On the Equation of Motion for a Fast Moving Small Object in the Strong Field Point Particle Limit

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We present a straightforward, divergence-free derivation of the equation of motion for a small but finite object in an arbitrary background using the strong field point particle limit. The resulting equation is that of a generalized geodesic for a non-rotating spherical object. This is consistent with the results of previous studies.

Equations of motion in general relativity have attracted much interest recently because of the belief that it may be possible to use gravitational waves as an astronomical tool in the near future. In fact, several gravitational wave detectors are now operating around the world, and there are plans to build the next generation detectors. Gravitational waves may open a new window to the universe, just as radio waves and X-rays did. Gravitational waves should allow us to study entirely new regimes in the universe, such as the surfaces of black holes and the very beginning of the universe, that no other type of radiation can reveal, and they will probably allow us to find new astronomical objects that we have not imagined to exist. However, in order to realize this, not only technology but also theory must be developed further. Theoretically, it is necessary to have accurate predictions of the nature of the waves emitted by various astrophysical sources. For this purpose, we need to have a good understanding of equations of motion with radiation reaction in a wide variety of situations. In particular, at present we have only a very poor understanding of the equation of motion for a fast moving source, such as small but extended objects (e.g., neutron stars or black holes), in an arbitrary external field. In such a situation, the object does not necessarily move along a geodesic of the external field, because the self-field of the object and the gravitational waves generated by the orbital motion are not negligible.

In this paper, we study the equation of motion describing such a situation. It should be mentioned that Mino et al. and others have derived an equation of motion for a point particle with mass $m$ that is represented by a delta function source in an arbitrary background, and the equation is interpreted as a geodesic equation on the geometry determined by the external field and the so-called tail part of the self field of the particle to first order in $m$. Furthermore, Mino et al. used another approach, the matched asymptotic expansion, to obtain an equation of motion without employing the concept of a point particle and thus avoided divergence in their derivation.
Our approach is different from the previously used approaches for the reason that we take the extended nature of the object into account and thus avoid using a singular source. This is done by using the point particle limit developed by one of the present authors.\textsuperscript{5) We believe that our approach simplifies the proof given by Mino et al. (employing the matched asymptotic expansion) that the Mino-Sasaki-Tanaka equation of motion is applicable to a nonsingular source.}

Let us start by explaining the situation we consider and introduce the strong field point particle limit,\textsuperscript{5}) which plays an essential role in our formulation. A small compact object with mass $m$ moves with arbitrary speed near a massive body with mass $M$. We assume that the object is stationary, except for higher-order tidal effects, so that we can safely ignore emission of gravitational waves from the object itself. However, of course, we cannot ignore the gravitational waves emitted by the orbital motion of the object. We denote the world line of the center of mass of the object by $z(\tau)$ and define the body zone of the object as follows. We imagine a spherical region around $z(\tau)$ whose radius scales as $\epsilon$. Also, we stipulate that the linear dimension of the object scale as $\epsilon^2$, so that the boundary of the body zone is located at the far zone of the object. Thus, we are able to carry out a multipole expansion of the field generated by the object at the surface of the body zone. We also implicitly assume that the mass of the object scales as $\epsilon^2$, so that the compactness of the object remains constant in the point particle limit $\epsilon \to 0$. This is why we call this limit the strong field point particle limit. Then we calculate the metric perturbation induced by the small object in this limit. The smallness parameter $\epsilon^2$ has the dimension of length, and this length characterizes the smallness of the object. One may regard it as the ratio of the physical scale of the object and the characteristic scale of the background curvature. We assume that the background metric $g_{\mu\nu}$ satisfies the Einstein equations in vacuum. Therefore, the Ricci tensor of the background vanishes. Since we have assumed that mass scale of the small object is much smaller than the scale of gravitational field of the background geometry, we approximate the metric perturbation by the linear perturbation of the small particle, $h_{\mu\nu}$.

