The Complex Scaling Method for Many-Body Resonances and Its Applications to Three-Body Resonances

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Resonance phenomena in quantum physics are very familiar in many fields of physics, but we have not yet obtained a complete physical understanding, mathematical description or computational treatment, especially in the case of many-body resonances. Recently, in experimental developments concerning unstable nuclear physics and heavy-ion nuclear reactions, much interest has been concentrated on many-body resonance problems. In the last quarter century, theoretical and mathematical treatments of many-body resonances have experienced great development through application of the complex scaling method (CSM). We can now treat resonant states of three-body systems in the same way as those of two-body systems. In this article, starting from the definition of a resonant state and discussion of its norm, we present a summary of recent studies of CSM to treat many-body resonances and applications to three-body resonant states in two-neutron halo nuclei and three-cluster systems.

§1. Introduction

In the last quarter century, a remarkable development in the description of resonances in quantum many-body systems has been realized through application of the complex scaling method (CSM).1), 2) Here we refer to quantum phenomena of many-body systems decaying into \( n \) bodies \((n \geq 3)\) as “many-body resonances”. A significant development in the treatment of resonances from two-body systems to many-body systems has been brought about by the use of “direct methods” like CSM. It is expected that CSM will play a more vital role in investigations of the problems of three-body and many-body resonances. With this in mind, the aim of this article is to give a brief review of CSM and its well-suited applications to three-body resonant states, which have been studied as an important problem in light nuclear systems, and to extremely neutron-rich nuclear systems, which have recently

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been studied as one of the most interesting problems in recent nuclear physics.

CSM was proposed by Aguilar, Combes and Balslev in 1971,\cite{aguilar1971,combes1971} and Simon advocated it as a direct method to derive resonances in many-body systems.\cite{simon1971} Here, "direct" means that we can directly obtain the resonance energy and the decay width of a resonance in the quantum many-body system by solving the eigenvalue problem for the complex scaled Schrödinger equation with a scaling angle $\theta$, $H_\theta \Psi_\theta = W \Psi_\theta$, where the boundary condition of the outgoing wave\cite{humber1969} is automatically satisfied for the resonance, which is an eigenstate with a complex eigenvalue, $W (|\arg W| \leq 2\theta)$. It has been shown theoretically that solutions in the form of bound and resonant states correspond to poles of the $S$-matrix: For a bound state, the binding energy $E_B$ is obtained as the real eigenvalue which is not changed by the complex scaling, and for a resonant state, the resonance energy $E_r$ and the decay width $\Gamma$ are obtained as the complex eigenvalues $W = E_r - i\Gamma/2$. The resonant eigenstates are described by square-integrable wave functions, whose norms are definable, like the bound states.

To this time, CSM has been applied to atomic and molecular physics extensively, and there is a review paper by Ho.\cite{ho1969} Furthermore, CSM has been developed for obtaining not only the resonance parameters but also scattering cross sections.\cite{ho1971} In nuclear physics, the same transformation ($r \rightarrow re^{i\theta}$, $p \rightarrow pe^{-i\theta}$) that plays a basic role in complex scaling was introduced in calculations of three-body Green functions by Nuttall and Cohen\cite{nuttall1969} in 1969, prior to the paper of Aguilar, Combes and Balslev. Gyarmati and Vertse\cite{gyarmati1971,gyarmati1972} showed in 1971 that the complex scaled wave functions for resonances have the same matrix elements as those calculated using the convergence factor method\cite{humber1969,ho1970,ho1971} introduced to regularize the asymptotic singular behavior of resonant wave functions. Even though various resonant phenomena\cite{gamow1928} have been studied since Gamow’s pioneering work\cite{gamow1928} for $\alpha$ decay, applications of CSM in nuclear physics were not so common until recent years, unlike in the case of atomic and molecular physics.

Interesting resonances appear at positive energies above thresholds. Thus, resonant states are usually described as scattering solutions of the Schrödinger equation, and their properties have been studied in the framework of the theory of nuclear reactions, as seen in a series of early papers by Humblet and Rosenfeld.\cite{humblet1969} In a different approach, Berggren attempted to describe the resonance as an extension of the bound state\cite{berggren1969,berggren1970} and proposed an extended completeness relation\cite{berggren1971,berggren1972} to include the resonant poles of the $S$-matrix by taking deformed contours in the complex momentum plane. Recently, the complex contour deformation method has been employed for single particle orbits in the shell model approach (the so-called Gamow shell model) to describe weakly bound states coupled strongly with continuum states.\cite{berggren1971,berggren1972,berggren1973} CSM also provides one kind of contour deformation in the sense of describing the poles of bound and resonant states in the same domain of the complex momentum plane. Therefore, we understand that CSM, which treats bound states and resonant states equally, stands on the same footing as the approach of Berggren for resonant states regarded as extensions of the bound states.

The merits of CSM can be found in studies of many-body systems, including continuum states. Various kinds of bound states and resonances of the many-body Hamiltonian $H$ can be derived by diagonalizing the complex scaled Hamiltonian $H_\theta$
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Fig. 1. Schematic spectrum of the complex scaled Hamiltonian $H_\theta$. Here, symbols $(b_1, b_2, \cdots), (c_1 c_2, \cdots), (r_1, r_2, r_3, \cdots), (t_1, t_2, \cdots)$ and $(d_1, d_2, \cdots)$ represent eigenvalues of bound states, bound states embedded in the continuum, resonances, real thresholds and resonant thresholds, respectively. The lines of angles of $2\theta$ from the thresholds are the rotated branch cuts.

within the space of $L^2$ basis functions. In Fig. 1, we present a schematic eigenvalue distribution (spectrum) of the many-body Hamiltonian $H_\theta$. In addition to the bound states below thresholds, the bound states embedded in continuum states are obtained, if they exist. Above the thresholds, resonances are obtained as eigenstates with complex eigenvalues: In the case of a three-body system, two-body and three-body resonances can be obtained above the two-body and three-body thresholds, respectively. Note that we need not employ directly boundary conditions for two-body and three-body asymptotic resonant wave functions, as such asymptotic behavior of the resonances are properly described by the complex scaled $L^2$ wave functions. The similar eigenvalue distributions are also obtained in the cases in which $n \geq 3$ for $n$-body systems.

Another important advantage concerns continuum states. As seen in Fig. 1, the continuum spectra of $H_\theta$ start from two types of thresholds. First, there are real thresholds for decaying into two clusters (the two-body threshold), three clusters (the three-body threshold), and so on. Second, there are resonant thresholds for decaying into resonating clusters. For the original Hamiltonian $H$, all continuum states starting from these thresholds are degenerate on the real energy axis. However, for the complex scaled Hamiltonian $H_\theta$, they are separately obtained on the rotated cuts starting from different thresholds with a common angle of $2\theta$. The merit of using the same scaling parameter $\theta$ for every relative coordinate and conjugate momentum is that this results in all cuts being rotated by the same angle. This allows the unique identification of all types of continuum states on the $2\theta$-rotated lines displayed in Fig. 1. These rotated continuum states do not include any resonant contribution from the resonant poles uncovered by the rotated branch cuts.

The importance of CSM in nuclear physics is increasing as the focus of studies moves from stable to unstable nuclei. While in stable nuclei, the resonances appearing in the excited energy region above thresholds emit a nucleon (proton or neutron) or an $\alpha$ particle, in unstable nuclei, bound, resonant and continuum states are energetically degenerate and must be treated simultaneously. This is illustrated in Fig. 2. Many-body resonances in stable nuclei as well have been studied by classifying them into potential scattering, doorway and compound resonances on the basis of active degrees of freedom. In light nuclei, resonances are related to those in quantum few-body systems, such as few-nucleon and cluster systems. In unstable
nuclei, phenomena observed in light nuclei appear generally in every nucleus, where two- and/or three-body resonances at very low energies co-exist with bound states.

The remaining sections are organized as follows:

§2. Resonances in the complex scaling method
  2.1. Two-body resonances
  2.2. The ABC theorem
  2.3. Three-body resonances

§3. Eigenvalue problems for many-body resonances with a finite number of basis functions
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  5.4. Three-alpha resonant states in \(^{12}\)C

§6. Summary
§2. Resonances in the complex scaling method

2.1. Two-body resonances

In the case of a central potential $V(r)$, assuming a finite range potential, the Schrödinger equation can be written as

$$\left\{ \frac{\hbar^2}{2\mu} \left[ -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \right] + V(r) \right\} \psi_l(k, r) = E \psi_l(k, r).$$  \hspace{1cm} (2.1)

The asymptotic behavior of its solutions that are regular at $r = 0$ is given by

$$\psi_l(k, r) \xrightarrow{r \to \infty} \frac{1}{2 l^{l+1}} \left[ f^+_l(k)e^{-ikr} - (-)^l f^-_l(k)e^{ikr} \right],$$  \hspace{1cm} (2.2)

where $f^\pm_l(k)$ are Jost functions for the momentum $k = \sqrt{2\mu E / \hbar^2}$. Since $f^+_l(k)$ and $f^-_l(k)$ represent asymptotic amplitudes of incoming and outgoing waves, respectively, the $S$-matrix is defined as

$$S_l(k) = \frac{f^-_l(k)}{f^+_l(k)},$$  \hspace{1cm} (2.3)

and therefore its poles are given by zeros of the Jost function $f^+_l(k)$. In Fig. 3, we depict the general case of the pole distribution in (a) the complex momentum plane and (b) the corresponding energy plane. The poles of bound states are plotted as solid circles, anti-bound (or virtual) states as crosses, resonances as open circles and anti-resonances as squares. The energy plane consists of two Riemann sheets, because the momentum takes two sets of values. The first and second sheets correspond to the upper (so-called “physical”) and the lower (“unphysical”) halves of the momentum plane, respectively. The poles of the bound states exist on the upper half of the momentum plane (the first energy sheet), while the poles of the conjugate anti-bound (virtual) states appear on the lower half of the momentum plane (the second energy sheet). Resonant and anti-resonant poles appear on the lower half of the momentum plane (the second energy sheet).

We classify the discrete solutions into bound states, anti-bound (virtual) states, resonances and anti-resonances, according to the positions of the $S$-matrix poles in the complex momentum plane as follows:

- **Bound states:** $k_B = i\gamma_b$
- **Anti-bound states:** $k_{AB} = -i\gamma_{ab}$
- **Resonances:** $k_R = \kappa_r - i\gamma_r$
- **Anti-resonances:** $k_{AR} = -\kappa_r - i\gamma_r$

(The quantities $\kappa_r$ and $\gamma_r, b, ab$ are all positive.)

