The TCLE method (a method in which the admittance of a system is directly calculated from time-convolutionless equations with external driving terms) formulated in terms of thermo-field dynamics, is applied to a boson system interacting with its surroundings, referred to as the “heat reservoir”, and with an external driving field. For a boson system interacting with its heat reservoir, the fluctuation-dissipation theorem, a dispersion relation, and sum rules are studied through application of the TCLE method. The effects of the interference between the external driving field and the heat reservoir, which give the effects of the initial correlation and memory, as well as the effects of the boson-boson interaction, are numerically investigated for a non-degenerate boson system interacting with its heat reservoir, up to first order in powers of the boson-boson interaction.

§1. Introduction

In a previous paper,\(^1\) we formulated the TCLE method, in which the admittance of a system is directly calculated from time-convolutionless (TCL) equations with external driving terms,\(^1\)\(^{-12}\) in terms of thermo-field dynamics (TFD). If we can derive the admittances of a quantal system interacting with its surroundings, referred to as the “heat reservoir” in this paper, and with an external driving field using the TCLE method, we can investigate the effects of the interference between the external driving field and the heat reservoir, which give the effects of the initial correlation and memory, and which are sometimes called the “interference effects” in this paper. We also applied the TCLE method formulated in terms of TFD to interacting boson system, derived a form of the admittance for boson systems, and expanded it up to first order in powers of the boson-boson interaction.\(^1\)

The interference effects, which give the effects of the initial correlation and memory, have been numerically studied for a quantal oscillator\(^9\) and for a quantal spin of magnitude 1/2,\(^10\)\(^{-12}\) each interacting with a heat reservoir. In those works, it was shown that the interference effects increase the heights of the peaks of the power spectra in the resonance region, and decrease the power spectra in the small frequency region. For many-body systems interacting with a heat reservoir, the effects of the internal interactions must be considered. It is interesting to study the effects of the internal interactions and the interference effects in many-body systems interacting with a heat reservoir. It is also interesting to investigate linear response theory,\(^13\),\(^14\) for many-body systems interacting with a heat reservoir, using the TCLE method, as done for a quantal spin of magnitude 1/2 in Ref. 12).

In the present paper, we consider an interacting boson system in contact with a heat reservoir, and examine the fluctuation-dissipation theorem, a dispersion relation, and sum rules, using the TCLE method. We also consider a non-degenerate
M. Saeki

M. Saeki

boson system that consists of weakly interacting bosons in contact with a heat reservoir, and numerically study the effects of the boson-boson interactions and the interference effects up to first order in powers of the boson-boson interaction for the case of a boson-like reservoir. We use the same notation and units as in Ref. 1).

In §2, we investigate the fluctuation-dissipation theorem, a dispersion relation, and sum rules for an interacting boson system in contact with a heat reservoir, using the TCLE method. In §3, we report the results of a numerical study of the admittance obtained with the TCLE method for a non-degenerate boson system with a negative chemical potential, up to first order in powers of the boson-boson interaction. In §4, we give a short summary and some concluding remarks.

§2. Linear response theory for a boson system interacting with a heat reservoir

We consider an interacting boson system in contact with a heat reservoir and with an external driving field which is a classical field described by a c-number function of time \( t \). We investigate the linear stationary response of this boson system to the external driving field analytically, using the TCLE method formulated in terms of TFD.\(^1\) We consider the case that the external driving field is a periodic function of the frequency \( \omega \). We take the Hamiltonians of the boson system (with boson mass \( m \)) to be

\[
\mathcal{H}_{S0} = \sum_k \hbar \epsilon_k b_k^\dagger b_k, \quad \left( \hbar \epsilon_k = \frac{\hbar^2 k^2}{2m} - \mu \right)
\]

\[
\mathcal{H}_{S1} = \frac{1}{2} \sum_q \sum_k \sum_{k'} \hbar V_q b_{k+q}^\dagger b_{k'}^\dagger b_k b_{k'},
\]

and take the interactions of the boson system with the reservoir and with the external driving field, respectively, to be

\[
\mathcal{H}_{SR} = \sum_k \hbar g_k \left( b_k R_k^\dagger + b_k^\dagger R_k \right),
\]

\[
\mathcal{H}_{ed}(t) = - \sum_k \{ b_k F_k^\ast(t) + b_k^\dagger F_k(t) \} = - \sum_k \{ b_k F_k^\ast(\omega) e^{i \omega t} + b_k^\dagger F_k(\omega) e^{-i \omega t} \},
\]

where \( b_k \) and \( b_k^\dagger \) are the boson operators of wave-number \( k \), \( R_k \) and \( R_k^\dagger \) are the reservoir operators of wave-number \( k \), and \( F_k(t) \) is the Fourier transform of wave-number \( k \) for the external driving field. Here, \( \mu \) is the chemical potential of the boson, \( \hbar V_q \) is the Fourier transform of wave-number \( q \) for the boson-boson interaction, and \( \hbar g_k \) are the coupling constants between the boson system and the reservoir. In Ref. 1), we assumed that the external driving field is turned on adiabatically at the initial time \( t = 0 \), and derived the following expression for the admittance of the boson system:

\[
\chi_{b_k b_k^\dagger}(\omega) = \frac{1}{\hbar} \int_0^{\infty} dt \langle 1_S | b_k S(t) \exp \{ -i \int_0^t d\tau \mathcal{H}_{S1}(\tau) \} (b_k^\dagger - \bar{b}_k) | \rho(0) \rangle \{ i + X_k(\omega) \} e^{i \omega t},
\]
with

\[ \mathcal{H}_S \equiv (\mathcal{H}_S - \mathcal{H}_S^\dagger)/\hbar = \mathcal{H}_{S0} + \mathcal{H}_{S1}, \quad \mathcal{H}_S = \mathcal{H}_{S0} + \mathcal{H}_{S1}, \]

where \( \ket{\rho(0)} \) is given by \(^1\)

\[ \ket{\rho(0)} = (1_R|\rho_{TE}) = (1_R|\exp(-\beta \mathcal{H})|1_R|\exp(-\beta \mathcal{H})); \quad \mathcal{H} = \mathcal{H}_S + \mathcal{H}_R + \mathcal{H}_{SR}. \] (8)

In the above expression for the admittance \( X_{b_k b_k^\dagger}(\omega) \), the collision superoperator \( C^{(2)} \) is given by \(^{15}\)

\[ C^{(2)} = -\sum_k \{(b_k^\dagger - b_k^\dagger)(b_k^\dagger \phi_k^+(\epsilon_k) - b_k \phi_k^+(\epsilon_k)^*) + (b_k^\dagger - b_k^\dagger)(b_k \phi_k^+(\epsilon_k) - b_k^\dagger \phi_k^+(\epsilon_k)^*)\}, \] (9)

and \( X_k(\omega) \) corresponds to the terms of the interference between the external driving field and the heat reservoir, which represent the effects of the initial correlation and memory, and is given by \(^1\)

\[ X_k(\omega) = [(\phi_k^+(\omega) - \phi_k^+(\epsilon_k) - (\phi_k^+(\omega)^* - \phi_k^+(\epsilon_k)^*))]/(\omega - \epsilon_k), \] (10)

where \( \phi_k^+(\epsilon) \) and \( \phi_k^+(\epsilon) \) are given by

\[ \phi_k^+(\epsilon) = g_k^2 \int_0^\infty d\tau \langle 1_R|R_k^\dagger(\tau)R_k(0)|\rho_R \rangle e^{-i\epsilon \tau}, \] (11)

\[ \phi_k^+(\epsilon) = g_k^2 \int_0^\infty d\tau \langle 1_R|R_k(\tau)R_k^\dagger(0)|\rho_R \rangle e^{i\epsilon \tau}. \] (12)

In the calculations of the collision superoperator and interference thermal state, we have ignored the terms that contain the boson-boson interaction \( \mathcal{H}_{S1}. \)^1

2.1. Fluctuation-dissipation theorem

The Fourier transform of the symmetrized correlation function \( \langle 1|[b_k^H(t), b_k^\dagger]|\rho_{TE}\rangle \) with the Heisenberg superoperator \( b_k^H(t) = \exp(i\mathcal{H}t)b_k\exp(-i\mathcal{H}t) \) for the boson system, can be expressed using the anti-symmetrized correlation function \( \langle 1|[b_k^H(t), b_k^\dagger]|\rho_{TE}\rangle \) as \(^{13,14}\)

\[ \int_{-\infty}^{\infty} dt \langle 1|[b_k^H(t), b_k^\dagger]|\rho_{TE}\rangle e^{i\omega t} = \coth \frac{\beta \hbar \omega}{2} \int_{-\infty}^{\infty} dt \langle 1|[b_k^H(t), b_k^\dagger]|\rho_{TE}\rangle e^{i\omega t}, \] (13)

\[ = \coth \frac{\beta \hbar \omega}{2} \int_0^\infty dt \{ \langle 1|[b_k^H(t), b_k^\dagger]|\rho_{TE}\rangle e^{i\omega t} + \langle 1|[b_k, b_k^H(t)]|\rho_{TE}\rangle e^{-i\omega t} \}, \]

\[ = \coth \frac{\beta \hbar \omega}{2} \int_0^\infty dt \{ \langle 1|[b_k^H(t), b_k^\dagger]|\rho_{TE}\rangle e^{i\omega t} + (\text{complex conjugate}) \}, \] (14)

where \( \rho_{TE} \) is the normalized, time-independent density superoperator for the boson system and reservoir.\(^1\) [See (8).] In the derivation of the above relation, the basic
requirements (8) and (9) of TFD given in Ref. 1) have been used. In Appendix A, it is shown that the one-sided Fourier transform of the anti-symmetrized correlation function can be represented using the admittance obtained with the TCLE method as (A.13). Applying the relation (A.13) to the boson system, we have the relation

\[
\int_0^\infty dt \langle [b_k^\dagger(t), b_k^\dagger] | \rho_{TE} \rangle e^{i \omega t} = -i \hbar \chi_{b_k b_k^\dagger}(\omega).
\]  

