Non-negative depth reconstruction for a two-dimensional partial inertial inundation model
Hongbin Zhang, Yueling Wang, Qiuhua Liang, Luke S. Smith and Chris G. Kilsby

ABSTRACT
This paper proposes a non-negative depth reconstruction method for improving the numerical performance of a partial inertial model (PIM) for applications involving steep-slope and low-friction conditions. The PIM solves the continuity equation of two-dimensional (2D) shallow water equations (SWEs) with the interface fluxes evaluated by a simplified momentum equation that partially restores the inertial terms. In applying the PIM to flood simulations, a practical challenge is to represent complex topography and to track the moving wet–dry interface without resulting in negative water depths. Another challenge is to avoid the numerical issue caused by the lack of physical diffusive terms when it is applied to low-friction cases. To cope with these difficulties, the PIM is improved by introducing a non-negative depth reconstruction method, featuring two different ways for calculating the interface fluxes. The performance of the improved PIMs is investigated through applications to several theoretical and practical benchmark test cases. The comparison of the numerical results against analytical solutions or predictions from the original PIM and a full 2D finite-volume hydrodynamic model shows that the proposed reconstruction method can avoid non-negative water depth predictions, and improve the numerical performance of the original PIM when applied to steep-slope and low-friction conditions.

Key words | complex topography, flood inundation, hydraulic model, non-negative depth reconstruction, partial inertial model, wetting and drying

ABBREVIATIONS AND NOTATION

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1D | One-dimensional |
2D | Two-dimensional |
3D | Three-dimensional |
CFL | Courant–Friedrichs–Lewy |
CPU | Central processing unit |
DTM | Digital terrain model |
FDM | Finite difference method |
FEM | Finite element method |
FVM | Finite volume method |
GPU | Graphics processing unit |
PIM | Partial inertial model |
RMSE | Root mean square error |
SWE | Shallow water equation |
ZIM | Zero inertial model |

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INTRODUCTION

Numerous flood modelling tools have been developed and applied in practice during the last three decades with the increasing availability of high-resolution geographic data (e.g., airborne laser altimetry data) and more advanced computing resources (Bates & de Roo 2000; Stoesser et al. 2003; Hunter et al. 2005; Vidal et al. 2005; Yu & Lane 2006; Liang et al. 2008). One-dimensional (1D) models (e.g., Lai & Khan 2015) are not suitable in accurately simulating floodplain inundations and flows in meandering channels. While three-dimensional (3D) models (e.g., Chau & Jiang 2001, 2004; Haun et al. 2011) can produce more detailed predictions due to the more complete representation of flow processes, the complicated model structures result in quite high computational burdens, which limits their practical applications. In fact, for most inundation events over floodplains, the horizontal domain scale is significantly larger than the water depth, and thereby the vertical water particle acceleration of the flow can be ignored without affecting the modelling accuracy. Among the two-dimensional (2D) models, hydrodynamic models that solve the full 2D SWEs have proved to be particularly effective and represent the state-of-the-art (Fraccarollo & Toro 1995; Anastasiou & Chan 1997; Liang et al. 2008; Ai & Jin 2009; Kesserwani & Liang 2010). However, the full 2D SWEs, better representing the underlying physics of flood waves, generally require sophisticated computational methods to provide reliable and stable numerical solutions. Even if the resulting models are more efficient than 3D models, they are still computationally demanding for large-scale simulations.

Considerable effort has been made in order to balance model performance between computational cost and
solution accuracy (Hubbard & Dodd 2002; Neal et al. 2010; Liang 2011; Bravo et al. 2012; Liang & Smith 2013; Zhang et al. 2015b). One popular pathway towards increasing computational efficiency is to derive simplified models by omitting certain terms in the momentum equations of the full 2D formulation (Hunter et al. 2007). A common simplified form is the diffusion-wave approximation or zero inertial model (ZIM) achieved by neglecting the dynamic terms to give a momentum equation that represents steady but non-uniform open channel flows (Cunge et al. 1980; Bates & de Roo 2000; Horritt & Bates 2002; Bradbrook et al. 2004; Hunter et al. 2008; Wang et al. 2011). However, this method requires very small time steps to maintain stable simulations. It is suggested that the inertial terms should be included in order to overcome this problem (Hunter et al. 2008). Following this idea, Bates et al. (2010) derived a formulation that neglects the convective acceleration terms but retains the local acceleration term in order to improve numerical stability. The resulting model, referred to as the partial inertial model or PIM, has been validated through a number of test cases. It has since been demonstrated that partly restoring the inertial effects in the simplified model can greatly enhance numerical stability and hence save computational cost in simulating non-breaking and slowly varying flows (Dottori & Todini 2011; Fewtrell et al. 2011; Neal et al. 2011; Schumann et al. 2011; Chen et al. 2012; de Almeida et al. 2012; Stephens et al. 2012; de Almeida & Bates 2013). However, there are two main issues which may still lead to numerical problems in using the PIM. The first one is that the model may generate negative water depths in certain conditions where the water surface gradient is large. The original PIM only reconstructs the water depth at the interface by taking the difference between the maximum water level and the maximum bed elevation of either side of a cell interface, which maintains non-negative water depths in most conditions but not the case when a large water surface gradient exists. Large water surface gradient may result in large fluxes through the cell interface and therefore cause a negative water depth. The second numerical problem is caused by low friction when applying the original PIM as the scheme lacks diffusive terms and hence the stabilising effect that is offered at low-friction conditions (de Almeida et al. 2012).