We employ the harmonic gauge, in which we have

\begin{equation}
\bar{h}^{\mu\nu} = 0,
\end{equation}

where the semicolon represents the covariant derivative with respect to the background metric, and the trace-reversed variable is defined as usual:

\begin{equation}
\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\lambda} h_{\rho\lambda}.
\end{equation}

Then the linearized Einstein equations take the form

\begin{equation}
-\frac{1}{2} \bar{h}^{\mu\nu,\xi} (x) - R_{\xi \rho}^{\mu \nu} (x) \bar{h}^{\xi \rho} (x) = 8\pi T^{\mu\nu} (x).
\end{equation}

This can be solved formally as

\begin{equation}
\bar{h}^{\mu\nu} (x) = 8\pi \int d^4 y \sqrt{-g} G^{\mu\nu}_{\alpha\beta} (x, y) T^{\alpha\beta} (y),
\end{equation}
where we have used the retarded tensor Green function defined by

$$G^{\mu \nu \alpha \beta}(x,y) = \frac{1}{4\pi} \theta(\Sigma(x) - y) \left[ u^{\mu \nu \alpha \beta}(x,y) \delta(\sigma(x,y)) + v^{\mu \nu \alpha \beta}(x,y) \theta(-\sigma(x,y)) \right].$$  (5)

For a general tensor Green function and the definitions of $\sigma(x,y)$, $\bar{g}^{\mu \alpha}(x,y)$, $u^{\mu \nu \alpha \beta}(x,y)$, $v^{\mu \nu \alpha \beta}(x,y)$, and $\Sigma(x)$, please refer to Refs. 2) and 9).

Now we take the point particle limit. In this limit, the above metric perturbation contains terms with different $\epsilon$ dependences. This simplifies the calculation of the equation of motion. For example, terms with negative powers of $\epsilon$ appear from the delta function part of the Green function,

$$s\bar{h}^{\mu \nu}(x) = 2 \int d^4 y \sqrt{-g} u^{\mu \nu \alpha \beta}(x,y) \delta(\sigma(x,y)) T^{\alpha \beta}(y).$$  (6)

As explained below, we only need the field on the boundary of the body zone, which is the far zone of the body itself, and thus we may make use of the multipole expansion for the field. For this purpose, we choose our coordinates as

$$y^\alpha = z^\alpha(\tau) + \delta y^\alpha,$$

$$z^0(\tau) = \tau, \quad \delta y^0 = 0,$$

where $z^\alpha(\tau)$ is the world line of the center of the object. The center is assumed always to be inside the body in the point particle limit, and thus there is no ambiguity regarding the choice of the center. In these coordinates, the volume element satisfies $d^4 y = d\tau d^3 \delta y$. Thus we have

$$s\bar{h}^{\mu \nu}(x) = 2 \int d^3 \delta y \sqrt{-\bar{g}} u^{\mu \nu \alpha \beta}(x,z(\tau_y) + \delta y) \frac{1}{\bar{\sigma}(\tau=\tau_y)} T^{\alpha \beta}(y),$$  (7)

where $\tau_y$ is the retarded time of each point $y$. Then the multipole expansion is obtained by expanding the above expression at the retarded time of the center of the object $\tau_z$, defined by $\sigma(x,z(\tau_z)) = 0$. This can be done easily by using the condition $\sigma(x,z(\tau_y) + \delta y) = 0$. Then the difference between $\tau_z$ and $\tau_y$ is given by

$$\delta \tau = -\frac{\sigma_{,\alpha}(x,z(\tau_z)) \delta y^\alpha}{\bar{\sigma}(x,z(\tau_z))} + O(\delta y^2).$$  (8)

