The terms $1/N$ and Δ in the transcendental Equation [25] are given in full by Equations [23] and [24]. Expanding the Bessel functions, Δ may be written in the form

$$\Delta = \frac{\left[1 - \frac{a^2}{4}\left(\frac{\Omega^2}{c^2} - \xi^2\right) + \frac{a^4}{64}\left(\frac{\Omega^2}{c^2} - \xi^2\right)^2 + \ldots\right]}{\frac{a^2}{4}\left(\frac{\Omega^2}{c^2} - \xi^2\right)\left[1 - \frac{a^2}{8}\left(\frac{\Omega^2}{c^2} - \xi^2\right) + \frac{a^4}{192}\left(\frac{\Omega^2}{c_2} - \xi^2\right)^2 + \ldots\right]} \quad [72]$$

In order to be able to clear Equation [25] of fractions, the quotient of the two infinite series in Equation [72] is written as a single series

$$\Delta = \frac{1}{\frac{a^2}{4}\left(\frac{\Omega^2}{c^2} - \xi^2\right)}\left[1 - \frac{a^2}{8}\left(\frac{\Omega^2}{c^2} - \xi^2\right) - \frac{a^4}{192}\left(\frac{\Omega^2}{c^2} - \xi^2\right)^2 + \ldots\right] \quad [73]$$

Using this value the transcendental Equation [25] may be put in the form

$$\frac{2\rho_0\Omega^2}{ma}\left(\frac{\Omega^2}{c_0{}^2} - \xi^2\right)\left[1 - \frac{a^2}{8}\left(\frac{\Omega^2}{c^2} - \xi^2\right) - \frac{a^4}{192}\left(\frac{\Omega^2}{c^2} - \xi^2\right)^2 + \ldots\right] + \Omega^2\left(\frac{\Omega^2}{c^2} - \xi^2\right)\left(\frac{\Omega^2}{c_0{}^2} - \xi^2\right)\left[-1 - \frac{h^2\xi^2}{12}\right]$$
$$+ \left(\frac{\Omega^2}{c^2} - \xi^2\right)\left(\frac{\Omega^2}{c_0{}^2} - \xi^2\right)\frac{h^2c_0{}^2}{12}\left[\frac{1}{a^4} + \xi^4\right] + \left(\frac{\Omega^2}{c^2} - \xi^2\right)\frac{1}{a^2}\left[\Omega^2 - c_0{}^2 - c_0{}^2(1 - \nu^2)\xi^2 + \frac{\nu h^2 c_0{}^2\xi^4}{6} + \frac{c_0{}^2h^4\xi^6}{144}\right] = 0 \quad [74]$$

Now the assumed expansion given by Equation [71] is substituted in the foregoing for Ω. The result is a series of the form

$$b_1\xi^4 + b_2\xi^6 + b_3\xi^8 + \ldots = 0 \quad [75]$$

Each coefficient b_n of this series must be zero since the value Ω is a root of the transcendental Equation [25]. Equating b_1 to zero and neglecting h^2/a^2 compared to unity yields an equation involving $\bar{c}$ but not d. This is interpreted as an equation on the yet unknown $\bar{c}$. The roots are

$$c_1, c_2 = c\left[\frac{2AR + R + R^2(1 - \nu^2) \mp \sqrt{[2AR + R + R^2(1 - \nu^2)]^2 - 4R^2(1 - \nu^2)(2A + R)}}{2(2A + R)}\right] \quad [76]$$

where

$$R = \frac{c_0{}^2}{c^2}, \qquad A = \frac{\rho_0 a}{m} \quad [77]$$

Equating b_2 to zero yields an equation involving both $\bar{c}$ and d. This equation is considered to determine two values of d, one for $\bar{c} = c_1$ and one for $\bar{c} = c_2$. The solution of the equation $b_2 = 0$ for d yields

$$d = ca^2\left[\frac{(A + 4)\left[\left(\frac{\bar{c}}{c}\right)^5 - \left(\frac{\bar{c}}{c}\right)^3(1 + R) + \left(\frac{\bar{c}}{c}\right)R\right] - \left(\frac{\bar{c}}{c}\right)\frac{h^2}{12a^2}(1 + 2R^2\nu)}{-16\left(\frac{\bar{c}}{c}\right)^2(2A + R) + 8R(2A + 1) + 8R^2\ (1 - \nu^2)}\right] \quad [78]$$

Discussion

John Parmakian.[11] This paper presents a "closer look" at the principles underlying basic water-hammer theory. For practical purposes it confirms previously established theory which has been verified experimentally on many occasions. However, it indicates some interesting possibilities with reference to very fast precursor waves in the pipe wall which may have a very minor effect on the pressure changes inside the pipe. It also estimates the length over which a sharp water-hammer wave front may extend as it moves along the axis of the pipe.