This paper aims to tackle the aforementioned numerical issues in the context of using the PIM. The non-negative depth reconstruction method introduced by Liang & Marche (2009) is applied to improve wet–dry interface tracking in PIM, which directly reconstructs the water level instead of water depth as in the traditional PIM. Two different ways are adopted to calculate the interface fluxes and tested in the current improved PIM. One takes the maximum value of discharge at either side of the interface as proposed in the original PIM; the other adopts the average value of the discharge from the two sides of the interface to improve the performance of the model in handling flows over low-friction beds. Then the non-negative water depth condition is proved theoretically and validated against several benchmark test cases. Meanwhile, the results of the two improved PIMs are compared with the analytical solutions or the results obtained from the original PIM and a full 2D shallow flow model to demonstrate their capabilities and limitations for simulating flows under different conditions.

**METHODOLOGY AND MODEL DESCRIPTION**

The methodology framework of the PIM and the proposed numerical improvements are described in this section.

**Governing equations**

The PIM solves the following continuity equation based on mass conservation

\[
\frac{\partial \eta}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0
\]

(1)

where \(t\) denotes time, \(x\) and \(y\) are the Cartesian coordinates, \(\eta\) represents water level, and \(q_x = uh\) and \(q_y = vh\) are the unit width discharges, with \(h\), \(u\) and \(v\) being the water depth and velocity components in the two Cartesian directions, respectively. The discharges are approximated using a simplified formula derived from the full momentum equation. Diffusion-wave approximation indicates that the dynamic terms of the full momentum equations become insignificant and may be neglected without significantly affecting the
physical description of many slow-varying overland flows (Hunter et al. 2007). In this work, the simplified momentum formula proposed by Bates et al. (2010) is adopted, which is derived from the above diffusion-wave assumption by retaining some of the inertial effects. The main motivation for using the partial inertial formulation is to avoid very small time steps that result from the original diffusion-wave approximation.

Assuming rectangular channel cross-sections and negligible convection acceleration terms, the momentum equation can be simplified to become

$$\frac{\partial q_x}{\partial t} + gh \frac{\partial (h + z_b)}{\partial x} + \frac{gn^2q_x|q|}{h^{7/3}} = 0$$

(2)

where $g$ is the acceleration due to gravity, $z_b$ is the bed elevation and $n$ is the Manning coefficient. In the above equation, $\partial q_x/\partial t$ is the local acceleration term, $gh(\partial h/\partial x)$ is the pressure force term, $gh(\partial z_b/\partial x)$ is the slope source term representing the gravity effect, and $gn^2q_x|q|/h^{7/3}$ is the friction source term. An explicitly discretised equation for $q_x$ may be derived as follows:

$$q_x^{k+1} = q_x^k - gh^t \Delta t \left[ \Delta \frac{\partial q_x}{\partial x} + \frac{\partial n^2q_x|q|}{h^{7/3}} \right]$$

(3)

where $\Delta$ is the time step, $\Delta x$ is the cell size in $x$-direction, $\Delta \eta$ calculates the local difference of water level and $\|q\|$ returns the magnitude of $q$. Replacing $q_x^k$ in the friction term with $q_x^{k+1}$ to form a linear equation for the unknown $q_x^{k+1}$ creates a semi-implicit scheme to achieve better numerical stability (Bates et al. 2010); Equation (3) may then be rearranged to give

$$q_x^{k+1} = \frac{q_x^k - gh^t \Delta t \frac{\Delta \eta}{\Delta x} + \frac{\partial n^2q_x|q|}{h^{7/3}}}{1 + gh^t \Delta t \frac{\partial n^2q_x|q|}{h^{7/3}}}$$

(4)

This semi-implicit scheme can prevent the flow from changing direction under the condition of extremely high friction force, as an explicit scheme does. It also avoids the heavy computational burden of an iterative full-implicit scheme.

From Equation (4), the static equilibrium of the numerical scheme, or the so-called $C$-property (Bermúdez & Vázquez 1994), can be preserved because zero velocities and invariant water surface gradient in still water can keep the water motionless at the next time step. Future work is required to extend the scheme to also preserve the dynamic equilibrium (Valiani & Begnudelli 2006; Creaco et al. 2010).