Because $f^+_l(k) = 0$ at all of these poles, the asymptotic wave functions of Eq. (2.2) are proportional to $e^{ik_p r}$, where the momentum $k_p$ at every pole is given by $k_p = k_B, k_{AB}, k_R, k_{AR}$. Therefore, we see that only the bound states have the damping form $\psi_l^B \sim \exp (-\gamma_b r)$. This means that only the bound states can be obtained as solutions of the eigenvalue problem if we use the basis consisting of $L^2$ functions, with energy eigenvalues appearing on the first Riemann sheet.
Let us consider the complex scaling method proposed by Aguilar, Balslev and Combes.\cite{3,4} They introduced the following transformation $U(\theta)$ for the radial coordinate $r$ and its conjugate momentum $k$:

$$U(\theta)r = re^{i\theta}, \quad U(\theta)k = ke^{-i\theta}. \quad \text{(2.4)}$$

The Hamiltonian is transformed as

$$H(\theta) = U(\theta)HU(\theta)^{-1}, \quad \text{(2.5)}$$

where $U(\theta)^{-1} = U(-\theta)$, and the Schrödinger equation, $H\psi = E\psi$, is written

$$H(\theta)\Psi(\theta) = E(\theta)\Psi(\theta), \quad \text{(2.6)}$$

where $\Psi(\theta) = U(\theta)\Psi = e^{\frac{3}{2}i\theta}\Psi(re^{i\theta})$. The factor $e^{\frac{3}{2}i\theta}$ comes from the Jacobian of this coordinate transformation in the case of a three-dimensional space.

For the case of a central potential, the Schrödinger equation is transformed as

$$\left\{ \frac{\hbar^2}{2\mu} \left[ -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \right] e^{-2i\theta} + V(re^{i\theta}) \right\} \psi^\theta_l(k, r) = E(\theta)\psi^\theta_l(k, r). \quad \text{(2.7)}$$

The asymptotic behavior of its solutions, corresponding to Eq. (2.2), is also obtained by setting $V(re^{i\theta}) = 0$ as

$$\psi^\theta_l(k, r) \xrightarrow{r \to \infty} \left[ f^+_l(\theta)(k)e^{-ikre^{i\theta}} - (-1)^lf^-_l(\theta)e^{ikre^{i\theta}} \right], \quad \text{(2.8)}$$

where $f^\pm_l(\theta)(k)$ are the complex-scaled Jost functions.

The asymptotic wave functions for these poles, $k = k_p$, are described by the second term of the outgoing waves, $\exp(i k_p re^{i\theta})$. It is easily understood that the bound
state wave functions maintain the damping behavior for $\theta < \pi/2$. The wave functions of resonances, which had divergent behavior originally as $e^{ik_R r} = e^{i(k_r - i\gamma_r)r} = e^{\gamma r} \cdot e^{ik_r r}$, behave as

$$
e^{ik_R r} e^{i\theta} = e^{i(k_r - i\gamma_r) re^{i\theta}} = e^{i(r(k_r - i\gamma_r)(\cos \theta + i \sin \theta))} = e^{(-\kappa_r \sin \theta + \gamma_r \cos \theta)r}. \quad (2.9)$$

This equation shows that the divergent behavior of the resonant wave functions is regularized when we take the scaling angle $\theta$ to be larger than the angle $\theta_r = \tan^{-1}(\frac{\gamma_r}{\kappa_r})$ of the resonance position $\kappa_r - i\gamma_r$.

An intuitive understanding can be obtained from Fig. 3(a). Under the complex scaling, the real axis of the $k$-plane is rotated clockwise by $\theta$, so that the upper-half (physical) plane is inclined at an angle of $\theta$. This inclined upper half-plane contains, in addition to the bound state poles, the resonant poles characterized by angles $\theta_r = \tan^{-1}(\frac{\gamma_r}{\kappa_r})$, where $\theta_r < \theta$. The wave functions for these resonant poles become normalizable by integration along the complex scaled axis $re^{i\theta}$ in the same way as bound states. Therefore, we can obtain these resonances with $\theta_r < \theta$, together with the bound states, by the diagonalization of the Hamiltonian with a finite number of basis functions.

2.2. The ABC theorem

The properties of solutions of the complex scaled Schrödinger equation were elucidated (in the so-called ABC theorem) by Aguilar, Combes and Balslev\(^3\),\(^4\) as follows:

1) The resonant solutions are described by square-integrable functions, just like the normalizable bound states.

2) The energies of the bound states are not changed by the scaling.

3) If the scaling parameter $\theta$ is chosen to be larger than the angle $\theta_r = \tan^{-1}(\frac{\gamma_r}{\kappa_r})$, then its energy $E_r$ and half-width $\Gamma/2$ are obtained as the real and imaginary parts of the complex eigenvalue $E(\theta)$, e.g., $E(\theta) = E_r - i\Gamma/2$.

4) The continuum spectra are obtained along beams in the complex energy plane. They start at the threshold energies of decays of the system into its subsystems and are rotated clockwise by an angle of $2\theta$ from the positive real axis.

Concerning the norms of resonances, we should note that the complex scaled Hamiltonian $H(\theta)$ is not Hermitian, and hence the scalar product cannot be expressed in terms of Hermite-conjugate states.\(^10\),\(^24\) We must construct bi-orthogonal basis states, which are defined\(^2\) by $(\Psi| = \tilde{\Psi}$ and satisfy the following equation:

$$
H^\dagger(\theta)\tilde{\Psi}(\theta) = E^*(\theta)\tilde{\Psi}(\theta). \quad (2.10)
$$

If $\psi^\theta_l(k, r)$ is an eigen-solution of Eq. (2.7), its conjugate solution is given by $\tilde{\psi}^\theta_l(k, r) = \psi^\theta_l(-k^*, r)$, because $E^*(\theta) = \frac{\hbar^2}{2\mu}(-k^*)^2$. Then the scalar product, defined as

$$
(\Psi(\theta)|\Psi(\theta)) = \int d\tau\tilde{\Psi}^*(\theta)\Psi(\theta), \quad (2.11)
$$
for the eigen-solutions of Eq. (2.7) reads
\[
(\psi_1^\theta(k,r)|\psi_1^\theta(k,r)) \equiv \int dr \{\bar{\psi}_1^\theta(k,r)\}^* \psi_1^\theta(k,r) = \int dr \{\psi_1^\theta(-k^*,r)\}^* \psi_1^\theta(k,r) \\
= \int dr \psi_1^\theta(-k,r) \psi_1^\theta(k,r) = \int dr \{\psi_1^\theta(E^\theta,r)\}^2.
\] (2.12)

The equality between first and second lines in Eq. (2.12) is due to the analytic property of the resonances for complex momenta \( k \).

To solve Eq. (2.7), we expand the wave functions \( \psi_1^\theta(k,r) \) in a finite number of \( L^2 \) basis functions, \( u_l(i,r), i = 1, 2, \cdots, N \):
\[
\psi_1^\theta(k,r) = \sum_i c_i(E,\theta) u_l(i,r).
\] (2.13)

The expansion coefficients \( c_i(E,\theta) \) and the discrete spectrum \( E (= \frac{k^2 k^2}{2\mu}) \) are obtained by solving the eigenvalue problem
\[
\sum_i H_{ji}(\theta)c_i(E,\theta) = E(\theta)c_j(E,\theta),
\] (2.14)
where \( H_{ji}(\theta) \) are matrix elements of the complex scaled Hamiltonian given in Eq. (2.5). A schematic distribution of the eigenvalues of \( H(\theta) \) as asserted by the ABC theorem is shown in Fig. 4. In addition to the eigenvalues of the bound states and resonances, discretized spectra of the continuum states are obtained as eigenvalues on the \( 2\theta \)-line. The exact eigenvalues of the bound states and resonances are independent of the scaling parameter \( \theta \). Therefore, when the basis set is large enough to obtain accurate solutions for bound states and resonances, their eigenvalues are not expected to move significantly, even if the value of \( \theta \) increases.

In practical applications of CSM, the types of interactions are limited; in particular, \( V(re^{i\theta}) \) should be analytic. This restriction is a barrier to application to a nuclear force that has the singularity of a hard core at short distances. The Woods-Saxon potential, which is often used in nuclear physics, also has singularities at \( z_c \) (1 + \( \exp(\frac{z_c - R}{a}) = 0 \), \( z_c = R + na\pi i = re^{i\theta_c} \)), while there is no singularity for \( \theta < \theta_c = \tan^{-1}(\frac{a\pi}{R}) \). The Gaussian-type potential is transformed as \( V(re^{i\theta}) \sim e^{-r^2 \cos(2\theta)}e^{-ir^2 \sin(2\theta)} \). When \( \theta \) is larger than \( \frac{\pi}{4} \), this potential diverges. Thus, we cannot apply CSM with \( \theta > \theta_c = \frac{\pi}{4} \) to a Gaussian potential.

2.3. Three-body resonances

For the sake of simplicity, we explained the complex scaling method for the two-body problem. However, a great advantage of this method can be seen in studies of many-body systems. In this subsection, we give basic consideration of three-body resonances.

In order to illustrate what is asserted by the ABC theorem for the eigenvalues of the complex scaled Hamiltonian \( H(\theta) \), we show, in Fig. 5, a schematic diagram of the eigenvalues for a system of three clusters \( (a, b \text{ and } c) \). Generally speaking,
every two-body subsystem may have bound states and resonances. Measured from the three-body threshold energy $E_{abc}$, threshold energies of two-body decays into $a + (bc)$, $b + (ca)$ and $c + (ab)$ are described by the binding energies of $(bc)$, $(ca)$ and $(ab)$ subsystems, which are indicated in Fig. 5 as $E_a$, $E_b$ and $E_c$, respectively. These energies of the two-body bound states, together with those of the three-body bound states, are obtained on the real energy axis as eigenvalues of the complex scaled three-body Hamiltonian $H(\theta)$. Besides the three-body bound states, continuum spectra appear on the $2\theta$-lines starting from each real threshold. Furthermore, when the two-body subsystems have resonances, their energies are obtained as complex eigenvalues of $H(\theta)$, which are shown as $E_a^*$, $E_b^*$ and $E_c^*$ in Fig. 5. From these resonant thresholds, the continuum spectra are also obtained. The so-called resonant thresholds corresponding to these resonance energies are the origins of the straight lines describing the continuum. The isolated complex eigenvalues correspond to two-body and three-body resonances in the three-body systems. Although the two-
body resonances can arise above each kind of two-body decay threshold, three-body resonances occur only above the three-body threshold. The two-body resonances exhibit a variety of structures due to the complicated structure of the multi-fold Riemann sheets.