For the typical example shown in the paper at a distance of about 3 miles from the origin of the initial sharp wave form, the entire length of the extended wave front will pass a fixed point in the pipe line in about 0.01 sec. Therefore, in order to detect changes in the wave-front form, several very high-speed pressure pickups would be required at stations located many miles apart.

During the testing of the Grand Coulee Pumping Plant several years ago the writer had an opportunity to study high-frequency pressure-wave forms at two stations in the pump-discharge line.

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These stations were located about 640 ft apart. The source of the high-frequency pressure waves was the pump which since has been modified. An oscillogram of the pressure changes recorded by the high-speed pressure pickups is shown in Fig. 7, herewith. No noticeable difference in the pressure-wave forms at these two stations could be detected. Perhaps the 640-ft distance between the two stations was too short and the initial wave fronts not sharp enough. In any case this experience may be indicative of the difficulty in detecting the wave-distortion phenomena of the type described in the paper.

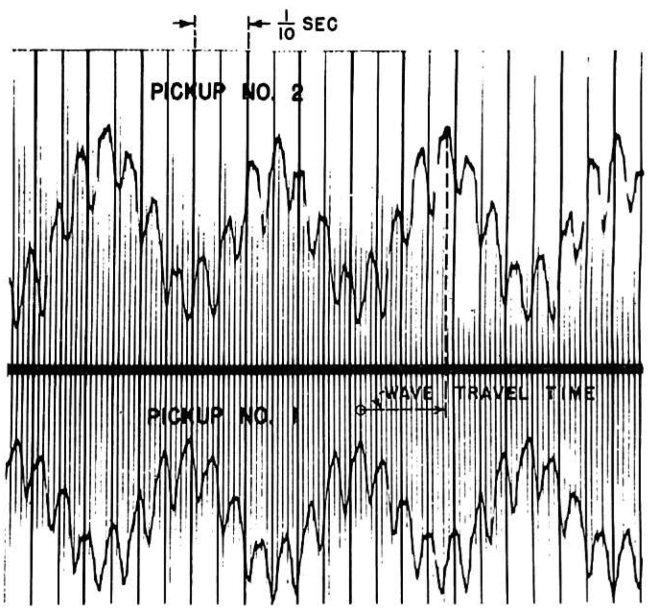

Fig. 7 Oscillogram of Pressure Changes

Although the earlier water-hammer theories have been developed for a pipe consisting of individual elements which are freely extensible as noted in the paper, some of the later writers[12] have included in the theory the effect of the longitudinal restraint of the pipe line.

Author's Closure

Mr. Parmakian correctly points out that the time it takes a water-hammer wave to pass by a fixed point is very short. This is due to the fact that water-hammer waves travel at high velocities. The important physical aspect which the paper describes is the form of the wave along the pipe at a given instant. An initially sharp wave front spreads out over a considerable length in terms of pipe diameters. This change would be quite noticeable after the wave had traveled a comparatively short distance of perhaps 50 or 100 pipe diameters.

The data of Fig. 7 are interesting and are in no way contrary to the results of the paper. With reference to Fig. 3, it may be said that for any wave length greater than five times the radius of the pipe the phase velocity of the first mode is very close to the water-hammer velocity c_1. A wave length equal to 5 radii corresponds to a frequency of about 644 cps for a pipe of 1-ft radius. This means that wave forms containing only frequencies which were lower than 644 cps would be expected to disperse very slowly. The wave forms in Fig. 7 probably do not contain much energy in the frequency range above 644 cps. Therefore little dispersion should be expected. The step wave assumed in the paper contains all frequencies. Hence the dispersion due primarily to the high-frequency short-wave length components is prominent.

The reference cited by Mr. Parmakian does take longitudinal restraint into account. But it omits the effects of the inertia of the pipe wall and the radial inertia of the fluid. Therefore no dispersion of the water-hammer waves and no precursor type waves are obtained.

[12] See "Water-hammer Analysis," by John Parmakian, Prentice-Hall, Inc., New York, N. Y., 1955.