At each time step during a simulation, $q_x$ and $q_y$ (which may be similarly derived) are updated using Equation (4) and combined with Equation (1) to update the water level.

Numerical scheme

In this work, Equation (1) is solved using a finite volume method (FVM) based on the structured grid, which results in the following time-marching formula for updating the water level at a new time step

$$\eta_{ij}^{k+1} = \eta_{ij}^k - \Delta \left( \frac{q_{ix}^{k+1} - q_{iw}^{k+1}}{\Delta x} + \frac{q_{jw}^{k+1} - q_{jN}^{k+1}}{\Delta y} \right)$$

(5)

The main advantage of FVM against other popular numerical approaches including finite difference method and finite element method is that FVM can maintain local mass and momentum conservations in each computational cell and can be flexibly formulated for unstructured grid systems with less computational cost (Hinkelmann 2005).

Flux calculation

Taking the eastern interface as an example, $q_{xE}$ is estimated using Equation (4) as

$$q_{xE}^{k+1} = \frac{q_{xE} - gh^t \Delta t \frac{n^2}{\Delta x} + \frac{\partial n^2|q|}{h^{7/3}}}{1 + gh^t \Delta t \frac{\partial n^2|q|}{h^{7/3}}}$$

(6)

Herein the values of flow variables on the right-hand side are obtained at time level $k$. The fluxes through the other three cell interfaces at time level $k$ can be evaluated in a similar way. Subsequently, the updated discharges at the centre of cell $(i,j)$ are simply evaluated by averaging...
the two associated fluxes:

\[ q_{xL}^{k+1} = \left( q_{xE}^{k+1} + q_{xW}^{k+1} \right)/2 \quad \text{and} \quad q_{xR}^{k+1} = \left( q_{xN}^{k+1} + q_{xS}^{k+1} \right)/2 \]  \( (7) \)

In order to estimate the interface values of the flow variables in Equation (6), the face values at the left and right sides of the interface may firstly be approximated to be the same as the corresponding cell-centred values, which leads to a first order numerical scheme (Zhang et al. 2013a). The left face values of the eastern interface are therefore given by

\[ h^L_{i+1,j} = h^L_{i,j}, \quad \eta^L_{i+1,j} = \eta^L_{i,j} - h_{i+1,j}^L, \quad q_{xEi}^L = q_{xEi}^j \]  \( (8) \)

Similarly, the corresponding right face values are

\[ h^R_{i+1,j} = h^R_{i,j}, \quad \eta^R_{i+1,j} = \eta^R_{i,j} - h_{i+1,j}^R, \quad q_{xRi}^R = q_{xRi}^{j+1} \]  \( (9) \)

The corresponding velocity components are then evaluated by

\[ u^L_{i,j} = \frac{q^L_{xE} - q^L_{xW}}{h^L_{i,j}}, \quad u^R_{i,j} = \frac{q^R_{xE} - q^R_{xW}}{h^R_{i,j}} \]  \( (10) \)

In a dry cell (defined as \( h < 10^{-10} \) m) (Brufau & García-Navarro 2003), the velocities are set to be zero directly and not evaluated by Equation (10).

The original PIM reconstructs water depth at the interface by taking the difference between the maximum water level and the maximum bed elevation at either side of the interface as

\[ h_i = \max (\eta^L_{i,j}, \eta^R_{i,j}) - \max (\eta^L_{i,j}, \eta^R_{i,j}) \]  \( (11) \)

This reconstruction maintains non-negative water depth in most situations. However, large velocity and negative water depth may still occur when the water surface gradient is large. For example, as shown in Figure 1(b), when the difference between water levels at the left and right side of an interface becomes too large, a large flux can be generated as a result of Equation (6), which in turn calculates a negative water depth from Equation (5). In order to ensure non-negative water depth, water level, instead of water depth, should be reconstructed.

A single value of bed elevation at a cell interface is defined (Audusse et al. 2004) as

\[ z_{bE} = \max (\eta^L_{bE}, \eta^R_{bE}) \]  \( (12) \)

The water depths at either side of the interface are then reconstructed as

\[ h^L_{i,j} = \max (0, \eta^L_{i,j} - z_{bE}), \quad h^R_{i,j} = \max (0, \eta^R_{i,j} - z_{bE}) \]  \( (13) \)

which obviously ensures non-negative water depth. Based on Equations (12) and (13), the corresponding water level and unit width discharge are then reconstructed as

\[ \eta^L_{i,j} = h^L_{i,j} + z_{bE}, \quad \eta^R_{i,j} = h^R_{i,j} + z_{bE} \]

\[ q_{xLj}^R = u^L_{i,j} h^L_{i,j}, \quad q_{xRj}^L = u^R_{i,j} h^R_{i,j} \]  \( (14) \)