Now let us consider excited unbound states of a so-called Borromean system, observed in neutron-rich nuclei as a typical three-body system. A Borromean system consists of three clusters, \(a\), \(b\) and \(c\), and has no bound states in any two-body subsystems. Therefore, the lowest real threshold of the Borromean system is necessarily the three-body one. The three-body continuum spectrum appears at the \(2\theta\)-line from the three-body threshold. Apart from the continuum lines, all states above this threshold (including the ground states of all subsystems) are resonances. Continuum spectra also start from two-body resonant thresholds.

Because the interactions between the clusters are so weak that there are no bound states between any two clusters, it is an interesting problem to understand the binding mechanism of the three-body bound states. All Borromean systems observed to this time have very few bound states (Fig. 2). Many excited states are observed as resonances, and it is conjectured the proper three-body can be observed at low energies. Due to the small binding energy of the ground state, a Borromean system is believed to break up easily by a weak perturbation. To this time, Coulomb fields have been used as such perturbations, and Coulomb breakup cross sections have been observed. These breakup experiments have yielded some interesting information regarding the structure of resonances and the continuum (see §5.2 for further discussion).

\section*{3. Eigenvalue problems for many-body resonances with a finite number of basis functions}

In this section, we explain the eigenvalue problem for the complex scaled Hamiltonian with a finite number of \(L^2\)-class basis functions. Since a many-body system has a large number of degrees of freedom, the number of basis functions becomes very large. Here we employ Gaussian functions as convenient basis functions for the description of few-body systems and present several methods to check the convergence of resonant solutions.

\subsection*{3.1. Complex scaling method with Gaussian basis functions}

In numerical calculations of nuclear many-body systems, we often use a variational method with appropriate bases, such as Gaussian, harmonic oscillator, and so on. For few-body problems of three- and four-body systems, the Gaussian expansion method is shown to be very powerful, with various kinds of matrix elements calculated with high precision.\(^{25}\) In the complex scaling method, calculations with high precision are needed to maintain the stability of the complex eigenvalue problem. Therefore, we employ Gaussian basis functions to solve the complex scaled Schrödinger equation for few-body systems.
The Gaussian basis functions are defined by

\[
\phi^{b_i}_{lJ}(r) = \frac{u_l(r, b_i)}{r^l} \left[ Y_l(\hat{r}) \chi_{1/2} \right]_J, \quad (i = 1 \cdots N)
\]

\[
u_l(r, b_i) = \sqrt{\frac{2}{\Gamma(l + \frac{3}{2})}} r^{l+1} \exp \left( -\frac{r^2}{2b_i^2} \right).
\]

The size parameters \(b_i\) are given by a geometrical progression of the form \(b_i = b_0 \gamma^{i-1}\), with the first term \(b_0\) and the common ratio \(\gamma\) chosen so as to obtain high numerical precision. (See Ref. 25) for details of the Gaussian basis method.) In Ref. 25), complex-range Gaussian basis functions are proposed to reproduce highly oscillatory wave functions. A complex-scaled resonant state has an oscillatory behavior in the large-distance region. However, the amplitude of the resonant wave function decreases when we take an appropriate value of the scaling angle \(\theta\), and hence we employ real-range Gaussian basis functions in the following calculations.

Under the transformation \(r \rightarrow re^{i\theta}\), the matrix elements of \(\frac{d^2}{dr^2}\) and \(\frac{1}{r}\) are factorized with \(e^{-2i\theta}\) and \(e^{-i\theta}\), respectively, while the overlap matrix elements are invariant. For other operators \(O\) which are non-linear functions of \(r\), we can obtain their matrix elements by replacing \(b_i \rightarrow b_i e^{i\theta}\), because we have

\[
\langle lJ; i|O(\theta)|lJ; j \rangle = \langle \phi^{b_i}_{lJ}(r)|U(\theta)OU^{-1}(\theta)|\phi^{b_j}_{lJ}(r) \rangle
\]

\[
= \langle U^{-1}(\theta)\phi^{b_i}_{lJ}(r)\rangle|O|U^{-1}(\theta)\phi^{b_j}_{lJ}(r) \rangle = \langle \phi^{b_i}_{lJ}(re^{-i\theta})\rangle|O|\phi^{b_j}_{lJ}(re^{-i\theta}) \rangle
\]

\[
(3.2)
\]

This technique is very convenient in cases of non-local or momentum-dependent operators as well.

3.2. The convergent solution with the \(\theta\)- and \(b\)-trajectory methods

Since the complex scaled Hamiltonian \(H(\theta)\) is not Hermitian, we must search for the stationary point of the complex energy for the parameters \((N, b_0, \gamma, \theta)\) used in Eq. (2.14) and the basis functions, while the expansion (linear) parameters \(c_i\) are determined by solving the eigenvalue problem. The parameters \(N\) and \(\gamma\) can be chosen so that the solutions converge, but this does not apply to \(b_0\) and \(\theta\). Therefore, we apply the so-called \(\theta\)- and \(b\)-trajectory methods to obtain stationary values for the parameters of resonances. Before presenting this technique, we demonstrate a generalized virial theorem for the resonant wave functions.

In the complex scaling method, we take the imaginary value \(i\theta\) as a parameter of the transformation. However, it can be generalized to a complex parameter as \(i\theta \rightarrow z = \alpha + i\theta\). Then we have \(U(\theta) \rightarrow U(z) = e^{zG}\), where \(G\) is the generator \(G = -\frac{1}{2\hbar}(p \cdot r + r \cdot p)\) of a scale (dilation) transformation. This generalized scale transformation is also expressed as

\[
U(z)r = re^{\alpha+i\theta}.
\]

Therefore, we can say that the generalized scale transformation consists of a real \((e^\alpha)\) and a complex \((e^{i\theta})\) scaling. The real scaling method is called a stationary method, and it is often used to obtain resonant solutions in bound state calculations.
For an infinitesimal variation $\delta z$, the scaled resonant wave function around the stationary point $z_0$ is expressed as

$$\Psi^R_z = \Psi^R_{z_0} + \delta \Psi^R_z. \quad (3.4)$$

The matrix element of the scaled Hamiltonian is calculated as

$$E^R(z) = (\Psi^R_z | H(z) | \Psi^R_z) = \int dr \Psi^R_z H(z) \Psi^R_z. \quad (3.5)$$

The first derivative, $\frac{\partial E^R(z)}{\partial z}$, should vanish at the stationary point. The variation of $E^R(z)$ with respect to $z$ is described using the Hellman-Feynman theorem\textsuperscript{26} as

$$0 = \frac{\partial E^R(z)}{\partial z} \bigg|_{z_0} \delta z = \left[ \int dr \Psi^R_z \frac{\partial H(z)}{\partial z} \Psi^R_z \right]_{z_0} \delta z. \quad (3.6)$$

From this result and the operator relation $\frac{\partial H(z)}{\partial z} |_{z=z_0} = \frac{\partial (e^{-(z-z_0)^G} H_e(z-z_0)^G)}{\partial z} |_{z=z_0} = [H(z_0), G]$, we have

$$0 = (\Psi^R_{z_0} | \frac{\partial H(z)}{\partial z} | \Psi^R_{z_0})_{z_0} = (\Psi^R_{z_0} | 2T_{z_0} - r \cdot \nabla V_{z_0} | \Psi^R_{z_0}). \quad (3.7)$$

Thus, the generalized virial (hypervirial) theorem,\textsuperscript{27}

$$2(\Psi^R_{z_0} | T_{z_0} | \Psi^R_{z_0}) = (\Psi^R_{z_0} | r \cdot \nabla V_{z_0} | \Psi^R_{z_0}), \quad (3.8)$$

is obtained for resonances at the stationary point of $z$. We can use this generalized virial theorem as a check of the resonance solutions.

We can also search for such a stationary point of $E^R(z)$ by plotting it as a function of $z$. The real part of $z$ represents the real scaling of the system described by the wave functions with a common size parameter $b_0$ of the Gaussian basis functions. Therefore, when we use the size parameter $b_0$ to describe the real part of $z$, the complex energy $E^R(z)$ is expressed as a function of $(b_0, \theta)$. We refer to two trajectories of $E^R(z)$ obtained by varying the values of $b_0$ and $\theta$ as $b$- and $\theta$-trajectories, respectively. Since $E^R(z)$ is an analytic function of $z$ or the variables $b_0$ and $\theta$, the $b$- and $\theta$-trajectories should be orthogonal to each other. Using this property, we can easily determine the stationary point of the resonance energy with high precision by drawing these trajectories for the obtained eigen-energies.

In Fig. 6, we show an example of $b$- and $\theta$-trajectories. Here, we used a schematic potential model\textsuperscript{28} and searched for the second $1^-$ resonance (see §4.2 and Ref. 12). We prepared $N = 30$ Gaussian basis functions with the length parameter $b_i = b_0 \cdot \gamma^{i-1}$ ($i = 1, \ldots, 30$). We fixed $\gamma$ to 1.25 and varied $\theta$ from 16° to 27° for the $\theta$-trajectory. The stationary point was found at approximately 20° in the $\theta$-trajectory. The $b$-trajectory was found to be roughly a circle. The true solution for $E^R$ should be
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3.3. Complex scaling method and analytical continuation in the coupling constant

In addition to CSM, the method of analytical continuation in the coupling constant (ACCC) was proposed as a useful way to calculate the resonance parameters by Kukulin et al.,\textsuperscript{29,30} and they studied its applicability to two-body systems in detail. Their results are summarized in their textbook ‘Theory of Resonance’.\textsuperscript{31} Recently, Tanaka, Suzuki and Varga\textsuperscript{32} pointed out that recent developments of computational power and techniques enable us to use ACCC in actual calculations of few-body systems. They showed that ACCC can be used to obtain resonant parameters of three-body systems by comparing with the solutions of CSM, and this method has been used in recent studies of resonances in unstable nuclei.\textsuperscript{32–39} The advantage of ACCC is that we can apply this method without any modification of the conventional techniques for calculating the bound state. Furthermore, the effectiveness of ACCC has been demonstrated in analyses of resonances with large decay widths and also of virtual states, which are very difficult or impossible within CSM. But it should be mentioned that the accuracy of the calculated resonant parameters even for sharp resonances is not as good as for CSM. For this reason, ACCC is regarded as a complementary method to CSM. We propose a method, called ACCC+CSM, which incorporates the merits of both methods and significantly ex-
pands the region of applicability.