(15)

Substituting the above relation into (14), we obtain

\[
\chi_{b_k b_k^\dagger}(\omega) = \frac{1}{2 \hbar} \tanh \frac{\beta \hbar \omega}{2} \int_{-\infty}^\infty dt \langle [b_k^\dagger(t), b_k^\dagger] | \rho_{TE} \rangle e^{i \omega t},
\]  

(16)

which shows that the imaginary part \(\chi_{b_k b_k^\dagger}(\omega)\), which represents the effect of dissipation for the boson system, is proportional to the Fourier transform of the symmetrized correlation function \(\langle [b_k^\dagger(t), b_k^\dagger] | \rho_{TE} \rangle\). This relation is a form of fluctuation-dissipation theorem.\(^{13,14}\) Thus, the fluctuation-dissipation theorem has been proved using the TCLE method for the boson system under consideration.

2.2. Dispersion relation

We first expand the eigenvector |\(\Psi_n\rangle\) or \(\langle \Psi_n |\) with eigenvalue \(\lambda_n\) or \(\lambda_n^*\) of the superoperator \(\hat{\mathcal{H}}_S + i C^{(2)}\) in the complete orthonormal basis \{|m\rangle, \langle m|\} as

\[
|\Psi_n\rangle = \sum_m a_{nm} |m\rangle, \quad \langle \Psi_n | = \sum_m a_{nm}^* \langle m|,
\]  

(17)

where \{|m\rangle\} and \{\langle m|\} are generated, respectively, by cyclic operations of the creation quasi-particle superoperators \(\beta_k^\dagger\) on the thermal vacuum ket-vector \(|0\rangle = |\rho(0)\rangle\), and by cyclic operations of the annihilation quasi-particle superoperators \(\beta_k\) on the thermal vacuum bra-vector \(\langle 0| = \langle 1_S|\). Here, \(\beta_k\) and \(\beta_k^\dagger\) are, respectively, the annihilation and creation quasi-particle superoperators defined in Refs. 1), 15) and 16). By virtue of the complete orthonormality of \{|\Psi_n\rangle, \langle \Psi_n |\}, the coefficients \(a_{nm}\) in (17) satisfy

\[
\sum_n a_{nm} a_{nm'}^* = \delta_{mm'}, \quad \sum_n a_{nm} a_{n'm}^* = \delta_{nn'}.
\]  

(18)

Representing the superoperator \(\hat{\mathcal{H}}_S + i C^{(2)}\) in terms of the complete orthonormal basis \{|\Psi_n\rangle\} as

\[
\hat{\mathcal{H}}_S + C^{(2)} = \sum_n \lambda_n |\Psi_n\rangle \langle \Psi_n |, \quad (\lambda_n = \lambda_n^* + i \lambda_n'\ )
\]  

(19)

the admittance \(\chi_{b_k b_k^\dagger}(\omega) [(5)]\) can be rewritten as follows,

\[
\chi_{b_k b_k^\dagger}(\omega) = \frac{1}{\hbar} \int_0^\infty dt \sum_n \exp\{-i(\lambda_n' - \omega) t + \lambda_n' t\} \langle 1_S| b_k |\Psi_n\rangle \langle \Psi_n | b_k^\dagger - b_k | \rho(0) \rangle \{i + X_k(\omega)\},
\]
where we have defined \(|\tilde{X}_k(\tau)|\) and \(\zeta(\tau)\) by writing \(X_k(\omega)\) given by (10)–(12) as

\[
X_k(\omega) = i \int_0^\infty d\tau \int_0^\tau ds \, |\tilde{X}_k(\tau)| \exp\{i \zeta(\tau) + i (\omega - \epsilon_k) s\}. \tag{21}
\]

In the expression (20) of the admittance \(\chi_{bb_kk'}(\omega)\), \(a_{n1k}\) is the coefficient of the one quasi-particle state \(|1_k\rangle\) of wave-number \(k\) in the expansion (17) for \(|\Psi_n\rangle\); that is, we have

\[
a_{n1k} = \langle 1_\| | \beta_k | \Psi_n \rangle \quad \quad a_{n1k}^* = \langle \Psi_n | \beta_k^\dagger | 0_\| \rangle. \tag{22}
\]

The eigenvalue \(\lambda_n\) of \(\hat{H}_S+iC(2)\) has a negative imaginary part in general, i.e., \(\lambda_n'' < 0\). [For example, see Refs. 4) and 6) for the cases of a quantal oscillator and of a quantal spin, respectively.] For a non-interacting boson system (\(\hat{H}_S = 0\)), the eigenvector \(|\Psi_n\rangle\) and eigenvalue \(\lambda_n\) of the superoperator \(\hat{H}_S+iC(2)\) become\(^1\)

\[
|\Psi_n\rangle = |n\rangle, \quad \quad \langle \Psi_n | = \langle n |; \quad \quad a_{nm} = \delta_{nm}, \quad \text{for} \quad \hat{H}_S = 0, \tag{23}
\]

\[
\lambda'_1k = \epsilon_k + \phi''_k, \quad \quad \lambda''_1k = -\phi'_k, \quad \text{for} \quad \hat{H}_S = 0, \tag{24}
\]

where \(\phi'_k\) and \(\phi''_k\) are defined by\(^1\)

\[
\phi'_k + i \phi''_k = \phi^{+}_k - \phi^{\dagger}_k = g_k^2 \int_0^\infty d\tau |1_\| \langle R_k(\tau), R^\dagger_k(0) | \rho_\| \rangle e^{i \epsilon_k \tau}. \tag{25}
\]

Then, the real and imaginary parts of \(\chi_{bb_kk'}(\omega)\) are, respectively, given by

\[
\chi'_{bb_kk'}(\omega) = -\frac{1}{\hbar} \sum_n |a_{n1k}|^2 \left\{ \int_0^\infty dt \sin\{(\omega - \lambda'_n) t\} \, e^{\lambda''_n t} \right. \\
+ \int_0^\infty dt \int_0^\infty d\tau \int_0^\tau ds \, |\tilde{X}_k(\tau)| \sin\{(\omega + \lambda'_n) t + \epsilon_k s + \zeta(\tau)\} \, e^{\lambda''_n t} \left. \right\}, \tag{26}
\]

\[
\chi''_{bb_kk'}(\omega) = \frac{1}{\hbar} \sum_n |a_{n1k}|^2 \left\{ \int_0^\infty dt \cos\{(\omega - \lambda'_n) t\} \, e^{\lambda''_n t} \right. \\
+ \int_0^\infty dt \int_0^\infty d\tau \int_0^\tau ds \, |\tilde{X}_k(\tau)| \cos\{(\omega + \lambda'_n) t + \epsilon_k s + \zeta(\tau)\} \, e^{\lambda''_n t} \left. \right\}. \tag{27}
\]