In the above reconstruction, it is evident that the water level coincides with the bed elevation in a dry cell. If the dry cell has a bed elevation higher than the water level in a neighbouring cell, as shown in Figure 1(c), a spurious flux will be calculated between the dry cell and the adjacent wet cell due to the higher water level on the dry side. To avoid this unphysical numerical flux, apart from reconstructing the bed elevation and water levels by following Equations (12)–(14), the difference between the fake and actual water level must be identified and subtracted from

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**Figure 1** | Three generalised configurations for advancing wet-dry front.
the reconstructed values. A general formula for specifying this difference is given as follows:

$$\Delta z = \max(0, (z_{BE} - \eta^E))$$  \hspace{1cm} (15)$$

The associated bed elevation and water levels are subsequently modified by subtracting $\Delta z$ from their original values as

$$z_{BE} - \Delta z \rightarrow z_{BE}, \eta^E - \Delta z \rightarrow \eta^E, \eta^R - \Delta z \rightarrow \eta^R$$  \hspace{1cm} (16)$$

These modified face values are then substituted into the right-hand side of Equation (6) to calculate the flux, for which the interface values of the water depth and discharge ($h_E$ and $q_{xE}$) are also required. Similar to the original PIM, the maximum water depth from either side of the interface is used in this work. When deciding the interface discharge $q_{xE}$, two options are available and tested herein.

(1) Take the maximum unit width discharge at either side of the interface

When $q^L_{xE}$ and $q^R_{xE} \geq 0$

$$q_{xE} = \max(q^L_{xE}, q^R_{xE})$$  \hspace{1cm} (17)$$

When $q^L_{xE} < 0$ and $q^R_{xE} < 0$

$$q_{xE} = \min(q^L_{xE}, q^R_{xE})$$  \hspace{1cm} (18)$$

This is essentially the same scheme as in the original PIM. Using the larger values in flux calculation may help to partly overcome the negative effect of neglecting the convective acceleration terms in the momentum equation. Together with the new non-negative depth reconstruction as proposed in the current work, this model is hereafter referred to as ‘improved PIM1’, which will be used to demonstrate the improvement due to the water level reconstruction.

(2) Take the average unit width discharge of both sides of the interface

$$q_{xE} = (q^L_{xE} + q^R_{xE})/2.$$  \hspace{1cm} (19)$$

A mathematical proof has shown that the original PIM is not able to produce a stable solution when friction is low, and the adoption of average face value of discharge may mitigate this problem by introducing extra numerical diffusion \cite{deAlmeida2002}. Combining with the current water level reconstruction, this new scheme is called ‘improved PIM2’ in the rest of the text, which will be tested for low-friction simulations.

Validity of the non-negative depth reconstruction

During a flood event, the wetting and drying process is commonly associated with both advancing and retreating wet–dry fronts. In a 1D manner, an advancing wet–dry front may fall into one of the three generalised configurations as illustrated in Figure 1.

Figure 1(a) represents a wet–dry front moving onto a dry horizontal bed. Since it is an advancing front, the flow directions at eastern and western cell interfaces are both positive, that is, from left to right. Assuming the worst case where there is no flux entering the cell under consideration from the western interface, the finite volume updating formula becomes (the temporally constant bed component has been subtracted from both sides of the formula):

$$h^k_{i+1} = h^k_i - \frac{\Delta t}{\Delta x} q^k_{xE} = h^k_i - \frac{\Delta t}{\Delta x} u^E_E h^k_E$$  \hspace{1cm} (20)$$

Defining the Courant number \cite{Courant1967} as

$$C_r = \frac{\Delta t}{\Delta x} u$$  \hspace{1cm} (21)$$

Equation (20) can be then rewritten as follows:

$$h^k_{i+1} = h^k_i - C_r h^k_E$$  \hspace{1cm} (22)$$

Since the Courant number is taken to be less than 1 for an explicit scheme and $h^k_E \leq h^k_i$ is always true from the reconstruction, we have $h^k_{i+1} \geq 0$, that is, non-negative water depth is predicted for the case as shown in Figure 1(a).

Figure 1(b) shows a wet–dry front moving onto an initially dry bed with a downhill step. If the step size is large enough, excessive water surface gradient term in
Equation (6) will be calculated, which may lead to a large updated discharge and consequently a negative water depth in the cell under consideration. In this work, following the aforementioned reconstruction, this case actually becomes that as illustrated in Figure 1(a), and therefore, negative water depth is avoided.