Here, we only briefly describe the basic idea of ACCC, because the details are given in Ref. 31). We consider an unbound system with Hamiltonian $H$. In ACCC, we introduce a parameter $\lambda = 1 + \delta$ (a coupling constant) into the Hamiltonian as

$$ H(\lambda) = H_0 + \lambda V = H + \delta V. $$

(3-9)

We express the attractive part of the potential in the original Hamiltonian as $V$. Note that $H(\lambda = 1)$ is the original Hamiltonian. When the strength of the attractive potential ($\lambda$) increases, the energy of an unbound state decreases, and it turns into a bound state. For two-body systems, it is known\(^{31}\) that the square root of the energy behaves as $k_l(\lambda) \sim \sqrt{\Lambda - \lambda_0}$ for $l \neq 0$ and $k_0(\lambda) \sim (\lambda - \lambda_0)$ for $l = 0$ around the branching coupling constant $\lambda_0$. Here, $l$ is the relative angular momentum. If $l \neq 0$, $\lambda_0 (> 1)$ is easily obtained as the coupling constant that gives zero energy ($E(\lambda_0) = 0$, $k_l(\lambda_0) = 0$), corresponding to the threshold. In the case of $l = 0$, it is known that $k_0(\lambda_0)$ is not zero and goes to a negative imaginary value ($k_0(\lambda_0) = i\chi_0$, $\chi_0 < 0$). For three-body systems, there have been no general studies of the branching coupling constant of an $S$-wave, but it was shown by Tanaka et al.\(^{33}\) that the branching energy of the $S$-wave for the Borromean system is nearly equal to zero, due to the presence of an effective barrier in the three-body system. We also numerically checked this property for the three-body $S$-wave resonances in the $^8\text{He}+n+n$ system by using an iterative method.\(^{31}\) The calculated result shows that the branching energy $E(\lambda_0)$ is nearly equal to zero, within numerical errors. The quantity $k(\lambda)$ is expressed as an analytic function of $x = \sqrt{\Lambda - \lambda_0} = \sqrt{\delta - \delta_0}$, and it is described using the Padé approximation.\(^{31}\)

$$
\begin{align*}
 k_i^{MN}(x) &= i\frac{c_0 + c_1x + c_2x^2 + \cdots + c_Mx^M}{d_0 + d_1x + d_2x^2 + \cdots + d_Nx^N}.
\end{align*}
$$

(3-10)

For the bound state ($\lambda > \lambda_0 > 1$), because $x = \sqrt{\Lambda - \lambda_0} > 0$, $k_i^{MN}(x)$ is purely imaginary on the positive imaginary axis. We can determine the coefficients, $c_i (i = 0, \cdots, M)$ and $d_j (j = 1, \cdots, N)$, by solving Eq. (3-10) for $k_i^{MN}(x_1)$, $k_i^{MN}(x_2)$, $\cdots$, $k_i^{MN}(x_{i_{M+N+1}})$. Inserting the obtained $c_i (i = 0, \cdots, M)$ and $d_j (j = 1, \cdots, N)$ into Eq. (3-10), the analytic function $k_i^{MN}(x)$ is defined for $\lambda > \lambda_0$ ($x > 0$). This function can be analytically continued into the region $1 \leq \lambda \leq \lambda_0$, where the bound state of Eq. (3-9) turns into an unbound one. The unbound region for $\lambda (< \lambda_0)$ is defined by the imaginary variable $x = i\sqrt{\lambda_0 - \Lambda}$, and thus $k_i^{MN}(x)$ is complex. For this region of $\lambda$, in Eq. (3-10) we simply set $\lambda=1$ and obtain the resonant solution of $k_i^{MN}(x = i\sqrt{\lambda_0 - \Lambda})$ that is analytically continued from the bound state.

By using CSM or ACCC alone, we could not obtain any resonance solution with large decay widths, such as the soft dipole resonance in the $^4\text{He}+n+n$ system, as discussed below. The reason is that the decay width might be very large. When the resonance pole of the $S$-matrix has a large imaginary part, we must solve a complex eigenvalue problem whose solution is very far from the real energy axis. It is very difficult to obtain numerical solutions for such a problem by using CSM or ACCC alone. The reason why we cannot get the solution with CSM is attributable to the
singular property of the complex scaled Hamiltonian for the large scaling angle, as mentioned above. However, with ACCC, we can obtain a solution in the region where CSM is not applicable. However, the numerical instability in ACCC is very large for a resonance far from the real energy axis. The origin of this instability is the Padé approximation for such a resonance. Since the coefficients of the Padé approximation are determined in the bound state region, this approximation fails numerically for states with the large decay widths. However, we have accurate resonance solutions obtained in addition to bound states solutions in CSM. The idea is to determine the coefficient of the Padé approximation in the resonant region instead of the bound state region, using CSM. By using ACCC+CSM, we can easily obtain resonance solutions of large decay widths, which cannot be obtained with conventional methods.

§4. Rotated continuum states in the complex scaling method

The complex scaling method was proposed as a useful tool to calculate resonant states. In this method, continuum states are described by solutions, obtained on the rotated branch cuts, of the complex-scaled Schrödinger equation. These rotated continuum states are believed to have properties that differ from the ordinary continuum ones obtained as scattering states. In this section, we show that the rotated continuum states play an important role in the construction of an extended completeness relation in the complex scaling method. The strength function of transitions to continuum states is discussed after applying the extended completeness relation to calculations of the Green function. Furthermore, we investigate the properties of the rotated continuum states through calculations of the phase shift.

4.1. The extended completeness relation in CSM

Bound and scattering (continuum) states form a complete set that is represented by the completeness relation

\[ 1 = \sum_b |\chi_b \rangle \langle \chi_b | + \int_0^\infty dE |\chi_E \rangle \langle \chi_E |, \]

\[ = \sum_b |\chi_b \rangle \langle \chi_b | + \int_{-\infty}^{+\infty} dk |\chi_k \rangle \langle \chi_k |, \] (4.1)

where \( \chi_b \) and \( \chi_E \) are the bound (the discrete negative part of the energy spectrum) and continuum (the continuous positive part) states, respectively, on the first Riemann sheet of the energy plane. The continuum states \( (\chi_k, \chi_{-k}) \) in the momentum representation belong to the states on the real \( k \) axis. Therefore, integration over the \( k \) axis corresponds to that along the rims of the cut of the first Riemann sheet of the energy plane, as shown in Fig. 7(a). In the case of a potential problem, the mathematical proof of the completeness relation (Eq. (4.1)) was given by Newton\(^{23}\) using the Cauchy theorem.

As a natural extension, the extended completeness relation (ECR) was derived by Berggren.\(^{18}\) He changed the path of integration for the continuum states as shown
in Fig. 8. His intention was to treat resonances similarly to bound states. The ECR was given in the form

\[ 1 = \sum_{b}^{n_b} |\chi_b^\theta \rangle \langle \chi_b^\theta | + \sum_{r}^{n_r(L)} |\chi_r^\theta \rangle \langle \chi_r^\theta | + \int_{L} dk |\chi_k^\theta \rangle \langle \chi_k^\theta |, \tag{4.2} \]

where \( \chi_b, \chi_r \) and \( \chi_k \) are bound, resonant and continuum states, respectively.

For the solutions of the complex scaled Hamiltonian \( H(\theta) \), it is also interesting to consider the completeness relation. Recently, a mathematical proof for the completeness relation in CSM was given by Giraud et al.\(^{40), 41)}\) for the single- and coupled-channel cases. In the case of the complex scaling, the momentum (real \( k \)) axis is rotated by \( \theta \), and the resonant poles enter the semicircle used in the Cauchy integration (Fig. 7(b)). Therefore, the resonances appear in the completeness relation for the complex scaled Hamiltonian \( H(\theta) \),

\[ 1 = \sum_{b} |\chi_b^\theta \rangle \langle \chi_b^\theta | + \sum_{r} |\chi_r^\theta \rangle \langle \chi_r^\theta | + \int_{L} dE |\chi_k^\theta \rangle \langle \chi_k^\theta |, \tag{4.3} \]

where \( \chi_b^\theta \) and \( \chi_r^\theta \) are the complex scaled bound states and complex scaled resonant states, respectively. Only those complex scaled resonant states that enter the semicircle rotated by \( \theta \) are taken into consideration, and their number is expressed by \( n_r^\theta \). Furthermore, continuum states \( \chi_E^\theta \) and \( \chi_k^\theta \) are located on the rotated cut \( L^E_\theta \) of the Riemann plane and on the rotated momentum axis \( L^k_\theta \), respectively.

Here, we note that the definition of the complex scaled bra- and ket-states for the non-Hermitian \( H(\theta) \) are different from the usual one for the Hermitian \( H \). In the latter case, the bra-state is the complex conjugate of the ket-state. On the other hand, in the case of \( H(\theta) \), bi-orthogonal states must be defined: The momentum of the bi-orthogonal bra-state, which is conjugate to the ket-state with momentum \( k \), is denoted as \( \tilde{k} \). For discrete states (bound and resonance), we have \( \tilde{k} = -k^* \), whereas
for continuum states, we have $\tilde{k} = k^*$. For the wave function of discrete states, we can use the same wave function for the bra- and ket-states, and for continuum states, the wave function of a bra-state is given by that of the ket-states divided by the $S$-matrix.

In this section, we show that CSM provides a natural way to separate resonance and continuum solutions satisfying ECR in CSM, which is governed by the scaling angle $\theta$ only. Based on this completeness relation, we comprehensively describe the three kinds of states. We investigate various physical quantities in terms of bound, resonant and continuum states, such as the transition strength (see the following section). We furthermore apply ECR in CSM to the three-body case, and some results are presented in §5.2.

4.2. The strength function with CSM

Applying ECR to calculations of physical quantities enables us to see the contributions from bound, resonant and continuum states. Berggren et al. have studied the separation of the strength function into these three terms by introducing various types of ECR with different patterns of this separation. However, they restricted their investigation to the validity of each type of pole expansion and did not explicitly examine the residual continuum term. Thus it is necessary to carry out a more thorough study, taking into account the continuum term.

The strength function $S(E)$ is expressed in terms of response function $R(E)$ as

$$S_\lambda(E) = \sum_\nu \langle \chi_\nu | \hat{O}_\lambda^\dagger | \chi_\nu \rangle \langle \chi_\nu | \hat{O}_\lambda | \chi_i \rangle \delta(E - E_\nu)$$

$$= -\frac{1}{\pi} \text{Im} R_\lambda(E),$$

$$R_\lambda(E) = \int dr dr' \chi_i^*(r) \hat{O}_\lambda^\dagger G(E, r, r') \hat{O}_\lambda \chi_i(r'),$$

where $E$ is the energy on the real axis, and $|\chi_i\rangle$, $|\chi_\nu\rangle$ and $\hat{O}_\lambda$ are the initial states, final states and an arbitrary transition operator of rank $\lambda$, respectively. The quantities $E_\nu$ are the energies of the final state. In this expression, we assume that the bound
The matrix elements of the scaled operator are independent of \( \theta \).