Using the familiar relation

\[
\text{P} \left( \frac{1}{Z} \right) = \frac{1}{2} \left( \frac{1}{Z + i \eta} + \frac{1}{Z - i \eta} \right) \bigg|_{\eta \to +0}, \tag{28}
\]
with the Cauchy principal part $P$, and performing the complex integrations, we have the principal part integrals
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' P \frac{1}{\omega' - \omega} e^{\pm i \omega' t} = \pm i e^{\pm i \omega t}, \quad (t > 0) \tag{29}
\]
which give
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' P \frac{1}{\omega' - \omega} \left\{ \sin \big( \omega' t + \cdots \big) \right\} = \left\{ \cos \big( \omega t + \cdots \big) \right\} \quad (t > 0) \tag{30}
\]
Thus, we obtain the dispersion relations
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \chi_{b_k b_k^\dagger}(\omega) = \frac{1}{\pi} \frac{\lambda_n}{\omega - \lambda_n' - i \lambda_n''}, \quad \int_{-\infty}^{\infty} d\omega \chi_{b_k b_k^\dagger}(\omega) = \frac{1}{\pi} \frac{\lambda_n}{\omega - \lambda_n' - i \lambda_n''} \tag{33}
\]
where we have assumed that the eigenvalue $\lambda_n$ of $\hat{H}_S + i C^{(2)}$ has a negative imaginary part, i.e., $\lambda_n'' < 0$. Noting that $X_k(\omega)$ given by (10)–(12) can be written as (21), we have
\[
\int_{-\infty}^{\infty} d\omega \frac{X_k(\omega)}{\omega - \lambda_n' - i \lambda_n''} = 0. \quad (\lambda_n'' < 0) \tag{34}
\]
Then, the $\omega$-integral for $\chi_{b_k b_k^\dagger}(\omega)$ becomes
\[
\int_{-\infty}^{\infty} d\omega \chi_{b_k b_k^\dagger}(\omega) = \frac{\pi}{i} \sum_n |a_{n1_k}|^2 = \frac{\pi}{i}, \tag{35}
\]
which implies the following:
\[
\int_{-\infty}^{\infty} d\omega \chi_{b_k b_k^\dagger}'(\omega) = 0, \quad \int_{-\infty}^{\infty} d\omega \chi_{b_k b_k^\dagger}''(\omega) = \frac{\pi}{i} \tag{36}
\]
Next, considering that in the limit $|\omega| \to \infty$, we have the relations
\[
X_k(\omega) = 0, \quad \omega \chi_{b_k b_k^\dagger}(\omega) = -1/\hbar, \quad (\omega \to \pm \infty) \tag{37}
\]
we calculate the integral
\[
\int_{-\infty}^{\infty} d\omega \left\{ \omega \chi_{b_k b_k^\dagger}^{(1)}(\omega) + \frac{1}{\hbar} \right\} = \frac{1}{\hbar} \sum_n |a_{n1}|^2 \int_{-\infty}^{\infty} d\omega \left\{ \omega - \frac{1}{\hbar} + i X_k(\omega) \left( \lambda'_n \right) \right\},
\]
\[
= \frac{1}{\hbar} \sum_n |a_{n1}|^2 \int_{-\infty}^{\infty} d\omega \left\{ i X_k(\omega) + \frac{i X_k(\omega) (\lambda_n' + i \lambda_n'') - \lambda_n' - i \lambda_n''}{\omega - \lambda_n' - i \lambda_n''} \right\}.
\]

By using (34) and by substituting the form of \(X_k(\omega)\) given by (10)–(12), the above integral can be calculated as
\[
\int_{-\infty}^{\infty} d\omega \left\{ \omega \chi_{b_k b_k^\dagger}^{(1)}(\omega) + \frac{1}{\hbar} \right\} = \frac{\pi}{\hbar} i \sum_n \lambda_n |a_{n1}|^2 + \frac{i}{\hbar} \int_{-\infty}^{\infty} d\omega X_k(\omega),
\]
\[
= \frac{\pi}{\hbar} i \sum_n \lambda_n |a_{n1}|^2 - \frac{g^2_k}{\hbar} \int_{-\infty}^{\infty} d\omega \int_{0}^{\tau} d\tau \int_{0}^{\tau} ds \delta(s) \langle R_k(\tau)|R_k(0)\rangle \rho_R e^{i \epsilon_k (\tau-s)} e^{i \omega s},
\]
\[
= \frac{\pi}{\hbar} i \sum_n \lambda_n |a_{n1}|^2 - \frac{2 \pi}{\hbar} g^2_k \int_{0}^{\infty} d\tau \int_{0}^{\tau} ds \delta(s) \langle R_k(\tau)|R_k(0)\rangle \rho_R e^{i \epsilon_k (\tau-s)},
\]
\[
= \frac{\pi}{\hbar} i \sum_n \lambda_n |a_{n1}|^2 - \frac{\pi}{\hbar} g^2_k \int_{0}^{\infty} d\tau \langle R_k(\tau)|R_k(0)\rangle \rho_R e^{i \epsilon_k \tau},
\]
where \(\delta(t)\) is the delta function:
\[
\delta(t) = \frac{1}{2 \pi} \int_{-\infty}^{\infty} d\omega e^{\pm i \omega t}.
\]

Thus, we obtain the sum rule
\[
\int_{-\infty}^{\infty} d\omega \chi_{b_k b_k^\dagger}^{(2)}(\omega) = \frac{\pi}{\hbar} \sum_n \lambda_n |a_{n1}|^2 - \frac{\pi}{\hbar} g^2_k \int_{0}^{\infty} d\tau \text{Im} \langle 1_{\mathbf{R}}| R_k(\tau), R_k^\dagger(0)\rangle \rho_R e^{i \epsilon_k \tau},
\]
\[
= \frac{\pi}{\hbar} \text{Re} \langle 1_{\mathbf{S}}| \beta_k(\hat{H}_S + i C^{(2)}) \beta_k^\dagger(0)\rangle \rho_R - \frac{\pi}{\hbar} g^2_k \int_{0}^{\infty} d\tau \text{Im} \langle 1_{\mathbf{R}}| R_k(\tau), R_k^\dagger(0)\rangle \rho_R e^{i \epsilon_k \tau},
\]
where we have substituted (22) and have used the completeness of the eigenvectors \(|\psi_n\rangle\) with eigenvalues \(\lambda_n\) of superoperator \(\hat{H}_S + i C^{(2)}\). The second term in the sum rule (41) comes from the effects of the interference between the external driving field and the heat reservoir.

### §3. Admittance of a weakly interacting boson system

In this section, we consider a non-degenerate boson system that consists of weakly interacting bosons with a negative chemical potential, and report the results of a numerical investigation of the admittance \(\chi_{b_k b_k^\dagger}(\omega)\) for this boson system. In Ref. 1, we derived the following form of the admittance \(\chi_{b_k b_k^\dagger}(\omega)\) for the boson system up to first order in powers of the boson-boson interaction \(\hat{H}_{S1}\):
The reservoir operators satisfy the commutation relations
\[ \omega \text{ where } \psi_{k_1}^{\dag} + \psi_{k_1}^{\ominus} \in \mathcal{R}_{k_1} (\bar{n}_{k_1} + 1) \]
\[ + \frac{i}{2 \hbar} \sum_{k_1} \left\{ i (\epsilon_k + \Phi_k^{\dag} - \omega) + \Phi_k' \right\} \left( i (\epsilon_k + \Phi_k'' - \omega) + \Phi_k' + 2 \Phi_k' \right) \]
\[ + \frac{i}{2 \hbar} \bar{n}_{k_1} \{ 2 V_0 + \bar{V}_{k_1} + \bar{V}_{k_1 - k} \} \{ i + X_k (\omega) \} \]
\[ + O(\mathcal{H}^2), \]  
(42)
where \( \bar{n}_k, \psi_k^{\dag} \) and \( \psi_k^{\ominus} \) are defined by
\[ \bar{n}_k = \{ \exp (\beta \hbar \epsilon_k) - 1 \}^{-1}, \]
(43)
\[ \psi_k^{\dag} = \hbar^2 g_k^2 \int_0^\beta d\beta' (\beta - \beta') \langle 1_R | R_k (\beta - i \hbar \beta') R_k (0) | \rho_R \rangle e^{-\beta' \hbar \epsilon_k}, \]
(44)
\[ \psi_k^{\ominus} = \hbar^2 g_k^2 \int_0^\beta d\beta' (\beta - \beta') \langle 1_R | R_k (-i \hbar \beta') R_k (0) | \rho_R \rangle e^{\beta' \hbar \epsilon_k}. \]
(45)
We take the boson-boson interaction to be given by
\[ \nu (\vec{x}) = V \delta (\vec{x}), \quad \hbar V_q = V / \Omega, \]
(46)
where \( \Omega \) is the volume of the boson system. We assume the density of states \( d(E) \) for the boson system to be given by
\[ d(E) = E^l / E_0^{l+1}, \quad \sum_k = \int_0^\infty dE d(E) = \frac{1}{E_0^{l+1}} \int_0^\infty dE E^l. \]
(47)
For the case of free particles in three dimensions, we have \( l = 1/2 \).