Figure 1(c) demonstrates a case when the flow hits a wall. In this case, the local bed modification avoids the calculation of fake water surface gradients and subsequent generation of spurious fluxes. At the eastern interface of the cell under consideration, the aforementioned reconstruction leads to \( z_{BE} = h_E^L, h_E^R = h_R^E = 0 \) and hence \( h_E^E = 0 \), which will lead to a zero flux across the interface. Since there is no outgoing flux, \( h_E^{E+1} \) will be equal to or greater than \( h_L^E \), depending on whether there is incoming flux from the western interface.

Therefore, the current non-negative depth reconstruction method ensures non-negative water depth for advancing wet–dry fronts. The analysis can be easily extended to prove the validity of the method for the retreating wet–dry fronts.

**Time step and boundary conditions**

The stability of current overall explicit numerical scheme is controlled by the Courant–Friedrichs–Lewy criterion, which may be used to predict an appropriate time step for the next iteration during a simulation. For example, on a 2D Cartesian uniform grid,

\[
\Delta t = C_t \min \left( \min_{ij} \Delta x \frac{|u_{ij}| + \sqrt{g h_{ij}}}{|u_{ij}| + \sqrt{g h_{ij}}}, \min_{ij} \Delta y \frac{|v_{ij}| + \sqrt{g h_{ij}}}{|v_{ij}| + \sqrt{g h_{ij}}} \right)
\]

Herein, the Courant number \( C_t \) is set to 0.75 for all test cases considered in this work.

In this work, two types of boundary conditions are used, that is, open and closed boundary conditions (Toro 2001). At open boundaries, the gradients of flow variables are assumed to be zero at the boundary, which may be achieved by imposing the following flow information in the ghost cells

\[
\begin{align*}
\eta_0 &= \eta_1, q_{x0} = q_{x1}, q_{y0} = q_{y1}, \eta_{m+1} = \eta_m, q_{xm+1} = q_{xm}, \eta_{m+1} = \eta_m, q_{ym+1} = q_{ym}
\end{align*}
\]

At closed boundaries, normal discharge, and the gradients of the water level and tangential discharge are required to be zero at the boundary, which may be achieved by

\[
\begin{align*}
\eta_0 &= \eta_1, q_{x0} = q_{x1}, q_{y0} = q_{y1}, \eta_{m+1} = \eta_m, q_{xm+1} = q_{xm}, q_{ym+1} = q_{ym}
\end{align*}
\]

**RESULTS AND DISCUSSION**

In this section, ‘improved PIM1’ and ‘improved PIM2’ are validated by applying them to simulate several carefully selected test cases. Results are compared with either analytical solutions or numerical predictions produced by the original PIM formulation and a first-order accurate full 2D shallow flow model (Liang 2010). Root mean square error (RMSE) and fit statistics (\( F^1 \) and \( F^2 \)) (Aronica et al. 2002) are employed to quantitatively analyse the results. In all of the simulations, \( g = 9.81 \text{ m/s}^2 \) is used. All simulations are performed on a standard personal desktop with an Intel(R) Core(TM) i5 central processing unit (CPU).

**Tidal flow over a beach with a varying slope**

The aim of this test case is to show the capability of the ‘improved PIM1’ in conserving non-negative water depth and the improvement over the original PIM by reproducing a moving shoreline triggered by a tidal cycle on a beach with a varying slope. As shown in Figure 2, the bed slope is defined as

\[
z_b(x) = \begin{cases} 
-0.001x + 1.4 & x \leq 100 \text{ m} \\
-0.01x + 2.3 & 100 \text{ m} < x < 200 \text{ m} \\
-0.001x + 0.5 & x \geq 200 \text{ m}
\end{cases}
\]

The computational domain is set up to be 500 m long along the flow direction and 100 m wide, with the bottom left corner being \((0, 0)\). It is discretised by a uniform grid of \(5 \times 5\) m resolution. Initially, the water body inside the domain is still at a water level of 1.75 m. A time-varying water depth is imposed at the eastern end of the domain.
as an inflow condition

\[ h(t) = 1.0 + 0.75 \cos \left( \frac{2\pi t}{T} \right) \]  

(27)

where the tidal period is assumed to be \( T = 3,600 \) s. In a tidal cycle, the water level decreases gradually from 1.75 m and begins to increase after reaching the minimum at \( T/2 \), as driven by the inflow. The water surface moves up slowly until the whole domain is submerged and the water level returns to the original 1.75 m. The western boundary is closed and all other boundary walls are transmissive. A constant Manning coefficient of \( 0.03 \) m\(^{-1/3}\) s is used throughout the whole domain.

Figure 2 | Tidal flow over a beach with a varying slope: water level profiles calculated by the full 2D shallow flow model, original PIM and ‘improved PIM1’ at different output times.
Profiles of the water level and flow velocity along the longitudinal central line of the domain are presented in Figures 2 and 3, at 720 s intervals throughout one tidal cycle. It is observed that the water level predicted by both of the original PIM and ‘improved PIM1’ agrees closely with that obtained by the full 2D shallow flow model and no negative water depth is predicted at the wet–dry front, thus confirming the capability of the original PIM and ‘improved PIM1’ for simulating slow-varying flows over irregular bed topography.