Using the complex scaled initial wave functions \( \chi_i^\theta \), the Hamiltonian \( H(\theta) \) and the transition operator \( \widetilde{O}_\lambda^\theta \), the response function is expressed as

\[
R_\lambda(E) = \int d\mathbf{r} d\mathbf{r}' \chi_i^{\theta*}(\mathbf{r}) (\tilde{O}_\lambda^{\theta}) (\mathbf{E}, \mathbf{r}, \mathbf{r}') \tilde{O}_\lambda^\theta \chi_i^{\theta}(\mathbf{r}'),
\]

where the complex scaled Green function is written

\[
G^\theta(\mathbf{E}, \mathbf{r}, \mathbf{r}') = \left( \mathbf{r} \left| \frac{1}{\mathbf{E} - H(\theta)} \right| \mathbf{r}' \right).
\]

Here, it is worth noting that \( R_\lambda(E) \) does not depend on the scaling. Substituting Eq. (4.3) into Eq. (4.9), we can separate the Green function into three terms, as

\[
G^\theta(\mathbf{E}, \mathbf{r}, \mathbf{r}') = \sum_b \frac{\chi^\theta(\mathbf{r}, k_B) \chi^{*\theta}(\mathbf{r}', k_B)}{\mathbf{E} - \mathbf{E}_B} + \sum_r \frac{\chi^\theta(\mathbf{r}, k_R) \chi^{*\theta}(\mathbf{r}', k_R)}{\mathbf{E} - \mathbf{E}_R} + \int_{L_\theta^k} dk_\theta \frac{\chi^\theta(\mathbf{r}, k_\theta) \chi^{*\theta}(\mathbf{r}', k_\theta)}{\mathbf{E} - \mathbf{E}_\theta},
\]

where \( \mathbf{E}_B \) and \( \mathbf{E}_R \) (= \( \mathbf{E}_r - \frac{i}{2} \mathbf{\Gamma} \)) are the energy eigenvalues of the bound states and resonances, respectively. From Eq. (4.8), we obtain the following relations for the response function:

\[
R_\lambda(E) = R_{\lambda, B}(E) + R_{\lambda, R}^\theta(E) + R_{\lambda, k}^\theta(E),
\]

\[
R_{\lambda, B}(E) = \sum_b \frac{(\chi_i^\theta|\tilde{O}_\lambda^\theta|\chi_b^\theta)(\chi_b^\theta|\tilde{O}_\lambda^\theta|\chi_i^\theta)}{\mathbf{E} - \mathbf{E}_B},
\]

\[
R_{\lambda, R}^\theta(E) = \sum_r \frac{(\chi_i^\theta|\tilde{O}_\lambda^\theta|\chi_r^\theta)(\chi_r^\theta|\tilde{O}_\lambda^\theta|\chi_i^\theta)}{\mathbf{E} - \mathbf{E}_R},
\]

\[
R_{\lambda, k}^\theta(E) = \int_{L_\theta^k} dk_\theta \frac{(\chi_i^\theta|\tilde{O}_\lambda^\theta|\chi_{k_\theta})(\chi_{k_\theta}|\tilde{O}_\lambda^\theta|\chi_i^\theta)}{\mathbf{E} - \mathbf{E}_\theta}.
\]

The strength function given in Eq. (4.4) is similarly separated as

\[
S_\lambda(E) = S_{\lambda, B}(E) + S_{\lambda, R}^\theta(E) + S_{\lambda, k}^\theta(E).
\]

The matrix elements of the scaled operator are independent of \( \theta \). The \( \theta \) dependence of \( R_{\lambda, R}^\theta(E) \) and \( R_{\lambda, k}^\theta(E) \) comes only from \( n_r^\theta \), \( L_\theta^k \) and \( E_\theta \), respectively. The strength function \( S_\lambda(E) \) is an observable, and it is positive definite for any energy and independent of \( \theta \).
CSM allows us to calculate each term of Eq. (4.15) easily, because bound states and resonant states are obtained under the same boundary conditions, and continuum states are obtained on the rotated contour determined by $\theta$ uniquely. In particular, due to such a decomposition of the unbound final states, we can unambiguously determine which state forms the structure of the strength function. These are prominent points of CSM. This framework is also applicable to three-body cases, including Borromean systems, as will be shown in §5.2.

Here, we investigate the strength function of the $E1$ transition in the simple potential model

$$H = -\frac{1}{2}\nabla^2 + V(r), \quad V(r) = -8e^{-0.16r^2} + 4e^{-0.04r^2}.$$  (4.16)

In this model, energy eigenvalues are obtained for $J^\pi = 0^+$ and $1^-$.\(^{12)}\) For $J^\pi = 1^-$, choosing a scaling angle of $3^\circ$, we obtain a bound state solution ($E_{bs} = -0.68$ MeV), a resonant solution ($E_1 = 1.17 - i0.49 \times 10^{-2}$ MeV) and continuum solutions. As seen in Fig. 9, for such a small angle, we cannot get the second resonant solution. However, by choosing $\theta = 10^\circ$, one more resonant solution ($E_2 = 2.02 - i0.49$ MeV) is separated from the continuum states, while $E_{bs}$ and $E_1$ are not changed.

In Fig. 10, we plot the strengths of the dipole transition between $1^-$ states and the ground state and compare these with the results (open circles) calculated by solving the scattering problem. A clear peak is seen just above 1 MeV. The dashed curve is the transition strength to the first resonance. Therefore, we conclude that the origin of the clear peak is the first resonance. In the left figure, corresponding to the small scaling angle of $3^\circ$, it seems that the continuum contribution (dotted curve) forms a second peak around 2 MeV. However, when we choose the larger scaling angle of $10^\circ$ in order to get the second resonant solution, the contribution from the second resonance (dash-dotted curve) becomes a major component, while the continuum contribution (dotted curve) becomes very small. Thus, we can conclude that the two major peaks of the strength function for the $E1$ transition originate from two lower resonances $1^-$. 

Fig. 9. Positions of the first and second resonances of $1^-$ states with rotated-continuum lines at $\theta=3^\circ$ and $10^\circ$. 

\[\text{Im(Energy) [MeV]}\] 
\[\text{Re(Energy) [MeV]}\] 
\[\text{rotated-continuum line ($\theta=3^\circ$)}\] 
\[\text{rotated-continuum line ($\theta=10^\circ$)}\] 
\[\text{1st res. ($\theta_R=0.12^\circ$)}\] 
\[\text{2nd res. ($\theta_R=6.78^\circ$)}\] 

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Fig. 10. The $E1$ transition strength function $(0^+ \rightarrow 1^-)$ at θ=3° (left) and 10° (right) in the schematic potential model. The dashed and dash-dotted curves represent the components of the first and second resonant states, respectively, and the dotted curve is the continuum term. The solid curves represent the sum of the contributions from resonant states and continuum states. The open circles represent the results obtained by solving the scattering problem without the complex scaling.

Hence, we find that the contributions from the embedded resonances are removed from the transition strength function of the rotated-continuum states if we employ a reasonably large scaling angle. In other words, since the two poles of the $S$-matrix are located inside the contour of integration in the momentum plane at $\theta = 10^\circ$, the $S$-matrix for the rotated-continuum states almost coincides with that for the pure continuum states without a sharp resonant contribution. This is also confirmed by checking some properties of the $S$-matrix, such as the phase shift on the rotated continuum states.

4.3. The phase shifts of the complex scaled continuum states

It is interesting to see the physical properties of the continuum states in CSM and the difference between the complex scaled and original continuum states. For this purpose, we investigate the behavior of the phase shift calculated for the scaled continuum states.

In accordance with the asymptotic wave function of the complex continuum states (Eq. (2.8)), the complex scaled $S$-matrix is given by

$$S^\theta_l(k) = \frac{f^{-\theta}l(k)}{f^{+\theta}l(k)} = e^{2i\delta^\theta_l(k)}. \quad (4.17)$$

Therefore, the phase shift $\delta^\theta_l(k)$ of the complex continuum states can be calculated using the Jost functions of the complex scaled Schrödinger equation (Eq. (2.7)). Because of the non-Hermitian nature of $H(\theta)$, the complex phase shift $\delta^\theta_l(k)$ is complex.

We plot the scaling angle dependence of the real part of the complex phase shift in Fig. 11. In the case that no scaling is made, we find, according to the Levinson theorem, $\delta_l(E = 0) - \delta_l(E \rightarrow \infty) = n_b \pi$. Because in this case we have $l = 1$, we can conclude from the phase shift behavior for $\theta = 0$ (the uppermost curve in Fig. 11) that there exists one bound state in this system. As discussed by Rittby,
the Levinson theorem can be extended to a complex phase shift.\textsuperscript{47) In CSM, the resonances found to exist in the wedge region of the energy plane by rotation of the branch cut behave as bound states. Therefore, for a complex phase shift, these resonances can be regarded as bound states, and the generalized Levinson theorem\textsuperscript{47) becomes

\begin{equation}
\text{Re}\left(\delta^0_l(|E_\theta| = 0)\right) - \text{Re}\left(\delta^0_l(|E_\theta| \rightarrow \infty)\right) = \left(n_b + n_r(L_\theta)\right) \cdot \pi, \tag{4.18}\end{equation}

where $n_r(L_\theta)$ is the number of resonances existing in the wedge region between the real energy axis and the rotated branch cut $L_\theta$.

In Fig. 11, $\text{Re}\left(\delta^0_l(|E_\theta|)\right)$ is plotted for several values of $\theta$ that are chosen so that $n_r(L_\theta)$ increases incrementally with a step size of 1. For example, the dotted line for $\theta = 7^\circ$, which reproduces two resonances ($E_1 = 1.17 - i 0.49 \times 10^{-2}$ MeV and $E_2 = 2.02 - i 0.49$ MeV) in the wedge region, spans the area $(n_b + n_r(L_\theta)) \cdot \pi = 3\pi$, as predicted by the generalized Levinson theorem Eq. (4.18). It should be noted that $\text{Re}\left(\delta^0_l(|E_\theta|)\right)$ crosses $\pi/2$ when the scaling angle is chosen so that the resonant pole lies just on $L_\theta$. In fact, it is a natural generalization of the principle that the phase shift crosses $\pi/2$ for a state with a sharp decay width.