We consider the case of a boson-like reservoir including a rapidly damped subsystem. The reservoir operators satisfy the commutation relations
\[ [R_k, R_k^\dag] = \delta_{kl}, \]  
(48)
while the other commutators vanish. The reservoir correlation functions are given by \( \langle 1_R | R_k (t) R_k (0) | \rho_R \rangle = \langle 1_R | R_k^\dag (0) R_k (0) | \rho_R \rangle \exp (i \omega_k^l t - \gamma_k^l t), \]
(49)
\[ \langle 1_R | R_k (t) R_k^\dag (0) | \rho_R \rangle = \langle 1_R | R_k (0) R_k^\dag (0) | \rho_R \rangle \exp (-i \omega_k^l t - \gamma_k^l t), \]
(50)
where \( \omega_k^l \) and \( \gamma_k^l \) are, respectively, the characteristic frequency and damping constant of the reservoir. Then, \( \Phi_k', \Phi_k'', \psi_k^{\dag} + \psi_k^{\ominus} \), and \( X_k (\omega) \) become
\[ \Phi_k' = \frac{g_k^2 \gamma_k^l}{(\epsilon_k - \omega_k^l)^2 + (\gamma_k^l)^2}, \quad \Phi_k'' = \frac{g_k^2 (\epsilon_k - \omega_k^l)}{(\epsilon_k - \omega_k^l)^2 + (\gamma_k^l)^2}, \]
\[ \psi_k^{\dag} + \psi_k^{\ominus} = g_k^2 \left\{ - \beta \hbar / (\epsilon_k - \omega_k^l) \right\} \]
\[ + [ \sinh \beta \hbar (\epsilon_k - \omega_k^l) - \coth (\beta \hbar \omega_k^l / 2) ] / (\epsilon_k - \omega_k^l)^2 \]
\[ + \coth (\beta \hbar \omega_k^l / 2) \cosh [\beta \hbar (\epsilon_k - \omega_k^l)] / (\epsilon_k - \omega_k^l)^2 \}, \]
\[ X_k (\omega) = g_k^2 \frac{\gamma_k^l (\epsilon_k + \omega - 2 \omega_k^l) + i \left[ (\gamma_k^l)^2 - (\epsilon_k - \omega_k^l)^2 \right] (\omega - \omega_k^l)}{[(\epsilon_k - \omega_k^l)^2 + (\gamma_k^l)^2] \left[ (\omega - \omega_k^l)^2 + (\gamma_k^l)^2 \right]} . \]
(53)
Fig. 1. (a) The power spectra $\mu^2 \chi''_{b_0b_0^\dagger}(\omega)/\hbar\omega$ for $\hat{\omega}_R=0.5$, $\hat{V}=0.03$, $\hat{\beta}=1$, and $\hat{\gamma}_R=1$. 
(b) The power spectra $\mu^2 \chi''_{b_0b_0^\dagger}(\omega)/\hbar\omega$ for $\hat{\omega}_R=0.5$, $\hat{V}=-0.03$, $\hat{\beta}=1$, and $\hat{\gamma}_R=1$.

In the derivation of (52), we have used the expectation values

\begin{align}
\langle 1_R | R_k R_k^\dagger | \rho_R \rangle &= 1/\{\exp(\beta \hbar \omega^k_R) - 1\} + O(g_R^2), \\
\langle 1_R | R_k R_k^\dagger | \rho_R \rangle &= \exp(\beta \hbar \omega^k_R)/\{\exp(\beta \hbar \omega^k_R) - 1\} + O(g_R^2),
\end{align}

where $g_R$ is the coupling constant between the reservoir bosons and the rapidly damped subsystem.

When the external driving field is uniform in space, the power loss of the boson system in the stationary state is proportional to $\omega \chi''_{b_0b_0^\dagger}(\omega)$ (as mentioned in Ref. 1)), where $\chi''_{b_0b_0^\dagger}(\omega)$ is the imaginary part of the admittance for wave-number $k=0$. In order to integrate the admittance $\chi_{b_0b_0^\dagger}(\omega)$ numerically, we define the dimensionless parameters scaled by $-\mu$ or $-\mu/\hbar$ as

\begin{align}
\hat{V} &= -V/(\mu \Omega), \\
\hat{\omega} &= -\hbar \omega/\mu, \\
\hat{\omega}_R &= -\hbar \omega^0_R/\mu, \\
\hat{g} &= -\hbar g_0/\mu, \\
\hat{\phi}' &= -\hbar \phi'_0/\mu, \\
\hat{\phi}'' &= -\hbar \phi''_0/\mu,
\end{align}

where $\hat{\phi}'$ and $\hat{\phi}''$ take the forms

\begin{align}
\hat{\phi}' &= \frac{\hat{g}^2 \hat{\gamma}_R}{(1-\hat{\omega}_R)^2 + (\hat{\gamma}_R)^2}, \\
\hat{\phi}'' &= \frac{\hat{g}^2 (1-\hat{\omega}_R)}{(1-\hat{\omega}_R)^2 + (\hat{\gamma}_R)^2}.
\end{align}

Using the above parameters, the admittance $\chi_{b_0b_0^\dagger}(\omega)$ for wave-number $k=0$ can be
Fig. 2. (a) The power spectra $\mu^2 x_{b\bar{b}}''(\omega)/\hbar \omega$ for $\hat{\omega}_R = 0.9$, $\hat{V} = 0.03$, $\hat{\beta} = 1$, and $\hat{\gamma}_R = 1$.  
(b) The power spectra $\mu^2 x_{b\bar{b}}''(\omega)/\hbar \omega$ for $\hat{\omega}_R = 0.9$, $\hat{V} = -0.03$, $\hat{\beta} = 1$, and $\hat{\gamma}_R = 1$.

written as

$$x_{b\bar{b}}(\omega) = \frac{1}{\mu} \frac{i + X(\hat{\omega})}{i (1 + \hat{\Phi}' - \hat{\omega}) + \hat{\Phi}'} - \frac{2i}{\mu} \frac{\hat{V}}{E_{0}^{l+1}} \int_{0}^{\infty} dE E^{l} \left\{ \frac{(\psi^+(E) + \psi^-(E)) \bar{n}(E) (\bar{n}(E) + 1)}{[i (1 + \hat{\Phi}' - \hat{\omega}) + \hat{\Phi}'] [i (1 + \hat{\Phi}' - \hat{\omega}) + \hat{\Phi}' + 2\Phi'(E)]} \right. $$

$$+ \left. \frac{\bar{n}(E)}{[i (1 + \hat{\Phi}' - \hat{\omega}) + \hat{\Phi}']^2} \right\} \{i + X(\hat{\omega})\}, \quad (58)$$

up to first order in powers of the boson-boson interaction, where $\bar{n}(E)$, $\Phi'(E)$, $\psi^+(E)$, and $\psi^-(E)$ are given by

$$\bar{n}(E) = \{\exp[\hat{\beta}(E + 1)] - 1\}^{-1}, \quad (59)$$

$$\Phi'(E) = \hat{g}^2 \hat{\gamma}_R / \{(E + 1 - \hat{\omega}_R)^2 + (\hat{\gamma}_R)^2\}, \quad (60)$$

$$\psi^+(E) + \psi^-(E) = \hat{g}^2 \left\{ -\hat{\beta}/(E + 1 - \hat{\omega}_R) \right. $$

$$+ \left\{ \sinh[\hat{\beta}(E + 1 - \hat{\omega}_R)] - \coth[\hat{\beta}(E + 1 - \hat{\omega}_R)/2] \right\} (E + 1 - \hat{\omega}_R)^2 $$

$$+ \coth[\hat{\beta}(E + 1 - \hat{\omega}_R)] \cosh[\hat{\beta}(E + 1 - \hat{\omega}_R)] / (E + 1 - \hat{\omega}_R)^2 \right\}, \quad (61)$$

$$X(\hat{\omega}) = \hat{g}^2 \frac{\hat{\gamma}_R \cdot (1 + \hat{\omega} - 2\hat{\omega}_R) + i \left( (\hat{\gamma}_R)^2 - (1 - \hat{\omega}_R)(\hat{\omega} - \hat{\omega}_R) \right)}{[1 - (\hat{\omega}_R)^2 + (\hat{\gamma}_R)^2] \left( (\hat{\omega} - \hat{\omega}_R)^2 + (\hat{\gamma}_R)^2 \right)} \right\}. \quad (62)$$

We calculate the admittance (58) numerically using the Simpson integral formula for the case $l = 1/2$, $E_{0} = 1$ and $\hat{g} = 0.3$. In order to elucidate the effects of the boson-boson interactions and the effects of the interference between the external
Fig. 3. (a) The real parts \((-\mu)\chi'_{b0b0}\)(\(\omega\)) for \(\hat{\omega}_R=0.5\), \(\hat{V}=0.03\), \(\hat{\beta}=1\), and \(\hat{\gamma}_R=1\).

