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Gyroscopes simply explained with Coriolis pseudotorques

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I. INTRODUCTION

One of the most amusingly counterintuitive encounters with physics is to watch a gyroscope that is supported only at one end as it slowly rotates in a horizontal plane and defies gravity. The configuration is shown in Fig. 1. The gyroscope is modeled as a negligible-mass shaft with a massive disk at one end, and is supported only at the other end. The weight of the disk results in a gravitational torque in the positive x direction. The intuitive expectation is that the unsupported disk will fall. The falling disk, after all, would create angular momentum about the positive x direction, a change in angular momentum required by the gravitational torque.

But the disk does not fall. Rather, the gyroscope, shaft, and disk[1] rotate ("precess") with angular velocity $\mathbf{\Omega}$. The standard explanation is given in terms of the gyroscope angular momentum $\mathbf{J} = I\boldsymbol{\omega}$, where I is the gyroscope moment of inertia ("rotational inertia") about its shaft, and $\boldsymbol{\omega}$ is the angular velocity of the disk about the shaft. The gravitational torque drives a rate of change of angular momentum that results not in the disk falling, but in $\mathbf{J}$ changing direction

$$\boldsymbol{\tau}_{\rm grav} = d\mathbf{J}/dt = \mathbf{\Omega} \times \mathbf{J} = \Omega\omega I\hat{\mathbf{x}}. \tag{1}$$

Here and below, boldface symbols indicate vectors, nonboldface equivalents are their magnitudes, and symbols like $\hat{\mathbf{x}}$ denote unit vectors in coordinate directions.

This analysis is simple and well known, but does not really give an intuitive explanation of the counterintuitive failure of the disk to fall. Such an explanation has, happily, already been given more than half a century ago, by Barker[2] in the pages of this journal; a much more recent version of this explanation has appeared in which VPYTHON simulations may offer addtional clarity.[3]

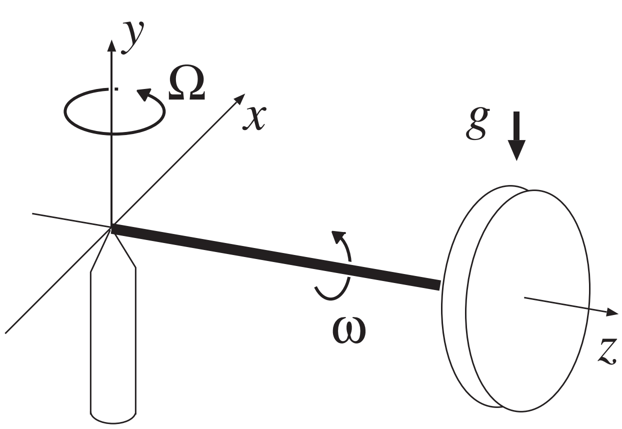

Fig. 1. A gyroscope with a shaft supported only at the end opposite the massive disk. Gravitational torque in the positive x direction drives a precession of the shaft with an angular velocity Ω in the positive y direction.

II. THE CORIOLIS PSEUDOTORQUE

Here, I would like to give another explanation, another viewpoint, that is not as direct as those cited above,[2,3] but that may be more appealing than the standard nonintuitive torque/angular momentum approach. It is, in any case, interestingly different.

This viewpoint requires only that we look at the configuration in a frame that is rotating with the precessing gyroscope. In this frame, the configuration differs from that in Fig. 1 only in the absence of precession; the only motion is the rotation of the disk around the shaft. In this reference frame, the gyroscope shaft remains in the z direction. The angular momentum $I\boldsymbol{\omega}$ is therefore constant in time and we face a new puzzle: Why doesn't the gravitational torque result in some change?

The answer starts with the expression for the acceleration $\mathbf{a}_{\rm rot}$ of a mass point in a frame rotating at angular velocity $\mathbf{\Omega}$, our new reference frame

$$\mathbf{a}_{\rm rot} = \mathbf{a}_{\rm in} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R}) - 2\mathbf{\Omega} \times \mathbf{v}_{\rm rot} - \frac{d\mathbf{\Omega}}{dt} \times \mathbf{R}. \tag{2}$$

Here, $\mathbf{R}$ is the vector from the coordinate origin to a mass point, $\mathbf{a}_{\rm in}$ is the acceleration of that mass point in the inertial frame, and $\mathbf{v}_{\rm rot}$ is the velocity of the mass point in the rotating frame. This equation has appeared many times in this journal[4–7] and is given in many mechanics texts.[8,9]

When acting on a mass point, the $\mathbf{\Omega}$ dependent terms on the right of Eq. (2) are the accelerations due to "pseudoforces" in the noninertial frame. The first of these is the most familiar, the centrifugal force. This pseudoforce, in the z direction, pushes mass away from the y axis and is opposed by tension in the gyroscope shaft so that one may (correctly) predict that these forces add to zero.[10] We also ignore the last term on the right, since $\mathbf{\Omega}$ is constant.

This leaves us with only the term $-2\mathbf{\Omega} \times \mathbf{v}_{\rm rot}$, the Coriolis acceleration, where $\mathbf{v}_{\rm rot}$ is the motion due to the $\boldsymbol{\omega}$ rotation about the z axis. For mass elements on the disk, the distribution of Coriolis pseudoforces is sketched in Fig. 2. It should now be evident that the Coriolis "pseudotorque" on the disk, in the negative x direction, must balance the gravitational torque in the positive x direction, so that there is no net torque, and the puzzle is solved. But to be sure of this, we should calculate the pseudotorque.