From this property of the complex phase shifts, we can see that the complex scaled continuum states do not have the effect of the resonances revealed in the wedge region. Thus, when we employ a large value of $\theta$ and a large number of resonances are revealed by the rotated branch cut $L_\theta$, the continuum states calculated on $L_\theta$ receive no contribution from these many resonances and approach the genuine continuum states. This understanding of the complex scaled continuum states is confirmed by observing the complex phase shift for a large value of $\theta$. From the results of Fig. 11, we can see that it is proportional to $-|k_\theta|$ (or $-\sqrt{|E_\theta|}$), and this indicates background behavior of the scattering from a hard sphere. The coefficient of this proportionality corresponds to the radius of the hard sphere.
§5. Examples of three-body resonances in nuclear systems

5.1. Three-body resonant states in two-neutron halo nuclei

The three-body resonant states can now be obtained by solving the eigenvalue problem of the complex scaled three-body Hamiltonian, as explained in previous sections. Here, we consider most typical three-body resonant states in a Borromean system. As such an example, the two-neutron halo nuclei $^6$He and $^{11}$Li have been studied, and here the results concerning three-body resonances are presented, focusing on the application of CSM to physical problems of the halo structure in those nuclei.

The nuclei $^6$He and $^{11}$Li have extremely small two-neutron separation energies, 0.975 MeV$^{48)}$ and 0.31 MeV,$^{49)}$ respectively. To describe the asymptotic $^4$He+$n+n$ and $^9$Li+$n+n$ properties of $^6$He and $^{11}$Li, respectively, we express the Hamiltonian as

$$H = H_c + t_{n_1} + t_{n_2} + t_c - T_{cm} + V_{cn_1} + V_{cn_2} + v_{nn},$$ (5.1)

where $H_c$ is the intrinsic Hamiltonian of a core-cluster ($^4$He or $^{11}$Li), and $t_{n_1}$, $t_{n_2}$, $t_c$ and $T_{cm}$ are the kinetic energy operators of two neutrons, a core-cluster and the center of motion, respectively. Furthermore, $V_{cn_1}$ ($V_{cn_2}$) and $v_{nn}$ are the interactions between the core-cluster and neutrons and between two neutrons, respectively.$^{50), 51)}$

When we can assume the core-cluster ($^4$He or $^{9}$Li) to be described by the ground state wave function of $H_c$, we have to solved a “three-body problem” for the above Hamiltonian $H$ described by the interactions $V_{cn_1}$ ($V_{cn_2}$) and $v_{nn}$ with no bound states among any two-body sub-system. However, this three-body problem is not so simple because the core-cluster is a composite particle consisting of the same nucleons as two valence neutrons above the core-cluster. Thus, the wave function of $A=^6$He or $^{11}$Li is described as

$$\Psi(A) = A'\{\Phi(A_c)\chi(n,n)\},$$ (5.2)

where $A_c = ^4$He or $^{9}$Li. Here, $A'$ represents antisymmetrization between nucleons in the core-cluster and the two valence neutrons, and in the present study we take into account this anti-symmetrization effect by the orthogonality condition model (OCM).$^{52)}$ In our calculations, we employ the pseudo-potential method of Kukulin et al.$^{53)}$ to exclude the Pauli-forbidden states from $\chi(n,n)$. The relative wave function, $\chi(n,n)$, of the core-cluster+$n+n$ three-body system can be accurately solved using the Hybrid T+V model$^{54), 55)}$ and the Gaussian expansion method of Hiyama, Kino and Kamimura.$^{25)}$ This model has been applied to the study of the resonant ground states in $^{10}$He with the $^8$He+$n+n$ model.$^{56)}$ The details of the present calculations are given in Refs. 54)–58).

In Fig. 12, we show the eigenvalue distribution of the $2^+$ solutions of the complex scaled OCM Hamiltonian for the $^4$He+$n+n$ system in the left panel,(a). There is no bound $2^+$ state solution in this system, and all eigenvalues are distributed in the fourth quadrant of the complex energy plane. We find two isolated eigen-solutions corresponding to the resonance poles $2^+_1$ and $2^+_2$. Experimentally, the $2^+$ resonance
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Fig. 12. (a) The eigenvalue distribution of the $2^+$ solutions for the complex scaled OCM Hamiltonian in the $^{4}\text{He}+n+n$ system with $\theta=0.61$ rad. The three rotated lines represent the $^{4}\text{He}+n+n$ three-body continuum state, and the two-body continuum states of $^{5}\text{He}(3/2^-)+n$ and $^{5}\text{He}(1/2^-)+n$, respectively. (b) Experimental and theoretical low-lying levels of $^{6}\text{He}$. The two numbers in parentheses are the resonance energy and width.

is observed at 0.822 MeV above the two-neutron decay threshold, but below the quasi-two-body decay threshold of $^{5}\text{He}(3/2^-)+n$, where $^{5}\text{He}$ is in a resonant state, and thus considered to be a genuine three-body resonant state. The first $2^+$ solution obtained with CSM exhibits good correspondence with not only the resonance energy but with the decay width of this observed three-body resonant state. Furthermore, the second $2^+$ resonance is calculated at $\sim 2$ MeV with a large decay width (4.22 MeV), but has not been observed experimentally.

In addition to the resonant eigen-solutions, many eigenvalues corresponding to continuum spectra were obtained, and their positions rotate with increasing $\theta$. They are separated into three groups in the present case of $^{4}\text{He}+n+n$. The first group consists of the three-body continuum solutions of $^{4}\text{He}+n+n$ obtained around the $2\theta$-line beginning from zero energy (corresponding to the three-body threshold) and rotated from the real axis by $-2\theta$. The continuum eigenvalues are distributed along the $2\theta$-line with some variance. The cause of this variance is believed to be the fact that the number of basis functions is not enough. In the present calculation, the basis functions are employed so as to solve the resonant states but not continuum states with enough accuracy. If a larger number of basis states can be used, the variance of the continuum solutions will be reduced. As shown below, however, this variance of the continuum solutions does not cause a serious problem in the investigation of the physical quantities concerning continuum states.

The Hamiltonian given in Eq. (5.1) for the three-body system includes terms describing two-body Hamiltonians of sub-systems. In this case, the sub-systems correspond to $^{5}\text{He}=^{4}\text{He}+n$ and $n+n$, and the former has resonances of $3/2^-$ and $1/2^-$, while the latter has no resonance. Therefore, as the second and third groups, we obtain continuum solutions distributed around the rotated lines beginning from the resonance positions of $3/2^-$ and $1/2^-$ in $^{5}\text{He}$, as shown in Fig. 12(a). Such resonances of sub-systems, from which the continuum states appear, are called “resonant
In Fig. 12(b), we plot the result of the low-lying resonant states of $J^\pi = 0^+, 1^+$ and $2^+$ calculated with CSM, where $\theta < 0.61$ rad., and compare them experimental data for $^6\text{He}$. In addition to the ground $0^+$ and the excited resonance $2^+$, three excited resonances ($2^+_2$, $1^+$ and $0^+_2$) are predicted. Because their widths are much larger than the energy intervals between neighboring resonances, it is thought that they are difficult to observe as isolated resonances. From analysis of the obtained wave functions of the resonant states ($0^+, 1^+, 2^+$), we can understand the dominant configuration of the valence neutrons in the $^4\text{He} + n + n$ system. They have dominant configurations of products of single neutron resonant wave functions $p_{3/2}$ and $p_{1/2}$ around $^4\text{He}$: $[(p_{3/2})(p_{3/2})]_{0^+}$ for $0^+_1$, $[(p_{1/2})(p_{1/2})]_{0^+}$ for $0^+_2$, $[(p_{3/2})(p_{1/2})]_{1^+}$ for $1^+$, $[(p_{3/2})(p_{3/2})]_{2^+}$ for $2^+_1$ and $[(p_{3/2})(p_{1/2})]_{2^+}$ for $2^+_2$. The Gamow shell model,\(^{20)-22}\) which was developed recently, yields similar results. It would be interesting to search for a resonant state caused by an excitation of the soft dipole mode, whose appearance in neutron halo nuclei has been predicted as a characteristic excitation arising from a coherent oscillation of protons with respect to spatially distributed neutrons.

In Fig. 13, we show the $1^-$ eigenvalue distribution obtained by diagonalizing the complex scaled Hamiltonian of the $^4\text{He} + n + n$ system with $\theta = 0.61$ rad. From this result, we see that all $1^-$ solutions are obtained along three lines of rotated branch cuts corresponding to two-body continuum states and three-body continuum states. There is no $1^-$ three-body resonance with a sharp decay width. Therefore, these results indicate that the $1^-$ unbound states above the $^4\text{He} + n + n$ threshold can be classified into two-body continuum states of $^5\text{He}(3/2^-,1/2^-) + n$ (squares and triangles) and three-body continuum states of $^4\text{He} + n + n$ (circles). The three-body $1^-$ resonant pole position with a large imaginary part in $^6\text{He}$ is discussed in the next subsection.
5.2. Coulomb breakup reactions of $^6$He and $^{11}$Li within CSM

We calculate the strength distribution from the ground state to $1^-$ states by using these solutions in the formalism presented in §4.2. In Fig. 14, the calculated strength distribution of the $E1$ transition is displayed. We can see a low energy peak in the strength at around 1 MeV. This peak energy is just above the two-body threshold (0.74 MeV) of $^5$He($3/2^-$)+$^n$\textsuperscript{54}) and the $E1$ transition strength slowly decreases with the excitation energy. It is interesting that the main part of the transition strength originates from the two-body continuum component of $^5$He($3/2^-$)+$^n$ (dashed curve), and that contributions from the other components are relatively very small (dotted curve and dash-dotted curve). This result indicates that the three-body Coulomb breakup strength of $^6$He is dominated by the sequential breakup of a $^6$He→$^5$He+$^n$→$^4$He+$^n$+$^n$ process. This is consistent with the observed invariant mass spectrum of the $^4$He+$^n$ subsystem.\textsuperscript{59)–61) The peak position of the calculated strength distribution of the $E1$ transition is similar to that of Danilin et al.\textsuperscript{62)}

Now, we investigate what kinds of final-state configurations give dominant contributions to the strength function. For this purpose, we calculate the transition contributions from various components of the $1^-$ configuration. When one of the valence neutrons is the $p$-wave resonance, the final-state configurations are expressed as $^5$He($3/2^-,1/2^-)+n_{sd}$, where the subscript indicates the orbital angular momentum of the last neutron: $[(p_{3/2})(s_{1/2})]_1^-$ and $[(p_{1/2})(s_{1/2})]_1^-$ channels for one valence neutron in the $s$-wave, and $[(p_{3/2})(d_{5/2})]_1^-$, $[(p_{3/2})(d_{3/2})]_1^-$ and $[(p_{1/2})(d_{3/2})]_1^-$ channels for one valence neutron in the $d$-wave.