(b) The real parts \((-\mu)\chi'_{b0b0}\)(\(\omega\)) for \(\hat{\omega}_R=0.5\), \(\hat{V}=-0.03\), \(\hat{\beta}=1\), and \(\hat{\gamma}_R=1\).

driving field and the heat reservoir, which are called the “interference effects”, we also calculate the admittance for the case of no boson-boson interaction, which is that without the second term in the admittance (58) and is denoted as “V=0”, and calculate the admittance in the case without interference terms, which is that without the term \(X(\hat{\omega})\) in the admittance (58) and is denoted as “IT=0”. In Figs. 1(a) and 2(a), the power spectra \(\chi''_{b0b0}\)(\(\omega\)/\(\omega\)) scaled by \(\hbar/\mu^2\) are displayed for \(\hat{\omega}_R=0.5\) and 0.9, respectively, with \(\hat{V}=0.03\), \(\hat{\beta}=1\), and \(\hat{\gamma}_R=1\). They are compared with the power spectra for the case in which there is no boson-boson interaction, and with the power spectra in the case without interference terms. In Figs. 1(b) and 2(b), the power spectra are displayed for \(\hat{\omega}_R=0.5\) and 0.9, respectively, with \(\hat{V}=-0.03\), \(\hat{\beta}=1\), and \(\hat{\gamma}_R=1\). These figures show that the boson-boson interaction and the interference terms between the external driving field and the heat reservoir, which are called the “interference terms”, produce effects that increase the heights of the peaks of the power spectra in the resonance region. In the resonance region, the interference effects are smaller than the effects of the boson-boson interactions when \(\hat{\omega}_R=0.5\), but they are comparable with the effects of the boson-boson interactions when \(\hat{\omega}_R=0.9\). Thus, these figures show that as \(\hat{\omega}_R\) increases, the interference effects become stronger in the resonance region. In the case that the boson-boson interaction yields a repulsive force (i.e., \(V>0\)), Figs. 1(a) and 2(a) show that as \(\hat{\omega}_R\) increases, the interference effects become stronger than the effects of the boson-boson interactions in the resonance region. Figures 2(a) and (b) show that the interference terms produce effects that cause the power spectra to decrease in the small frequency region.
In Figs. 3(a) and 4(a), the real parts $\chi'_{b_0b_0^\dagger}(\omega)$ scaled by $1/(-\mu)$ are displayed for $\hat{\omega}_R = 0.5$ and 0.9, respectively, with $\hat{V} = 0.03$, $\hat{\beta} = 1$, and $\hat{\gamma}_R = 1$. They are compared with those for the case in which there is no boson-boson interaction and with those in the case without interference terms. In Figs. 3(b) and 4(b), the real parts are displayed for $\hat{\omega}_R = 0.5$ and 0.9, respectively, with $\hat{V} = -0.03$, $\hat{\beta} = 1$, and $\hat{\gamma}_R = 1$. These figures show that in the resonance region, the interference effects are smaller than the effects of the boson-boson interactions when $\hat{\omega}_R = 0.5$, but they are comparable with the effects of the boson-boson interactions when $\hat{\omega}_R = 0.9$, and thus they become large as $\hat{\omega}_R$ becomes large. When the boson-boson interaction yields an attractive force (i.e., $V < 0$), Figs. 3(b) and 4(b) show that as $\hat{\omega}_R$ increases, the interference effects become stronger than the effects of the boson-boson interactions in the resonance region. Figures 1–4(a) and (b) show that if these line shapes can be measured experimentally, it should also be possible for the effects of the interference terms to be observed experimentally. In Figs. 5(a) and (b), the power losses $\omega \chi''_{b_0b_0^\dagger}(\omega)$ scaled by $1/\hbar$ are displayed for $\hat{V} = 0.03$ and $-0.03$, respectively, and for $\hat{\beta} = 1, 2, 4$, with $\hat{\omega}_R = 0.5$ and $\hat{\gamma}_R = 1$. Figures 5(a) and (b) show that the peak heights of the spectra decrease as the temperature decreases ($\hat{\beta}$ increases). This decrease of the peak heights is larger in the case of a repulsive boson-boson interaction ($V > 0$) than in the case of an attractive boson-boson interaction ($V < 0$). In Figs. 6(a) and (b), the power losses $\omega \chi''_{b_0b_0^\dagger}(\omega)$ scaled by $1/\hbar$ are displayed for $\hat{V} = 0 - 0.04$ and $-0.04 - 0$, respectively, with $\hat{\beta} = 1$, $\hat{\omega}_R = 0.5$ and $\hat{\gamma}_R = 1$. Figure 6(a) shows that when the boson-boson interaction yields a repulsive force ($V > 0$), the peak heights of the
Fig. 5. (a) The power losses $h\omega \chi''_{b_0 b_0} (\omega)$ for $\hat{V} = 0.03$, $\hat{\beta} = 1, 2, 4$, $\hat{\omega}_R = 0.5$, and $\hat{\gamma}_R = 1$.
(b) The power losses $h\omega \chi''_{b_0 b_0} (\omega)$ for $\hat{V} = -0.03$, $\hat{\beta} = 1, 2, 4$, $\hat{\omega}_R = 0.5$, and $\hat{\gamma}_R = 1$.

spectra increase in the direction of large frequency as the magnitude of $\hat{V}$ increases. Figure 6(b) shows that when the boson-boson interaction yields an attractive force ($V < 0$), the peaks heights of the spectra increase in the direction of small frequency as the magnitude of $\hat{V}$ increases.

§4. Summary and concluding remarks

We have considered an interacting boson system in contact with a heat reservoir and with an external driving field, and have analytically studied the admittance $\chi_{b_k b_k} (\omega)$ obtained with the TCLE method in Ref. 1. We have shown that the imaginary part $\chi''_{b_k b_k} (\omega)$ of the admittance, which represents the effect of dissipation for the boson system, is proportional to the Fourier transform of the symmetrized correlation function $\langle 1[|H_k^R(t), b_k^\dagger \rangle + |\rho_{TE} \rangle \rangle$ with the Heisenberg superoperator $b_k^H(t) = \exp(i\hat{H}t) b_k \exp(-i\hat{H}t)$. This relation is a form of the fluctuation-dissipation theorem.\textsuperscript{13,14} We have thus proved the fluctuation-dissipation theorem for the admittance obtained with the TCLE method. We have also derived the dispersion relations (31) and (32), and the sum rules (36) and (41) for the admittance $\chi_{b_k b_k} (\omega)$ obtained with the TCLE method. We have shown that the $\omega$-integral for the power loss $\omega \chi''_{b_k b_k} (\omega)$ includes terms representing the interference between the external driving field and the heat reservoir.

In Appendix B, we discuss the relation between the admittance obtained using
the relaxation method and that obtained using the TCLE method, for an interacting boson system in contact with a heat reservoir. The admittance \( \chi_{b_b \dagger} \) obtained with the relaxation method in the van Hove limit \(^{21}\) or in the narrowing limit,\(^{22}\) in which the heat reservoir is damped very rapidly, can be calculated using Feynman-type diagram methods, but it does not include the memory effects and the effects of the initial correlation of the boson system and the reservoir. The admittance \( \chi_{b_b \dagger} \) obtained with the relaxation method under thermal equilibrium initial conditions includes those effects, but it is not easy in general to calculate it using Feynman-type diagram methods. The memory effects and the effects of the initial correlation of the boson system and reservoir can be described by the terms representing the interference between the external driving field and the heat reservoir in the TCLE method, and can be elucidated by calculating the admittance \( \chi_{b_b \dagger} \) obtained with the TCLE method.

We have next considered a non-degenerate boson system that consists of weakly interacting bosons with a negative chemical potential, have numerically calculated the admittance \( \chi_{b_b \dagger} \) obtained with the TCLE method in Ref. 1) up to first order in powers of the boson-boson interaction \( \mathcal{H}_{\text{S1}} \), and have studied the effects of the boson-boson interactions and the effects of the interference between the external driving field and the heat reservoir. We have found that the boson-boson interaction and the interference between the external driving field and the heat reservoir, which are responsible for the effects of the initial correlation and memory, produce effects that increase the peak heights of the power spectra in the resonance region, and

\[ \omega \chi'' \]

Fig. 6. (a) The power losses \( h\omega \chi''_{b_b \dagger} \) for \( \hat{V} = 0 - 0.04, \hat{\beta} = 1, \hat{\omega}_R = 0.5, \) and \( \hat{\gamma}_R = 1. \)

(b) The power losses \( h\omega \chi''_{b_b \dagger} \) for \( \hat{V} = -0.04 - 0, \hat{\beta} = 1, \hat{\omega}_R = 0.5, \) and \( \hat{\gamma}_R = 1. \)
that as the characteristic frequency $\hat{\omega}_R$ of the heat reservoir increases ($\hat{\omega}_R \rightarrow 0.9$), the effects of that interference, which are called the “interference effects”, become large in the resonance region and become larger than the effects of the boson-boson interactions in that region for a repulsive boson-boson interaction (i.e., $V > 0$). The physical interpretation of these results is that the energy dissipation of the bosons increases in the resonance region through interactions of each boson with the other bosons and with the heat reservoir, and that in that region, the effects of the interaction of each boson with the heat reservoir become stronger as the characteristic frequency of the reservoir increases, and become stronger than the effects of the boson-boson interactions especially for a repulsive boson-boson interaction (i.e., $V > 0$). It has also been shown that the interference effects produce effects which cause the power spectra to decrease in the small frequency region ($0 < \hat{\omega} < 0.4$) when the characteristic frequency of the reservoir is large ($\hat{\omega}_R \sim 0.9$). Physically, we can understand this being due to the behavior that the energy dissipation of the bosons decreases in the small frequency region through interaction of the bosons with the heat reservoir when $\hbar \omega_0^R \sim -\mu$. The line shapes of the real parts show that the interference effects become larger than the effects of the boson-boson interactions in the resonance region, especially for an attractive boson-boson interaction (i.e., $V < 0$) as $\hat{\omega}_R$ becomes large ($\hat{\omega}_R \rightarrow 0.9$). We have also found that the heights of the peaks in the power losses decrease in the resonance region as the temperature decreases ($\hat{\beta}$ increases), and that the decrease of the peak heights in the case of a repulsive boson-boson interaction ($V > 0$) is larger than that in the case of an attractive boson-boson interaction ($V < 0$). In addition, we have found that as the temperature increases ($\hat{\beta}$ decreases) or as the magnitude of $\hat{V}$ increases, the heights of the peaks in the power losses increase in the direction of large frequency for a repulsive boson-boson interaction ($V > 0$), and increase in the direction of small frequency for an attractive boson-boson interaction ($V < 0$). This means that as the temperature increases or as the magnitude of $\hat{V}$ increases, the resonance frequency shifts in the direction of large frequency for a repulsive boson-boson interaction ($V > 0$), and shifts in the direction of small frequency for an attractive boson-boson interaction ($V < 0$).