Using this $\delta \tau$, we can expand $u^{\mu \nu \alpha \beta}(x,z(\tau_y) + \delta y)$ about $\tau_z$. In principle, we can calculate arbitrary higher multipole moments in this way. Here we only calculate the first two terms, namely the monopole and spin terms. Then we only need the following expressions:

$$u^{\mu \nu \alpha \beta}_{,\alpha \beta}(x,z(\tau_y) + \delta y) = u^{\mu \nu \alpha \beta}_{,\alpha \beta}(x,z(\tau_z)) + \left( u^{\mu \nu \alpha \beta;\gamma}_{,\alpha \beta}(x,z(\tau_z)) - \frac{\sigma_{,\gamma}(x,z(\tau_z))}{\bar{\sigma}(x,z(\tau_z))} \frac{d u^{\mu \nu \alpha \beta}_{,\alpha \beta}}{d\tau} \right) \delta y^\gamma,$$  (9)
\[
\dot{\sigma}(x, z(\tau_y) + \delta y) = \dot{\sigma}(x, z(\tau_2)) \left[ 1 + \frac{\dot{\sigma}(x, z(\tau_2)) - \dot{\sigma}(x, z(\tau_2))}{\dot{\sigma}^2(x, z(\tau_2))} \right] \delta y^\gamma. \tag{10}
\]

By defining the mass and the spin as
\[
m \ddot{z}^\alpha(\tau_2) \ddot{z}^\beta(\tau_2) = \int \delta y \sqrt{-g} T^{\alpha\beta}(y), \tag{11}
\]
\[
m S^{\gamma(\alpha(\tau_2) \ddot{z}^\beta(\tau_2)) = \int \delta y \sqrt{-g} \delta y^\gamma T^{\alpha\beta}(y), \tag{12}
\]
(see Ref. 4) for the definition of the spin tensor), we finally obtain the following expression for \( s \bar{h}^{\mu\nu} \):
\[
s \bar{h}^{\mu\nu}(x) = \frac{2m}{\dot{\sigma}(x, z(\tau_2))} u^{\mu\nu}_{\alpha\beta}(x, z(\tau_2)) \dot{z}^\alpha(\tau_2) \dot{z}^\beta(\tau_2)
\]
\[
+ \frac{2m}{\dot{\sigma}(x, z(\tau_2))} u^{\mu\nu}_{\alpha\beta}(x, z(\tau_2)) S^{\gamma(\alpha(\tau_2) \ddot{z}^\beta(\tau_2))}
\]
\[
+ \frac{2m}{\dot{\sigma}(x, z(\tau_2))} u^{\mu\nu}_{\alpha\beta\gamma}(x, z(\tau_2)) S^{\gamma(\alpha(\tau_2) \ddot{z}^\beta(\tau_2))}
\]
\[
- \frac{2m}{\dot{\sigma}^2(x, z(\tau_2))} \frac{d}{d\tau} \left( u^{\mu\nu}_{\alpha\beta}(x, z(\tau_2)) \sigma^\gamma(\tau_2) z^\beta(\tau_2) \right). \tag{13}
\]

Now we derive the equation of motion using this expression. First, we define the \( \epsilon \) dependent 4-momentum of the object as the volume integral of the effective stress energy tensor \( \Theta^{\mu\nu} \) over the body zone \( B(\tau) \):
\[
P^{\mu}(\tau) = -\int_{B(\tau)} \Theta^{\mu\nu} d\Sigma^\nu. \tag{14}
\]
Here we choose the Landau-Lifshitz form for the stress energy tensor,
\[
\Theta^{\mu\nu} = (-g)(T^{\mu\nu} + t_{LL}^{\mu\nu}), \tag{15}
\]
where \( T^{\mu\nu} \) is the stress-energy tensor of matter and \( t_{LL}^{\mu\nu} \) the Landau-Lifshitz(LL) pseudo-tensor, whose explicit expression can be found in their textbook.\cite{10} Because the effective stress energy tensor satisfies the conservation law \( \Theta^{\mu\nu}_{\mu\nu} = 0 \), the change of the 4-momentum defined above can be expressed as a surface integral over the boundary of the body zone \( \partial B \):
\[
\frac{dP^{\mu}}{d\tau} = -\oint_{\partial B} \epsilon^{\mu\nu}(1 + \epsilon a \cdot n) (-g) t_{LL}^{\mu\nu} d\Omega. \tag{16}
\]
Here, \( n^\mu \) is the unit normal vector to the surface and \( a^\mu = Dw^\mu/d\tau \) the 4-acceleration. The equation of motion is obtained by taking the point particle limit, \( \epsilon \to 0 \), in this expression. Thus, we need to calculate \( (-g) t_{LL}^{\mu\nu} \) on the body zone boundary (field
points), on which the multipole expansion of the self field and the Taylor expansion of the nonsingular part of the field are known, and to retain only terms of order $\epsilon^{-2}$ in the expression. All the other terms vanish in the limit or the angular integration.