The calculated strength distribution of the $E1$ transitions for various components of the final-state configuration are shown in Fig. 15. The left panel shows $(^4$He+$n_p)+n_s$ configurations with a clear low energy peak. The right panel shows $(^4$He+$n_p)+n_d$ configurations with a broad peak at a slightly higher energy. The subscripts $s$, $p$ and $d$ of the neutron $n$ indicate the orbital angular momentum of each
Fig. 15. Decomposition of the $E1$-transition strength distribution into contributions from different final state configurations of $^6\text{He}$. The left panel corresponds to ($^4\text{He}+n_p$)+$n_s$ configurations, and the right panel is for ($^4\text{He}+n_p$)+$n_d$ configurations. The subscript $n$ indicates the orbital angular momentum of each valence neutron.

valence neutron. From these results, it is found that the $^5\text{He}(3/2^-)+n_s$ component gives the main contribution and causes a large enhancement at low energies in the total strength. The $^5\text{He}(3/2^-)+n_d$ component also gives a large contribution and produces a broad peak around 2−3 MeV in the total strength. This property of the $^6\text{He}$ breakup cross section is strongly supported by recent analysis of experimental data.\(^{63}\)

Next, we investigate the strength distribution of the $E1$ transition in the $^9\text{Li}+n+n$ system. The $^9\text{Li}+n+n$ model is the same as that considered in Refs. 64) and 65). To analyze excited states in $^{11}\text{Li}$, we prepare three types of $^{11}\text{Li}$ wave functions, P-1, P-2 and P-3 (cf. Table I), which are characterized by the $(1s_{1/2})^2$ probability in the ground state. Here, we adjust the $(0p)^2-(1s)^2$ pairing coupling between valence neutrons to reproduce the observed binding energy of $^{11}\text{Li}$ (0.31 MeV).\(^{49}\) In Table I, we list some properties for the present wave functions: scattering lengths $a_s$ and energies $E$ for $^{10}\text{Li}$, and the $(1s_{1/2})^2$ probability $P$ and matter radius $R_m$ for the $^{11}\text{Li}$ ground state. As seen from this table, the large matter r.m.s. radius of $^{11}\text{Li}$

<table>
<thead>
<tr>
<th>Wave Function</th>
<th>$a_s(2^-)$ [fm]</th>
<th>$E(2^-)$ [MeV]</th>
<th>$a_s(1^-)$ [fm]</th>
<th>$E(1^-)$ [MeV]</th>
<th>$P[(1s_{1/2})^2]$ [%]</th>
<th>$R_m$ [fm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{10}\text{Li}$</td>
<td>$-12.7$</td>
<td>$-0.05$</td>
<td>$-6.5$</td>
<td>$-0.08$</td>
<td>$21.0$</td>
<td>$3.33$</td>
</tr>
<tr>
<td>$^{11}\text{Li}$</td>
<td>$-17.0$</td>
<td>$-0.03$</td>
<td>$-8.6$</td>
<td>$-0.06$</td>
<td>$29.4$</td>
<td>$3.58$</td>
</tr>
<tr>
<td>$^{11}\text{Li}$</td>
<td>$-21.7$</td>
<td>$-0.02$</td>
<td>$-10.7$</td>
<td>$-0.05$</td>
<td>$38.8$</td>
<td>$3.85$</td>
</tr>
<tr>
<td>$^{11}\text{Li}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$3.12\pm0.16^a$</td>
<td>$3.53\pm0.06^b$</td>
</tr>
</tbody>
</table>

\(^a\) Ref. 66), \(^b\) Ref. 67).
and the virtual state features of $^{10}$Li are reproduced.

In Fig. 16 we plot the calculated E1 transition strength distribution from the ground state to the continuum states. We can regard the total spin of the dipole ($1^-$) excitation from the ground state ($J^\pi = 3/2^-$) as $1^- \times 3/2^- = 1/2^+, 3/2^+, 5/2^+$. Here, we are summing up the contributions from the $1/2^+, 3/2^+$ and $5/2^+$ states for the three types of ground state wave functions. In Fig. 16 (a), the solid curve, the dashed curve and the dotted curve correspond to P-1, P-2 and P-3, respectively. We can easily see that the low energy enhancement of the strength for the P-3 case is significantly larger than the others. In other words, the enhancement is very sensitive to the $(1s_{1/2})^2$ probability of the two valence neutrons. This enhancement is interpreted as a threshold effect coming from the continuum states and reflects the halo structure of the $^{11}$Li ground state. We compare our results to the experimental data of MSU. Although the calculated peak position is similar to that of the MSU data, the shape is different. We also compare our results with the calculation (labeled by “Ga”) by Garrido et al.

In Figs. 16(b) and (c), we present the results for the cross sections of the $^{11}$Li breakup derived using the equivalent photon method, where the target is Pb. We find good agreement with the RIKEN data in the case of the P-2 wave function. The $(1s_{1/2})^2$ probability of P-2 is around 30%. Here, it is noted that the magnitude of the experimental cross section is not determined. For the GSI data, the P-1 wave function gives good correspondence, though the experimental error is still large.

Recently, Nakamura et al. reported a new data for the Coulomb breakup cross section of $^{11}$Li. They observed a strong low-energy peak in the distribution, which comprehensively explains three old data sets by accounting for the conditions of the experimental analyses for each data set. Theoretically, recently, we have studied the halo structure of $^{11}$Li, taking into account the tensor correlation in $^9$Li, in addition to the pairing correlation. These correlations naturally reproduce the large $(1s_{1/2})^2$ admixture in the $^{11}$Li ground state. It is also found that the calculated dipole strength overall explains the results of a recent experiment.

In Fig. 17, as in the case of $^6$He (Fig. 14), we show the decomposition of the E1-transition strength distribution of $^{11}$Li into the components of two-body and three-body continuum states for the three types of ground state wave functions, P-1, P-2
and P-3. The solid curves are the total strengths. The dotted curves, the dash-dotted curves and the dashed curves are the components contributed by the $^{10}\text{Li}(1^+)+n$, $^{10}\text{Li}(2^+)+n$ and $^9\text{Li}+n+n$ continuum states, respectively. It is found that the two-body continuum component of $^{10}\text{Li}(1^+)+n$ has a low energy enhancement in each panel, with the peak position just above the two-body threshold (0.42 MeV) of $^{10}\text{Li}(1^+)+n$. The enhancement above the lowest two-body threshold is very similar to that in the case of $^6\text{He}$, but the strength is not so large in comparison with the total strength. In the case of $^6\text{He}$, the contributions from two-body continuum states of $^5\text{He}(3/2^-)+n$ dominate the total strength. Another two-body continuum component of $^{10}\text{Li}(2^+)+n$ exhibits a broader structure, because of a larger decay width of the $2^+$ state of $^{10}\text{Li}$. The components of both two-body and three-body continuum states contribute equally to the Coulomb breakup of $^{11}\text{Li}$. Furthermore, the three-body continuum states depend strongly on the halo structure of the ground state ($[(1s_1/2)^2$ probability). In fact, in the case of the $^6\text{He}$ breakup, the contribution from the three-body continuum states is very small, as mentioned above. The $(1s_1/2)^2$ probability of two valence neutrons in the $^6\text{He}$ ground state is 2.4%, much smaller than that of $^{11}\text{Li}$. These results indicate that the mechanisms in the breakup reactions of $^6\text{He}$ and $^{11}\text{Li}$ are different. There may be broad resonances including virtual states located under the branch cuts of the present two-body and three-body continuum states. However, their effects seem to produce no remarkable structure, although they are seen to have considerable strengths. In the present calculation, we cannot conclude how much they could contribute to the strength. As a summary for the Coulomb breakup of $^{11}\text{Li}$, there are two comparable mechanisms of the $E1$ enhancement at very low energy: i) the resonant structure of $^{10}\text{Li}$ in the two-body continuum state of $^{10}\text{Li}+n$; ii) the response of the three-body continuum state of $^9\text{Li}+n+n$, which comes from the $s$-wave halo structure of the ground state.

5.3. **Soft-dipole resonance in $^4\text{He}+n+n$**

As mentioned above, the soft dipole resonance has been the focus of much investigation; it is a dipole mode in the oscillation of the core-nucleus to the halo.\textsuperscript{75)-77)} As a nucleus that may possess a soft dipole resonance, $^6\text{He}$ has been investigated by
many theoretical and experimental groups. However, theoretically, we cannot obtain such a resonant 1− solution for the 4He+n+n system with an accurate treatment of the three-body resonance.54, 78 For example, as we have already shown in Fig. 13, there is no three-body resonance in the 1− solutions. Experimentally, no 1− resonance has been observed (at least no clear, sharp resonance).59 Furthermore, we have seen that a continuum state causes the peak in the low energy region for the Coulomb breakup cross section.79–81 As discussed in §5.2, its origin is the two-body continuum states of 5He(p3/2)+n. However, there is a possibility that the 4He+n+n system has a three-body 1− resonant pole with a large imaginary part (a large decay width). In order to find such a three-body resonant pole, we proposed the new method called ACCC+CSM.36 By using ACCC+CSM, we can solve a long standing puzzle of the 1− state in 6He.

In Fig. 18, we plot the trajectories of the complex energies obtained by changing the potential strength parameter. The solid curve is the trajectory of the complex energy obtained from ACCC+CSM (M = 7, N = 7 in Eq. (3.10)). Here we determine the coefficients of the Padé approximation in ACCC by using the complex energies (momenta) of the resonance instead of the negative real energies of the bound state. These complex energies are obtained in CSM and represented by circles in Fig. 18 (δ = 0.66 – 0.94 with a step size of 0.02). We should mention that we use the so-called b- and θ-trajectory techniques of CSM in order to get an accurate solution (see §3.2 and Ref. 54)). In order to compare with the usual ACCC, we plot its results with the dotted curve. In the case δ=0.00, which corresponds to the solutions of the original Hamiltonian, the solution obtained using ACCC+CSM and that obtained using ACCC alone are very different. (The difference is about 8 MeV for the real part and about 7 MeV for the imaginary part.) In order to check the deviation for the present resonance solution with a large decay width against that obtained with CSM, we display the solutions for δ=0.50. Here, we mention that it is very difficult
to obtain a converged solution for a value of $\delta$ smaller than 0.50 (larger scaling angle $\theta$). The cross corresponds to CSM and the square to ACCT+CSM. We can see that the correspondence is very good. The calculated complex energy for $\delta = 0.00$ is $E = 3.0 - i15.6$ MeV. Its imaginary part multiplied by two, $15.6 \times 2 = 31.2$ MeV, is interpreted as the decay width, when it is small, which is not the case here. This is the reason why we could not determine the position of the resonance pole for the soft dipole resonance in $^6\text{He}$, as done in previous studies. Also, formulating a physical interpretation of such the large decay width state is difficult. We checked several parameter sets for $\delta$. Because the estimated numerical error is of order 1 MeV, we conclude that the $1^-\gamma$ resonance solution is a very large decay width state for which a physical interpretation is difficult to formulate.