We have shown that the increase of the peak heights of the power spectra due to the interference effects in the resonance region, is about 8% of the peak heights for a repulsive boson-boson interaction ($V > 0$) and is about 7.5% for an attractive boson-boson interaction ($V < 0$), when the characteristic frequency of the reservoir is large ($\hat{\omega}_R \sim 0.9$). We have also shown that the decrease of the power spectra due to the interference effects in the small frequency region ($0 < \hat{\omega} < 0.4$), is about 30% of the power spectra for a repulsive boson-boson interaction and for an attractive boson-boson interaction, when the characteristic frequency of the reservoir is large ($\hat{\omega}_R \sim 0.9$). Thus, the interference effects, which correspond to the effects of the initial correlation and memory, are significant in the resonance region and in the small frequency region, especially in the case that the characteristic frequency of the reservoir is large [$\hbar \omega_0^R \sim 0.9(-\mu)$], for a non-degenerate boson system which consists of weakly interacting bosons with a negative chemical potential. If the line shapes of the power spectra can be measured experimentally in that case, it should be possible to observe the interference effects in the resonance region and in the small frequency
region.

Appendix A

We consider a quantal system interacting with a quantal heat reservoir in an arbitrary external static field, and formally calculate the one-sided Fourier transforms of the correlation functions for physical quantities using the TFD formulated in Ref. 20. We use the same notation and units as in Ref. 1).

For a superoperator $A_j$ representing a physical quantity in the quantal system under consideration, a thermal state $A_j^H(t)\rho_{TE}$ satisfies the equation

$$\frac{d}{dt} A_j^H(t) \rho_{TE} = -i \hat{\mathcal{H}} A_j^H(t) \rho_{TE}, \quad (A.1)$$

owing to the basic requirement (9) of TFD given in Ref. 1), where $A_j^H(t)$ is the Heisenberg superoperator defined by

$$A_j^H(t) = \exp(i\hat{\mathcal{H}}t) A_j \exp(-i\hat{\mathcal{H}}t), \quad \tilde{A}_j^H(t) = \exp(i\hat{\mathcal{H}}t) \tilde{A}_j \exp(-i\hat{\mathcal{H}}t). \quad (A.2)$$

In Ref. 20), by eliminating the irrelevant part associated with the heat reservoir using the projection operator method, we derived the time-convolutionless (TCL) equation of motion for the reduced thermal state $|a_j(-t)\rangle$ defined by

$$|a_j(-t)\rangle = \langle 1_R | A_j^H(-t) \rho_{TE} \rangle = \text{tr}_R A_j^H(-t) \rho_{TE}. \quad (A.3)$$

The TCL equation for $|a_j(-t)\rangle$, with a second-order approximation for the system-reservoir interaction, reduces to

$$\frac{d}{dt} |a_j(-t)\rangle = -i \hat{\mathcal{H}}_S |a_j(-t)\rangle + C^{(2)}(t) |a_j(-t)\rangle + |I_j^{(2)}(t)\rangle, \quad (A.4)$$

where the collision superoperator $C^{(2)}(t)$ and the inhomogeneous term $|I_j^{(2)}(t)\rangle$ are given by

$$C^{(2)}(t) = -\int_0^t d\tau \langle 1_R | \hat{\mathcal{H}}_{SR} \exp(-i \hat{\mathcal{H}}_0 \tau) \hat{\mathcal{H}}_{SR} \exp(i \hat{\mathcal{H}}_0 \tau) |\rho_R\rangle, \quad (A.5)$$

$$|I_j^{(2)}(t)\rangle = i \langle 1_R | \hat{\mathcal{H}}_{SR} \exp(-i \hat{\mathcal{H}}_0 t) \int_0^\beta d\beta' A_j \rho_S \rho_R e^{\beta' \mathcal{H}_0} \mathcal{H}_{SR} e^{-\beta' \mathcal{H}_0} |1\rangle. \quad (A.6)$$

Here, we have adopted the first-order renormalization given by (21) or (22) in Ref. 1) for the Hamiltonian of the system and the system-reservoir interaction. The above equation can be formally solved as

$$|a_j(-t)\rangle = \exp_{-\{ -i \hat{\mathcal{H}}_S t + \int_0^t d\tau C^{(2)}(\tau) \}} A_j |\rho(0)\rangle + \int_0^t d\tau \exp_{-\{ -i \hat{\mathcal{H}}_S (t-\tau) + \int_\tau^t ds C^{(2)}(s) \}} |I_j^{(2)}(\tau)\rangle, \quad (A.7)$$

with $|\rho(0)\rangle = \langle 1_R |\rho_{TE} = \text{tr}_R \rho_{TE}$. Similarly, we have

$$\tilde{a}_j^{\dagger}(-t) = \text{[right-hand side of} (A.7) \text{with} \tilde{A}_j \text{in place of} A_j\text{].} \quad (A.8)$$
The expression for $|a_j(-t)|$ given by (A.7) can be rewritten and be expanded in powers of the system-reservoir interaction as

$$|a_j(-t)| = \langle 1_R | A_j^H(-t) | \rho_{TE} \rangle,$$

$$= \exp\{- i \hat{H}_S t + C^{(2)} t + \int_0^t d\tau \{ C^{(2)}(\tau) - C^{(2)} \} \} A_j | \rho(0) \rangle$$

$$+ \int_0^t d\tau \exp\{- i \hat{H}_S (t - \tau) + C^{(2)} (t - \tau) + \int_\tau^t ds \{ C^{(2)}(s) - C^{(2)} \} \} | I_j^{(2)}(\tau) \rangle,$$

$$= \exp\{- i \hat{H}_S t + C^{(2)} t \} \times \exp\{- i \int_0^\tau d\beta' \exp\{- i \hat{H}_S (t - \tau) + C^{(2)} (t - \tau) + \int_\tau^t ds \{ C^{(2)}(s) - C^{(2)} \} \} \times \langle 1_R | \hat{H}_{SR} \exp(- i \hat{H}_0 \tau) A_j | \rho(0) \rangle$$

$$+ \int_0^t d\tau \int_\tau^\infty ds \exp\{- i \hat{H}_S (t - \tau) + C^{(2)} (t - \tau) \} \times \langle 1_R | \hat{H}_{SR} \exp(- i \hat{H}_0 (s - \tau)) A_j | \rho(0) \rangle | \rho_R \rangle$$

$$- \int_0^t d\tau \int_\tau^\infty ds \exp\{- i \hat{H}_S (t - \tau) + C^{(2)} (t - \tau) \} \times \langle 1_R | \hat{H}_{SR} \exp(- i \hat{H}_0 \tau) A_j \exp(i \hat{H}_0(\tau - s)) \hat{H}_{SR} | \rho(0) \rangle | \rho_R \rangle$$

$$+ \cdots \text{(higher order terms)}, \quad \text{(A.9)}$$

with $C^{(2)} = C^{(2)}(\infty)$, where we have used the expressions (A.5) and (A.6) for $C^{(2)}(t)$ and $|I_j^{(2)}(t)|$ and have transformed the $\beta'$ integral into a time integral as

$$\int_0^\beta d\beta' \rho_S \rho_R e^{\beta' h \hat{H}_0} \hat{H}_{SR} | 1 \rangle = \rho_S \rho_R (h \hat{H}_0)^{-1} \{ e^{\beta h \hat{H}_0} - 1 \} \hat{H}_{SR} | 1 \rangle,$$

$$= i \lim_{\eta \to +0} \int_\tau^\infty ds e^{i \hat{H}_0(\tau - s)} \hat{H}_{SR} | \rho_S \rangle | \rho_R \rangle e^{-\eta s}. \quad \text{(A.10)}$$