For simplicity, we demonstrate the derivation of the equation of motion in the case of a non-rotating, spherical object. Recalling that the LL tensor is bilinear in the Christoffel symbol and that the Christoffel symbol is a derivative of the metric tensor, it is found that the only remaining terms come from the combination of the 0-th order of the smooth part of the metric and the part of the self field proportional to $\epsilon^{-1}$, which takes the following form:

$$(-1)^s h^{\mu\nu}(x) = \frac{2m}{\sigma(x, z(\tau))} u^{\mu\nu}_{\alpha\beta}(x, z(\tau)) \dot{z}^{\alpha}(\tau) \dot{z}^{\beta}(\tau)|_{\tau=\tau_0}. \quad (17)$$

The remaining part of the self field is the so-called tail part, which is regular in the limiting process. The field point $(x, \tau_x)$ is now on the body zone boundary, which is defined by $\sqrt{2\sigma(x, z(\tau_x))} = \epsilon$ and

$$[\sigma;_{\alpha}(x, z(\tau)) \dot{z}^{\alpha}(\tau)]_{\tau=\tau_x} = 0. \quad (18)$$

We have assumed a stationary spherical source, and in this case, $m$ is the Arnowitt-Deser-Misner (ADM) mass that the compact object would have if it were isolated,

$$m = \lim_{\epsilon \to 0} P^\tau. \quad (19)$$

Using the above expressions in the LL tensor, we have the following result in the point particle limit:

$$\frac{dP^\mu}{d\tau} = -m \Gamma^\mu_{\alpha\beta}(g_s) u^\alpha u^\beta - m \frac{1}{2} \Gamma^\alpha_{\alpha\beta}(g_s) u^\beta u^\mu. \quad (20)$$

Here the Christoffel symbol is evaluated using the 0-th order of the smooth part of the metric $g_s$. Using the fact that the ADM mass is related to the 4-momentum as

$$P^\mu = \sqrt{-g_s} mu^\mu \quad (21)$$

(which is suggested by the higher-order post Newton approximation\textsuperscript{11}), we finally have

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu_{\alpha\beta}(g_s) u^\alpha u^\beta, \quad (22)$$

which is the geodesic equation on the geometry determined by the smooth part of the metric around the compact object.

In fact, the spin effect on the equation of motion can be derived in a similar way, and the standard result is obtained.\textsuperscript{13}

We have proved that a small compact object moves on the geodesic determined by the smooth part of the geometry around an object. The equation describing this motion is the so-called MiSaTaQuWa equation. The smooth part contains gravitational waves emitted by the orbital motion, and therefore the equation contains the damping force due to the radiation reaction. It is necessary to explicitly calculate the
smooth part of the metric generated by the small object around the orbit. This is the heart of the problem, and it is not attempted in this paper. However, we point out that our method avoids use of a singular source, employing only the retarded Green function, by making use of the point particle limit, and all the quantities are evaluated at the surface of the body zone boundary. Thus we only need the dependence of the distance from the center of the object, namely, the $\epsilon$ dependence of the field. For these reasons we are able to avoid using any divergent quantities in any part of our calculation. One of the difficulties encountered when using a singular source is the ambiguity of the separation between the divergent and convergent parts at the singularity. With our method, it might be possible to avoid this kind of ambiguity and to obtain a unique equation for the fast motion with a radiation reaction. We would like to see this if this is indeed the case in a future work.

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