5.4. Three-alpha resonant states in $^{12}\text{C}$

For $^{12}\text{C}$, the 3$\alpha$ cluster model has been studied for a long time. Recently, much attention has been given to a three-$\alpha$ structure in the excited $0^+$ state above the 3$\alpha$ decay threshold since possibility of $\alpha$ condensation was proposed in nuclear systems. Although $^{12}\text{C}$ is a stable nucleus, it has only two bound states, the ground and excited $2^+$ states, and all the other states are resonant states. Therefore, it is interesting to study the spectroscopy of the excited states above the 3$\alpha$ threshold of $^{12}\text{C}$ by applying CSM to the 3$\alpha$ cluster model. To this time, there have been many studies of the 3$\alpha$ cluster model for excited states, but most of them have been carried out within a bound state approximation. The 3$\alpha$ system is a kind of Borromean systems, because the unique sub-system $\alpha + \alpha$ of the 3$\alpha$ system is unbound and has only resonances. To obtain reliable theoretical conclusions concerning spectroscopy above the 3$\alpha$ threshold, it is necessary to carry out the 3$\alpha$ calculation with the correct three-body boundary conditions for the resonant asymptotic wave functions.

Here, we applied CSM to the 3$\alpha$ OCM calculations and obtained the results shown in Fig. 19. The Hamiltonian $H(3\alpha)$ and details of the present 3$\alpha$ model calculation are given in Ref. 39). From Fig. 19(a), it is observed three $0^+$ resonant states are obtained above the 3$\alpha$ threshold at zero energy, in addition to one bound solution corresponding to the ground state of $^{12}\text{C}$. The 0$^+_2$ resonant state just above the threshold is called the Hoyle state. This state is very important to obtain an understanding of the $^{12}\text{C}$ production from three $^4\text{He}$ nuclei inside of stars. The distribution of eigenvalues around the $2\theta$-line beginning from the origin indicates that the 3$\alpha$ continuum states become almost degenerate with the $^8\text{Be}(0^+, 0.09 \text{ MeV}) + \alpha$ continuum states due to the small energy difference (0.09 MeV) from the 3$\alpha$ threshold. Furthermore, we have two kinds of continuum solutions, corresponding to $^8\text{Be}(2^+, 3.04 \text{ MeV}) + \alpha$ and $^8\text{Be}(4^+, 11.4 \text{ MeV}) + \alpha$.

Recently, an experimental group at RCNP has reported that excited $0^+$ and $2^+$ resonances coexist in the 10 MeV excitation energy region of $^{12}\text{C}$. This experiment suggests a key to solve a long-standing puzzle of disagreement between experiments and cluster model predictions: The 3$\alpha$ cluster calculations predict a $2^+$ state around $E_x = 10$ MeV as a member of a rotational band built upon the Hoyle state ($0^+_2$). Experimentally, it is believed that the observed state at $E_x = 10.3$ MeV, which has quite a large decay width, could be the $2^+_2$ state, but it has been concluded that the
spin of this state is $0^+$.\cite{85}) Therefore, for the cluster model, it is necessary to look for a broad $0^+$ resonant state in the same energy region as the $2^+_2$ energy. The $0^+_3$ and $0^+_4$ resonant solutions shown in Fig. 19(a) do not correspond to the $0^+$ state observed by the RCNP group, because their resonance energies are higher than 10 MeV and their widths are not so large. The scaling parameter $\theta = 15^\circ$ employed here seems too small to realize a broad $0^+$ state in the 10 MeV region.

We applied ACCC+CSM to look for such a broad $0^+$ state. As stated in Ref. 39), where the details of the calculations are given, it is necessary to introduce a three-$\alpha$ potential as an auxiliary coupling potential in the $3\alpha$ system in order to perform analytic continuation without any change of the resonance energies of the $2\alpha$ subsystem. As a result, we obtained a new $0^+$ resonant state of the $3\alpha$ system at the complex energy $E = 1.66 - i 0.74$ MeV, which corresponds to the excitation energy $E_x = 9.03$ MeV and the decay width $\Gamma = 1.48$ MeV. This broad resonant state, which has not been reproduced by any previous study of the cluster model, might account for the experimental conclusion of the RCNP group. In Fig. 19 (b), we show all the calculated resonant states, including the new broad $0^+$ candidate, and compare them with the experimental data.

§6. Summary

We have reviewed the complex scaling method (CSM) and its applications to many-body resonances, mainly three-body resonances of neutron halo systems and $\alpha$ cluster systems in the light nuclear mass region. CSM is expected to play an important role in studies of resonances for the following reasons. (i) We can directly obtain resonances of quantum many-body systems from the Schrödinger equation. It is possible to include the boundary condition of the resonances decaying at asymptotic distances in the basic equation, and to find the poles of the $S$-matrix, corresponding to the resonance energies and the decay widths, as its solutions. (ii) In CSM, the
resonance energies and the decay widths are obtained from energy eigenvalues of \( H(\theta) \), and their eigenstates are described within the \( L^2 \) function space. Since the norm of the resonance is uniquely defined, it has been shown that an extended completeness relation contains solutions of \( H(\theta) \) on bound states, resonant states and rotated continuum states. (iii) Not only we can easily describe the resonances in two-body scattering (decaying) processes in the same framework as the bound state calculations, but also we can directly extend these descriptions of the resonances for few-body cases, such as three- and four-body systems, with developments of their frameworks. Now, three-body problems can be studied on the same level as two-body ones.

From these points of view, we gave an explanation of the basic ideas and several techniques of CSM. We presented recent developments of theoretical studies of unstable nuclei in which bound states, resonant states and continuum states are energetically degenerate as an extreme system, and thus must be treated simultaneously. Much interest in these unstable nuclei has been concentrated on the so-called Borromean system, which is an extreme system having no bound states of any two-body subsystems. Besides a few bound states, we must treat three-body continuum states in addition to resonances. For such problems, CSM is the most suitable method. In the present review, we presented our studies of the core+\( n+n \) model and its extensions applying CSM. This model is a kind of coupled-channel method of rearrangement reaction channels,\(^{25}\) developed to describe few-body systems with high numerical precision.

As a typical system, \( ^6\text{He} \) has been studied on the basis of the \( ^4\text{He}+n+n \) model, because the assumption of a fixed doubly closed configuration is relatively reliable with \( ^4\text{He} \) as a core nucleus. The first excited \( 2^+ \) state of \( ^6\text{He} \) corresponds to a genuine three-body resonance, and, in addition, we obtained many three-body resonances predicted from \((0p)^2\) configurations for the valence neutrons above the \( ^4\text{He} \) core. Furthermore, we have searched for the resonance corresponding to the soft dipole excitation in \( ^6\text{He} \), but such a resonance with a broad decay width is not predicted by CSM. We proposed to combine CSM with ACCC. ACCC itself is a very useful method to find three-body resonances. Using ACCC+CSM, an interesting dipole resonance in \( ^6\text{He} \) is predicted at an energy far from the real axis because of a very broad decay width.

We also studied the Coulomb breakup cross sections of \( ^6\text{He} \) and \( ^{11}\text{Li} \) using CSM. In a breakup reaction of a three-body system, even if resonances can be described as poles of the \( S \)-matrix, it is difficult to investigate in a unified manner the effects of these states and continuum states on the strength function. CSM enables us to do this easily. The three-body resonances, three-body continuum states, and even two-body continuum states in which two constituents form a resonance, can be clearly distinguished in the total strength, and each contribution was examined in detail. In contrast to the case of \( ^6\text{He} \), we extended the three-body model of \( ^{11}\text{Li} \) to allow the excitation of the \( ^9\text{Li} \) core in \( ^{11}\text{Li} \) by carrying out a configuration mixing for \( ^9\text{Li} \), in particular paying attention to the neutron pairing correlation. It has been shown that the coupling between the pairing correlation among the \( p \)-shell neutrons in the \( ^9\text{Li} \) core and the di-neutron correlation of the valence neutrons plays a very important
role in understanding the exotic halo structure of $^{11}$Li. We performed the extended core+$n$+$n$ model calculations by taking into account these couplings explicitly. It is found that the result successfully describes the ground state properties of $^{11}$Li, and on the basis of this result, we performed an analysis of the three-body Coulomb breakup reaction, which is one of the hottest topics in the study of unstable nuclei in search of soft dipole resonances. Our results show that continuum responses are dominant, and the result accurately reproduces the experimental data.

Furthermore, we also discussed a long-standing problem of the 3$\alpha$ resonant states in excited states of $^{12}$C as another example of the three-body resonances. The three-$\alpha$ system is a kind of Borromean systems. Therefore, many excited states above the three-$\alpha$ threshold have been observed as three-body resonant states. The recent experiment indicates that the broad $0^+$ state around $E_x = 10$ MeV, above the Hoyle state, is degenerate with the $2^+$ state.\(^{84}\) The three-$\alpha$ cluster model calculations have predicted the $2^+$ state in this energy region, but the broad $0^+$ state has not been reproduced by calculations with a bound state approximation. We presented this problem to be solved by the three-body resonance calculation with CSM.

Finally, we summarize important consequences and prospects which we obtain from the studies on three-body resonances in nuclear many-body systems through the application of CSM: (1) We can now obtain three-body resonant states as well as three-body bound states in the same framework, and can study their structures and properties based on the understanding that the resonance is an extension of a bound state. (2) It has been revealed that remaining continuum states from which resonant states are separated have rich structure such as (two-body resonance)+(one-body continuum state) and genuine three-body continuum states. (3) As seen from discussions of strength functions, response functions and phase shifts in CSM, the decomposition of continuum states into resonant states and remaining continuum states provides prospects of studying structures and reactions of a nucleus in a common framework. (4) It is now feasible, as the need arises, to carry out analysis of $n$-body resonances with $n \geq 4$ by direct application of CSM with experiences and the technique developed in three-body cases. (5) We can easily apply the achievement of the resonance studies presented here to other fields associated with the resonances in the quantum many-body systems.

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