III. CALCULATING THE PSEUDOTORQUE

To facilitate the calculation, we introduce cylindrical coordinates r, θ, z illustrated in Fig. 3, to locate each mass element dm. In these coordinates, the Coriolis acceleration in Eq. (2) gives us the pseudoforce

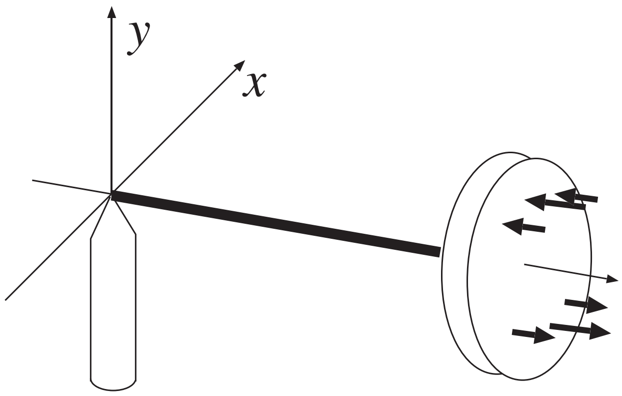

Fig. 2. The Coriolis forces on the disk in the rotating frame. The figure shows the sign difference in the Coriolis force on the upper and lower half of the disk.

$$d\mathbf{F} = -2\boldsymbol{\Omega} \times \mathbf{v}_{\rm rot}\, dm = -2\Omega\, \hat{\mathbf{y}} \times (\omega r \hat{\boldsymbol{\theta}}) dm$$
$$= -2\Omega\, \omega r \sin\theta\, \hat{\mathbf{z}}\, dm. \tag{3}$$

To calculate the torque about the coordinate origin, for comparison with Eq. (1), we denote the distance from the coordinate origin to the center of the disk by R_c, so that the vector $\mathbf{R}$ from the origin to a mass point is

$$\mathbf{R} = R_c\, \hat{\mathbf{z}} + r(\hat{\mathbf{x}} \cos\theta + \hat{\mathbf{y}} \sin\theta). \tag{4}$$

We now integrate $\mathbf{R} \times d\mathbf{F}$ to get the total pseudotorque

$$\boldsymbol{\tau}_{\rm pseudo} = \int r(\hat{\mathbf{x}} \cos\theta + \hat{\mathbf{y}} \sin\theta) \times d\mathbf{F}$$
$$= -2\Omega\omega \int r^2 (\hat{\mathbf{x}} \sin^2\theta - \hat{\mathbf{y}} \sin\theta \cos\theta)\, dm. \tag{5}$$

The term involving R_c has disappeared due to the cross product. The integration over θ makes the $\hat{\mathbf{y}}$ term vanish and gives a factor of 1/2 for the $\hat{\mathbf{x}}$ term. (The average of $\sin^2\theta$ over all theta is 1/2.) The result is then

$$\boldsymbol{\tau}_{\rm pseudo} = -\Omega\omega\, \hat{\mathbf{x}} \int r^2 dm = -\Omega\omega\, I\hat{\mathbf{x}}. \tag{6}$$

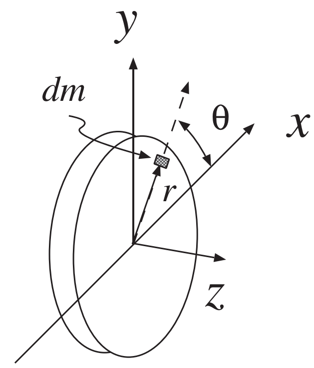

Fig. 3. The coordinates for calculating the pseudotorque.

This pseudotorque in the rotating frame does, indeed, perfectly balance the gravitational torque in Eq. (1).

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[1]For the gyroscope to rotate this way, it must initially be put into motion carefully, otherwise in addition to the horizontal precession there will be vertical nutational oscillations.

[2]Ernest F. Barker, "Elementary analysis of the gyroscope," Am. J. Phys. **28**(10), 808–810 (1960).

[3]Harvey Kaplan and Andrew Hirsch, "Gyroscopic motion: Show me the forces!," Phys. Teach. **52**(1), 30–33 (2014).

[4]Jacob Neuberger, "Coriolis force revisited," Am. J. Phys. **49**(8), 782–784 (1981), Eq. (1).

[5]Harold A. Daw, "Coriolis lecture demonstration," Am. J. Phys. **55**(6), 1010–1014 (1987), Eq. (2).

[6]Guy Vandegrift, "On the derivation of Coriolis and other noninertial accelerations," Am. J. Phys. **63**(7), 663 (1995), Eq. (7).

[7]L. Filipe Costa and José Natário, "The Coriolis field," Am. J. Phys. **84**(5), 388–395 (2016), Eq. (10).

[8]L. D. Landau and E. M. LIfschitz, *Mechanics*, 3rd ed. (Elsevier Butterworth-Beinemann, Burlington, MA, 1976), Vol. 1, Eq. (39.7) in p. 128.

[9]Matthew J. Benacquista and Joseph D. Romano, *Classical Mechanics* (Springer, Cham, 2018), Eq. (1.73) in p. 22.

[10]This statement is a bit of an oversimplification. The centrifugal pseudoforceforce acts on the mass distributed in the disk. The cancelation of the stress in the shaft and the centrifugal force on the disk therefore involve stresses in the disk. Note that the symmetry of this pseudoforce about the *xz* plane means that there can be no contribution to the torque that "holds up" the disk.