However, as is evident from Figure 3, the velocity produced by the original PIM shows a clear deviation from the results obtained by the ‘improved PIM1’ and the full 2D shallow flow model around $x = 100$ m at $t = 1,440$, $2,160$ and $2,880$ s. This is because the bed slope changes abruptly at this point and the water level changes

![Figure 3](https://iwaponline.com/jh/article-pdf/16/5/1158/387430/1158.pdf)

Figure 3 | Tidal flow over a beach with a varying slope: flow velocity profiles calculated by the full 2D shallow flow model, original PIM and ‘improved PIM1’ at different output times.
accordingly when the water is very shallow. As previously mentioned, when the water surface gradient is large, a large velocity may be predicted, which in turn may lead to negative depth during a simulation. Therefore, for extreme conditions, negative water depths may arise in the framework of the original PIM. However, after the water level reconstruction as introduced in the ‘improved PIM1’, a relatively smaller water surface gradient is calculated and the risk of producing a large velocity and negative water depth is thus minimised.

In terms of computational cost, the original PIM requires 2.8 s CPU time to finish the simulation for one tidal cycle, while the ‘improved PIM1’ and the full 2D shallow flow model costs 3.1 and 4.3 s, respectively. Therefore, the computational efficiency of the ‘improved PIM1’ is similar to the original PIM and the PIMs save about 28% of computational cost in comparison with the full 2D shallow flow model.

The temporal change in the water level and flow velocity at $x = 100$ m is also recorded during the simulation and plotted in Figure 4. The results of the original PIM and ‘improved PIM1’ are observed to agree very well with the outputs from the full 2D shallow flow model in terms of the water level, but the original PIM is less satisfactory in reproducing the velocity when compared with the ‘improved PIM1’. In order to quantify the results, the relative RMSE against the full 2D shallow flow model is calculated and illustrated in Figure 5. With a maximum error smaller than 5%, both simulations are satisfactory in reproducing the water level. The error in velocity is much less satisfactory due to the exclusion of the convective acceleration terms, but the error calculated by the ‘improved PIM1’ is apparently smaller than the original PIM between $t = 720$ and 1,440 s, which again confirms its numerical improvement as a result of using the water level reconstruction.

The effect of temporal and spatial resolutions on the performance of the ‘improved PIM1’ has also been investigated. The results from the full 2D shallow flow model with Courant number 0.1 and cell size 2.5 m × 2.5 m are utilised as the reference for calculating relative errors. The relative RMSEs increase with larger time step and cell size, as presented in Table 1, which appear to be consistent.

The Manning coefficient in this work is chosen to be the same as those used in the literature (Liang & Marche 2009; Wang et al. 2011) for the purpose of model comparison. A series of simulations have also been conducted using the ‘improved PIM1’ to investigate the sensitivity of model results to Manning’s $n$. As seen in Figure 6, the flow velocity generally decreases with the increasing Manning coefficient, which thereby results in increasing water level. However, when the Manning coefficient is very small, for example, 0.03 m$^{-1/3}$, less stable flow velocities occur, which also affects the water level prediction. This will be further examined in the next test case of ‘moving shorelines over parabolic topography’.

**Moving shorelines over parabolic topography**

Analytical solutions of the non-linear shallow water equations were derived for the perturbed flow over a
parabolic fictional bed topography (Sampson et al. 2006). This benchmark test is used herein to validate the capability of the original PIM and ‘improved PIM1’ in handling wetting and drying over non-uniform topography and also to investigate the limitation of these models in simulating flows on low-friction beds. The bed profile is described by

\[ z_b(x) = h_{00}(x/a)^2 \]  \hspace{1cm} (28)

The analytical solution for this test depends on a bed friction parameter \( \tau \) and a hump amplitude parameter.
\[ p = \sqrt{8gh_0}/a^2. \] The projection of the moving shorelines (two parallel straight lines on the x-y plane) is given by

\[ x_{11,22} = \frac{a^2 e^{-\tau t/2}}{2gh_0} \left( -Bs \cos st - \frac{\tau B}{2} \sin st \right) \pm a \] (29)

As \( t \to \infty \), \( x_{11} \to -a \) and \( x_{22} \to +a \), which indicates that the momentum of the oscillatory flow is gradually dissipated by bed friction.