Then, the one-sided Fourier transform of the correlation function $\langle 1 | A_i^{H}(t) A_j | \rho_{TE} \rangle$ for the physical quantities $A_i$ and $A_j$ can be expressed as

$$\int_0^\infty dt \langle 1 | A_i^{H}(t) A_j | \rho_{TE} \rangle e^{i \omega t} = \int_0^\infty dt \langle 1 | A_i A_j^{H}(-t) | \rho_{TE} \rangle e^{i \omega t},$$

$$= \int_0^\infty dt \langle 1_s | A_i a_j(-t) \rangle e^{i \omega t},$$

$$= \int_0^\infty dt \langle 1_s | A_i \exp\{- i \hat{H}_S t + C^{(2)} t \} A_j | \rho(0) \rangle e^{i \omega t}$$
The expectation value of the collision superoperator $\langle 1|A_j(t)A_j|\rho_{TE}\rangle$ can be written as

$$
\int_0^\infty dt \int_0^t d\tau \int_0^\infty ds \langle 1_S|A_i \exp\{-i\hat{H}_S(t - \tau) + C'(t - \tau)\} e^{i\omega t} \times \{ \langle 1_R|\hat{H}_{SR}\exp(-i\hat{H}_0 s)\hat{H}_{SR}\exp(i\hat{H}_0 (s - \tau)) A_j |\rho(0)\rangle |\rho_R\rangle 
- \langle 1_R|\hat{H}_{SR}\exp(-i\hat{H}_0 \tau) A_j \exp(i\hat{H}_0 (\tau - s)\) \hat{H}_{SR} |\rho(0)\rangle |\rho_R\rangle \},
$$

$$
= \int_0^\infty dt \langle 1_S|A_i \exp\{-i(\hat{H}_S - \omega) t + C'(t) t\} A_j |\rho(0)\rangle 
+ \int_0^\infty dt \int_0^\infty d\tau \int_0^\infty ds \langle 1_S|A_i \exp\{-i(\hat{H}_S - \omega) t + C'(t) t\} e^{i\omega s} \times \{ \langle 1_R|\hat{H}_{SR}\exp(-i\hat{H}_0 \tau) \hat{H}_{SR}\exp(i\hat{H}_0 (s - \tau)) A_j |\rho(0)\rangle |\rho_R\rangle 
- \langle 1_R|\hat{H}_{SR}\exp(-i\hat{H}_0 s) A_j \exp(i\hat{H}_0 (s - \tau)\) \hat{H}_{SR} |\rho(0)\rangle |\rho_R\rangle \},
$$

(A-8)

where we have performed some integral transformations and have omitted the higher-order terms of (A.9). Considering the relation (A.8), the one-sided Fourier transform of the anti-symmetrized correlation function $\langle 1||A_i^H(t), A_j ||\rho_{TE}\rangle$ can be written as

$$
\int_0^\infty dt \langle 1||A_i^H(t), A_j ||\rho_{TE}\rangle e^{i\omega t} = \int_0^\infty dt \langle 1||A_i, A_j^H(-t) ||\rho_{TE}\rangle e^{i\omega t},
$$

$$
= \int_0^\infty dt \langle 1||A_i \{ A_j^H(-t) - \hat{A}_j^H(-t)\} ||\rho_{TE}\rangle e^{i\omega t},
$$

$$
= \int_0^\infty dt \langle 1||A_i \{ a_j(-t) - \hat{a}_j^\dagger(-t)\} e^{i\omega t},
$$

$$
= \int_0^\infty dt \langle 1||A_i \exp\{-i(\hat{H}_S - \omega) t + C'(t) t\} \{ A_j - \hat{A}_j^\dagger\} |\rho(0)\rangle 
+ \int_0^\infty dt \int_0^\infty d\tau \int_0^\infty ds \langle 1_S|A_i \exp\{-i(\hat{H}_S - \omega) t + C'(t) t\} e^{i\omega s} \times \{ \langle 1_R|\hat{H}_{SR}\exp(-i\hat{H}_0 \tau) \hat{H}_{SR}\exp(i\hat{H}_0 (s - \tau)) \{ A_j - \hat{A}_j^\dagger\} |\rho(0)\rangle |\rho_R\rangle 
- \langle 1_R|\hat{H}_{SR}\exp(-i\hat{H}_0 s) \{ A_j - \hat{A}_j^\dagger\} \exp(i\hat{H}_0 (s - \tau)\) \hat{H}_{SR} |\rho(0)\rangle |\rho_R\rangle \},
$$

(A-9)

where $|D_j^{(2)}(\omega)\rangle$ is the interference thermal state defined in Ref. 1). Considering that the expectation value of the collision superoperator $C'(t)$ has a negative real part in general, the one-sided Fourier transform of the anti-symmetrized correlation function can be represented in terms of the admittance $\chi_{ij}(\omega)$ obtained with the TCLE method\(^1\) as

$$
\int_0^\infty dt \langle 1||A_i^H(t), A_j ||\rho_{TE}\rangle e^{i\omega t} = -i \hbar \chi_{ij}(\omega).
$$

(A-10)

**Appendix B**

In this appendix, we study the relation between the form of the admittance obtained using the relaxation method and that obtained using the TCLE method,
for the boson system (of volume $\Omega$) interacting with a heat reservoir considered in §2, using the TFD formulated in Ref. 20). We use the same notation and units as in Ref. 1).

The admittance $\chi_{b_kb^*_k}(\omega)$ obtained using the relaxation method is given by\(^{13}\)

$$
\chi_{b_kb^*_k}(\omega) = \frac{i}{\hbar} \int_0^\infty dt \langle [b^H_k(t), b^*_k] | \rho_{TE} \rangle e^{i \omega t},
$$

(B.1)

with the Heisenberg superoperator $b^H_k(t) = \exp(i\hat{H}t) b_k \exp(-i\hat{H}t)$ for the boson system. The admittance $\chi_{b_kb^*_k}(\omega)$ can be expressed in terms of the retarded Green function as

$$
\chi_{b_kb^*_k}(\omega) = \frac{i}{\hbar} \cdot \frac{1}{\Omega} \int_0^\infty d(t-t') \int d\vec{x} d\vec{x}' \langle [\psi^H(\vec{x}, t), \psi^H(\vec{x}', t')] | \rho_{TE} \rangle e^{i \omega (t-t') - i\vec{\kappa} \cdot (\vec{x} - \vec{x}'),
$$

(B.2)

where $G(\vec{k}, \omega)$ is the Fourier transform of the retarded Green function

$$
G(\vec{x} - \vec{x}', t - t') = \left\{ \begin{array}{ll}
-i \langle [\psi^H(\vec{x}, t), \psi^H(\vec{x}', t')] | \rho_{TE} \rangle, & (t > t') \\
0, & (t < t')
\end{array} \right.
$$

(B.3)

with

$$
\psi^H(\vec{x}, t) = \frac{1}{\sqrt{\Omega}} \sum_k b^H_k(t) e^{i\vec{k} \cdot \vec{x}}, \quad b^H_k(t) = \frac{1}{\sqrt{\Omega}} \int d\vec{x} \psi^H(\vec{x}, t) e^{-i\vec{k} \cdot \vec{x}}.
$$

(B.4)

In Ref. 20, by eliminating the irrelevant part associated with the heat reservoir using the projection operator method, we derived the time-convolutionless (TCL) equation of motion for the reduced thermal state defined by (A.3), and found a general form of the admittance obtained using the relaxation method under thermal equilibrium initial conditions in terms of TFD. The admittance $\chi_{b_kb^*_k}(\omega)$ [(B.1)] takes the form

$$
\chi_{b_kb^*_k}(\omega) = \frac{i}{\hbar} \int_0^\infty dt \langle 1_S | b_k \langle 1_R | \{ b^H_k(-t) - b^H_k(-t) \} | \rho_{TE} \rangle e^{i \omega t},
$$

$$
= \frac{i}{\hbar} \int_0^\infty dt \langle 1_S | b_k \exp \left\{ i (\omega - \hat{\mathcal{H}}_S) t + \int_0^t d\tau \mathcal{C}(\tau) \right\} (b^*_k - \bar{b}_k) | \rho(0) \rangle
$$

$$
+ \frac{1}{\hbar} \int_0^\infty dt \int_0^t d\tau \langle 1_S | b_k \exp \left\{ i (\omega - \hat{\mathcal{H}}_S)(t - \tau) + \int_\tau^t ds \mathcal{C}(s) \right\}
$$

$$
\times \langle 1_R | \hat{\mathcal{H}}_{SR} \left\{ 1 + \int_\tau^t ds \exp(-i \hat{\mathcal{Q}} \hat{\mathcal{H}} \hat{\mathcal{Q}} s) Q \hat{\mathcal{H}}_{SR} P \exp(i \hat{\mathcal{H}} s) \right\}^{-1}
$$

$$
\times \exp(-i \hat{\mathcal{Q}} \hat{\mathcal{H}} \hat{\mathcal{Q}} \tau) Q (b^*_k - \bar{b}_k) | \rho_{TE} \rangle e^{i \omega \tau},
$$

(B.5)

with the projection operators $\mathcal{P}$ and $\mathcal{Q}$ defined by\(^{15}\)

$$
\mathcal{P} = | \rho_R \rangle \langle 1_R | = \rho_R | 1_R \rangle \langle 1_R |,
$$

$$
\mathcal{Q} = 1 - \mathcal{P},
$$

(B.6)
where $C(t)$ is the collision superoperator given by\(^{20}\)

$$C(t) = -i \langle 1_R | \hat{H}_{SR} \{ 1 + i \int_0^t d\tau \exp(-i \hat{Q} \hat{Q} \tau) \hat{Q} \hat{H}_{SR} \mathcal{P} \exp(i \hat{H} \tau) \}^{-1} \} | \rho_R \rangle. \quad (B.7)$$

Here, we have adopted the first-order renormalization given by (21) or (22) in Ref. 1) for the Hamiltonian of the boson system and the boson-reservoir interaction. In the expression for the admittance $\chi_{b_k b_k^\dagger}^r(\omega)$ given by (B-5), the first term is the admittance obtained using the conventional relaxation method under decoupled initial conditions, and the second term comes from the inhomogeneous term in the TCL equation for the reduced thermal state (A-3) and represents the effects of the initial correlation of the boson system and the reservoir.

