$

(B.5)

$$
w \approx \pi - \frac{\sqrt{1 + 2K_1}}{1 + K_1}.
$$

(B.6)
Substituting these results into Eq. (3-7), we get Eq. (3-11). Similarly, from Eqs. (B-2), (B-3), and (3-11), we also get Eq. (3-12). These results give the order estimations $K'_n \sim O(K_1^{1/2})$ and $\epsilon_n \sim O(K_1^{1/2})$ for $K_1 \gg 1$, and certainly confirm the validity of $K'_n T/2 \rightarrow O(K_1^z)$ ($z > 0$) mentioned above.

References

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