Herein, we consider a case with \( \tau < p \). The temporal evolution of the water surface and velocity for \( x_{11} \leq x \leq x_{22} \) is given by

\[
\eta(x, t) = h_0 + \frac{a^2 e^{-\tau t}}{8gh_0} \left( (-s \tau \sin 2st + (\tau^2/4 - s^2) \cos 2st) - \frac{B^2 e^{-\tau t}}{4g} - \frac{e^{-\tau t/2}}{g} \left( Bs \cos st + \frac{\tau B}{2} \sin st \right) x \right)
\] (30)

\[
u(t) = Be^{-\tau t/2} \sin st
\] (31)

where \( s = \sqrt{p^2 - \tau^2/2}. \) As \( t \to \infty \), \( \eta(x, t) \to h_0 \) and \( \nu(t) \to 0. \)

During the simulation, the closed 10,000 m \( \times \) 15 m computational domain with the origin defined at the domain centre is discretised by 2,000 \( \times \) 3 cells. The relevant coefficients are set to be \( a = 3 \) km, \( h_0 = 10 \) m, \( \tau = 0.001 \) s\(^{-1}\) and \( B = 5 \) m/s. Figure 7 demonstrates the profile of the water surface produced by the original PIM and ‘improved PIM1’ along the x-axis at different output times throughout the simulation, compared with the analytical solution. The original PIM and ‘improved PIM1’ produce similar but less satisfactory results when compared with the analytical solution, especially during the early stage of the oscillating flow, due to their inability to simulate flows with large velocities. The numerical solutions agree well with the analytical solution at a later stage when the oscillating flow is approaching steady state owing to the effects of friction. This indicates that, while they do not excel in representing the detailed processes, the PIMs are able to simulate the final steady state of an event with high accuracy. Furthermore, the ‘improved PIM1’ does not predict negative water depths at any point during the simulation, which confirms the effectiveness of the presented non-negative depth reconstruction method.

In order to further evaluate the performance of the aforementioned models for low-friction beds, simulations are evaluated with \( \tau = 0.0005 \) s\(^{-1}\). Although no negative water depth is predicted, both of the models are not able to produce stable solutions due to the lack of diffusive terms and the insufficient stabilising effect provided by low friction forces. The ‘improved PIM2’ is then employed to investigate whether using the average face value of discharge when calculating fluxes can improve numerical stability. From Figure 8, it can be observed that although the ‘improved PIM2’ is also not able to reproduce the detailed processes very accurately, it improves the numerical stability and produces stable solution during the simulation due to the extra numerical diffusion introduced by the scheme (de Almeida et al. 2012).

**Thamesmead flood inundation**

Finally, the original PIM, ‘improved PIM1’ and ‘improved PIM2’ are applied to simulate a hypothetical inundation event at Thamesmead, located at the south bank of the River Thames in the UK. The results are compared with those produced by the full 2D shallow flow model. A processed 10 m bare-earth DTM as shown in Figure 9 is used to represent the selected 9,000 m \( \times \) 4,000 m floodplain. A 150 m wide breach in the embankment centred at (545,855, 181,040) is assumed to generate an inundation event, where the coordinates at the bottom left corner are (545,000, 178,000). The flow hydrograph through the breach is shown in Figure 10. During the 10-hour simulation, the whole floodplain with open boundary conditions is discretised by a uniform grid with 900 \( \times \) 400 cells and a constant Manning coefficient \( n = 0.035 \) m\(^{-1/3}\) s is applied (Liang et al. 2008).

Figure 11 displays the inundation maps at \( t = 10 \) hours produced by the four models, respectively. With regard to flood extents, the original PIM and the ‘improved PIM1’ are in close agreement with the full 2D shallow flow model, but the ‘improved PIM2’ produces a smaller flood extent. This is related to the different ways that these models calculate the interface fluxes. In practical applications, the convective acceleration is generally in the same direction as the velocity and hence accelerates the...
flow, so the omission of convective acceleration terms usually slows down the flow propagation. Therefore taking the maximum face value of discharge to calculate the interface fluxes can compensate, to some extent, the effect as a result of omitting the convective acceleration terms, and hence generates more satisfactory results than taking the average face value in this case. In addition, for practical applications the bed friction is normally large enough to stabilise the solution, and taking the average face value as the interface flux may introduce unwanted numerical diffusion and slow down further flow propagation. Therefore, the flux calculation scheme as implemented in ‘improved PIM2’ may not be recommended for realistic flood simulations.

Figure 7 | Moving shorelines over parabolic topography: profiles of the water surface along the x-axis at different times for \( \tau = 0.001 \text{s}^{-1} \).
In order to further compare the simulation results, the temporal changes in the water depth and flow velocity are recorded at four gauge points as indicated in Figure 9 and compared in Figures 12 and 13. Generally, the water depth and velocity predicted by the three PIMs follow the trends as predicted by the full 2D shallow flow model, but deviation between the results is also evident, especially for the ‘improved PIM2’. Excellent agreement can be found between the original PIM and ‘improved PIM1’ in both water depth and flow velocity predictions, which implies that for practical conditions where the bed slope and flow hydrodynamics change gently, original
PIM may be capable of producing non-negative water depths without the water level reconstruction. But obviously, this conclusion can only be drawn case by case.