The TCL equation is reduced to (A-4) in the second-order approximation for the system-reservoir interaction. Then, the admittance $\chi_{b_k b_k^\dagger}^r(\omega)$ [(B-5)] takes the form

$$\chi_{b_k b_k^\dagger}^r(\omega) = \frac{i}{\hbar} \int_0^\infty dt \langle 1_S | b_k \exp\{i (\omega - \hat{H}_S) t + \int_0^t d\tau C^{(2)}(\tau) \} (b_k^\dagger - \tilde{b}_k) | \rho(0) \rangle$$

$$+ \frac{i}{\hbar} \int_0^\infty dt \int_0^t d\tau \langle 1_S | b_k \exp\{-i \hat{H}_S \cdot (t - \tau) + \int_\tau^t ds C^{(2)}(s) \} | I_k^r(\tau) \rangle e^{i \omega t}, \quad (B.8)$$

with the second-order collision superoperator $C^{(2)}(t)$ given by (A-5), where $| I_k^r(\tau) \rangle$ corresponds to the inhomogeneous term in the second-order TCL equation and is given by

$$| I_k^{(2)}(t) \rangle = i \int_0^\beta d\beta' \langle 1_R | \hat{H}_{SR} \exp(-i \hat{H}_0 t) (b_k^\dagger - \tilde{b}_k) \rho_S \rho_R \exp(\beta' \hbar \hat{H}_0) \hat{H}_{SR} | 1 \rangle. \quad (B.9)$$

When the interaction of the boson system with the reservoir is taken as (3), $C^{(2)}(t)$ and $| I_k^{(2)}(t) \rangle$ become

$$C^{(2)}(t) = - \sum_k g_k^2 \int_0^t d\tau \{(b_k - \tilde{b}_k^\dagger) [b_k^\dagger(-\tau) R_k^-(-\tau) - \tilde{b}_k(-\tau) R_k^+(-\tau)] \}$$

$$+ (b_k^\dagger - \tilde{b}_k) [b_k(-\tau) R_k^+(\tau) - \tilde{b}_k^\dagger(-\tau) R_k^-(-\tau)] \}, \quad (B.10)$$

$$| I_k^{(2)}(t) \rangle = i \hbar \sum_k g_k^2 \int_0^\beta d\beta' \{ R_k^{(-t - i\hbar \beta')}^+(b_l - \tilde{b}_l)(b_k^\dagger(-t) - \tilde{b}_k^\dagger(-t)) \rho_S b_l^\dagger(-t - i\hbar \beta')$$

$$+ R_k^+(t - i\hbar \beta')(b_l - \tilde{b}_l)(b_k^\dagger(-t) - \tilde{b}_k^\dagger(-t)) \rho_S b_l(-t - i\hbar \beta') \} | 1 \rangle, \quad (B.11)$$

with the Heisenberg superoperators $b_k(t)$ and $\tilde{b}_k^\dagger(t)$ defined by

$$b_k(t) = \exp(i \hat{H}_S t) b_k \exp(-i \hat{H}_S t), \quad \tilde{b}_k^\dagger(t) = \exp(i \hat{H}_S t) \tilde{b}_k \exp(-i \hat{H}_S t), \quad (B.12)$$
where $R_k^+(t)$ and $R_k^-(t)$ are the reservoir correlation functions given by

$$
R_k^+(t) = \langle 1_R | R_k^2(t) | R_k(0) \rangle \rho_R ; \quad R_k^1(t) = \exp(i \mathcal{H}_R t / \hbar) R_k^1 \exp(-i \mathcal{H}_R t / \hbar), \quad (B.13)
$$

$$
R_k^-(t) = \langle 1_R | R_k(t) R_k^1(0) | \rho_R \rangle ; \quad R_k(t) = \exp(i \mathcal{H}_R t / \hbar) R_k \exp(-i \mathcal{H}_R t / \hbar). \quad (B.14)
$$

In the case of a rapidly damped reservoir (i.e., the reservoir correlation time approaches 0, $\tau_c \to 0$), the collision superoperator $C^{(2)}(t)$ and the inhomogeneous term $|I_{j}^{(2)}(t)|$ in Eq. (A.4) become

$$
C^{(2)}(t) \to C^{(2)} = C^{(2)}(\infty), \quad |I_{j}^{(2)}(t)| \to 0. \quad (\tau_c \to 0) \quad (B.15)
$$

In this limit, the admittance $\chi_{b_k b_k}^{r}(\omega)$ becomes

$$
\chi_{b_k b_k}^{r}(\omega) = (i / \hbar) \langle 1_S | b_k \{ i (\hat{\mathcal{H}}_S - \omega) - C^{(2)} \}^{-1} (b_k^+ - \tilde{b}_k) | \rho(0) \rangle, \quad (\tau_c \to 0) \quad (B.16)
$$

which is valid only in the van Hove limit\textsuperscript{21} or in the narrowing limit\textsuperscript{22} in which the heat reservoir is damped very rapidly. The admittance $\chi_{b_k b_k}^{r}(\omega) \quad [(B.16)]$ can be derived from Eq. (A.4) in the limit $t \to \infty$. The admittance $\chi_{b_k b_k}^{r}(\omega) \quad [(B.8)]$ can be expanded as

$$
\chi_{b_k b_k}^{r}(\omega) = \chi_{b_k b_k}^{r}(\omega) + \langle 1_S | b_k \{ i (\hat{\mathcal{H}}_S - \omega) - C^{(2)} \}^{-1} D_k^{(2)}(\omega) \rangle + \cdots = \chi_{b_k b_k}^{r}(\omega) + \cdots, \quad (B.17)
$$

as done in Refs. 9 and 11), where $|D_k^{(2)}(\omega)\rangle$ is the interference thermal state in the TCLE method\textsuperscript{1} and takes the form

$$
|D_k^{(2)}(\tau)| = - \frac{1}{\hbar} \int_0^\infty d\tau' \int_0^\beta d\beta' \langle 1_R | \hat{\mathcal{H}}_{SR} e^{-i \hat{\mathcal{H}}_0 \tau} (b_k^+ - \tilde{b}_k) \rho_R e^{i \beta' \mathcal{H}_0} \hat{\mathcal{H}}_{SR} e^{-i \beta' \mathcal{H}_0} | 1 \rangle e^{i \omega \tau}
$$

$$
+ \frac{i}{\hbar} \int_0^\infty d\tau \int_0^\tau ds \langle 1_R | \hat{\mathcal{H}}_{SR} e^{-i \hat{\mathcal{H}}_0 \tau} \hat{\mathcal{H}}_{SR} e^{i \hat{\mathcal{H}}_0 (\tau - s)} (b_k^+ - \tilde{b}_k) \rho_R | \rho(0) \rangle e^{i \omega s}.
$$

(B.18)

The first term of $|D_k^{(2)}(\omega)\rangle$ comes from the second term of $\chi_{b_k b_k}^{r}(\omega) \quad [(B.8)]$ and represents the effects of the initial correlation of the boson system and the reservoir. The second term of $|D_k^{(2)}(\omega)\rangle$ comes from the first term of $\chi_{b_k b_k}^{r}(\omega) \quad [(B.8)]$ and represents the memory effects.\textsuperscript{1} The expansion given by (B.17) shows that the admittance $\chi_{b_k b_k}^{r}(\omega) \quad [(B.8)]$ obtained using the relaxation method includes the admittance $\chi_{b_k b_k}^{r}(\omega)$ obtained using the TCLE method,\textsuperscript{1} and that the former admittance includes many more higher-order terms than the latter.

The admittance $\chi_{b_k b_k}^{rv}(\omega) \quad [(B.16)]$, which does not include the memory effects and the effects of the initial correlation of the boson system and the reservoir, can be calculated using Feynman-type diagram methods. However, it is not easy in general to calculate the admittance $\chi_{b_k b_k}^{r}(\omega) \quad [(B.5), \quad (B.8)]$, which includes those effects,
using such methods. The memory effects and the effects of the initial correlation of the boson system and the reservoir, which can be described by the terms representing the interference between the external driving field and the heat reservoir in the TCLE method, can be elucidated by calculating the admittance $\chi_{b_k b_k^\dagger}(\omega)$ obtained with the TCLE method.

References

21) L. van Hove, Physica **23** (1957), 441.