To further quantify the performance of the three PIMs, the relative RMSE and fit statistics ($F^1$ and $F^2$) are calculated for the whole simulation and plotted in Figures 14 and 15, with the numerical solution from the full 2D shallow flow model given as a reference. As can be seen in Figure 14(a), the relative RMSE of the water depth for the original PIM and ‘improved PIM1’ is below 25%. The behaviour of the ‘improved PIM2’ is slightly less satisfactory with the maximum relative RMSE reaching nearly 30%. Meanwhile, the capability of all three PIMs is not satisfactory in resolving the flow velocity, as shown in Figure 14(b), where the error may sometimes reach above 100%. Again, the omission of convective acceleration terms is the main reason why these PIMs cannot give reasonable predictions in the flow velocity. From the fit statistics ($F^1$ and $F^2$) presented in Figure 15, it can be seen that correct state of wet–dry cells predicted by the original PIM can match up to 96% with the full 2D shallow flow model, and the flood extents predicted by these two models agree over 70% for most of the simulation. The ‘improved PIM1’ is able to produce equally good results as the original PIM, but the results of the ‘improved PIM2’ are generally less impressive,
compared with the other two PIMs. As a whole, these results confirm that the performance of the original PIM and ‘improved PIM1’ are satisfactory in terms of predicting water depths and flood extents, when compared with the full 2D shallow flow model. With respect to the computational time, the original PIM, ‘improved PIM1’, ‘improved PIM2’ and full 2D shallow flow model cost 3,491, 3,574, 3,576 and 4,717 s, respectively, for the 10-hour simulation. The ‘improved PIM1’ and ‘improved PIM2’ require 2.3% more computational time than the original PIM and save 24.2% when compared with the full 2D shallow flow model.

**CONCLUSIONS**

In order to balance the computational cost and solution accuracy, a number of numerical or mathematical techniques have been implemented to improve full 2D shallow flow models for flood simulations. The PIM represents one of these techniques, and solves the continuity equation with interface fluxes evaluated by a momentum formula neglecting the convective acceleration term from the full equation. In this work, a non-negative depth reconstruction method with two ways for evaluating the interface fluxes has been proposed to improve the numerical performance of the original PIM for simulations over steep-slope and low-friction domains. The proposed non-negative depth reconstruction technique ensures the prediction of non-negative water depth for simulations over complex domain topographies, which has been proved theoretically. The two approaches of evaluating interface fluxes have been compared for low-friction simulations.

The capabilities and limitations of the improved PIMs have been evaluated through applications to simulate two theoretical test cases and one practical flood inundation...
The validity of the non-negative depth reconstruction method has been confirmed in all of the tests. By applying to benchmark test cases, the two PIMs proposed in this work and the original PIM reported by Bates et al. (2010) are demonstrated to be not able to provide accurate predictions for rapid-varying flow with large flow velocity due to the omission of convective acceleration terms. Therefore, the PIM models considered in this work are only proposed for applications related to slow-varying flood inundation events.

Figure 13 | Thamesmead inundation: temporal change in flow velocity at four gauge points.

Figure 14 | Thamesmead inundation: time histories of the relative RMSE for the water depth (a) and flow velocity (b).
In practical conditions with a gentle bed slope and water gradient, the original PIM and ‘improved PIM1’ featuring different face value reconstruction techniques can produce similar simulation results without causing negative depths. However, the ‘improved PIM1’ has been proved to be a better choice for more accurate and stable simulations when handling applications associated with steep bed slopes, due to the use of the proposed water level reconstruction technique.

For flows over low-friction beds, neither the original PIM nor the ‘improved PIM1’ can provide stable solutions. The ‘improved PIM2’ has shown to be effective for this type of application due to the use of a more diffusive way in calculating interface fluxes, which essentially helps stabilise the flow on low-friction beds. However, as shown in the Thamesmead inundation test case, the ‘improved PIM2’ may introduce unwelcome numerical diffusion and lead to inaccurate results in the practical inundation simulations, where the bed roughly is generally high enough. de Almeida et al. (2012) proposed a method within the original PIM for addressing this problem by using a weighing factor to adjust the effect of the numerical diffusion term, but the selection of the weighing factor is entirely case-dependent. Future work may be needed to design a criterion in setting the weighing factor automatically according to the bed friction.

In terms of computational cost, due to the use of much simplified governing equations and corresponding simple numerical scheme (Liang 2010). Future work will be to explore the use of modern high-performance computational techniques, for example, adaptive grids, and computing hardware, e.g., graphics processing units (GPUs), to further accelerate this type of model.

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