The Role of Interference in the Semiclassical Approximation to Chaotic Motion

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Interference of distinct contributions to a given amplitude is a familiar aspect of quantum mechanics. Semiclassically, such interference arises from distinct classical paths connecting the appropriate initial and final points. In chaotic systems the number of distinct contributions grows very rapidly with the length or the duration of the trajectory. After a short time hundreds or millions of distinct classical paths are contributing. We discuss the accumulation of errors and the sometimes surprising result of many interfering contributions.

§ 1. Introduction

Interference is the essence of quantum mechanics. In the Feynman path integral approach, for example, all arbitrary paths contribute with equal weight to any given amplitude, making constructive or destructive interference the entire basis of the result. Any quantum amplitude of the form \( \langle il | j \rangle \) and be written, e.g.,

\[
\langle il | j \rangle = \sum_n \langle i | n \rangle \langle n | j \rangle \tag{1·1}
\]

where \( |n\rangle \) is another orthonormal basis; we may view \( \langle il | j \rangle \) as an amplitude resulting from the interference of terms in Eq. (1·1). Quantum mechanically, an amplitude may legitimately be the result of the sum of an uncountable infinity of interfering terms.

The quantum propagator \( G(x, x' ; t) \) gives the amplitude for going from \( x' \) to \( x \) in time \( t \). This is the task also of the approximate Van Vleck Green function propagator, which in somewhat incomplete form dates back to 1928.\(^1\) The semiclassical Van Vleck Green function propagator in coordinate space, suitably modernized with Maslov-type phase corrections,\(^2\) is a sum over as many classical paths as exist between \( x' \) and \( x \) in time \( t \):

\[
G(x, x' ; t) \approx G_{sc}(x, x' ; t) = \left( \frac{1}{2\pi i\hbar} \right)^d \sum_j \left| \text{Det} \left( \frac{\partial^2 S_j(x, x' ; t)}{\partial x \partial x'} \right) \right|^{1/2} \exp \left( iS_j(x, x' ; t) / \hbar - \frac{i\pi \nu_j}{2} \right). \tag{1·2}
\]

In this expression, the sum over \( j \) is for all trajectories connecting \( x' \) to \( x \) in time \( t \). The determinant plays the role of the square root of a classical probability, and the
phase is determined by the classical action $S_i(x, x'; t)$. An index $\nu_i$ is determined by conjugate points (focal points). $S_i(x, x'; t)$ is specified by the time integral of the Lagrangian $\mathcal{L}$

$$S_i(x, x'; t) = \int_0^t dt' \mathcal{L} = \int_0^t dt' \{ p(t') \cdot \dot{x}(t') - H(p(t'), x(t')) \}$$

along the $i^{th}$ classical path. $H$ is the classical Hamiltonian.

The semiclassical Green function is an example of the "universal semiclassical form"

$$\phi(x) = \sum_n \sqrt{P_n(x)} \exp(i\phi_n(x)/\hbar) ,$$

where

$$P_n(x) = \text{classical probability for the } n^{th} \text{ way of reaching } x ,$$

$$\phi_n(x) = \text{classical action along } n^{th} \text{ path reaching } x .$$

The sum over $n$ is needed because there may be $0, 1, 2, ...$ ways in general of reaching $x$. A similar expression can be substituted for any other representation in terms of eigenstates of Hermitian operators. The Born interpretation, namely that $\phi(x)$ is a probability amplitude, dictates that the wavefunction should go as the square root of the classical probabilities in the correspondence limit. The phase is the action integral

$$\phi_n(x) = \int p_n(x') \cdot dx' .$$

Under suitable assumptions the Van Vleck Green function is the basis of an exact formulation of quantum mechanics, by "time slicing" or multiplying $N$ Van Vleck propagators for time $t/N$ together to give a propagator for time $t$, taking the limit $N \to \infty$:

$$G(x, x', t) = \int dx'' \cdots \int dx'''' \cdots G_{sc}(x, x'', t/N) \cdots G_{sc}(x'''', x', t/N) .$$

For a potential with finite walls there is no possibility of trajectories arriving by more than one path in arbitrarily short time, and it is easily verified that the limit of Eq. (1.7) reduces to the Feynman path integral as $N \to \infty$. (Even if infinite walls are present it really makes no difference.) Constructing an approximate approach to quantum mechanics in which the times are left finite and a finite number of time slices are used is an attractive goal but a more nearly numerical subject than we wish to examine here.

§ 2. Exact cases

There are some circumstances where the semiclassical Green function is exact. If the underlying classical dynamics is linear, as in a free particle or harmonic oscillator, then the semiclassical Green function is trivially exact. This happens because the quadratic expansion of the exponent is a global, rather than local,
approximation to the dynamics. There are other cases that are not so simple at first. Exact cases give insight into the intricacies that arise in the inexact applications, including chaotic systems.

**Slits**

An instructive starting point is the one, two, ... slit interference experiment (see Fig. 1). The semiclassical and quantum versions of this are the identical provided we agree to initiate the classical mechanics at the slits. We make the Kirchoff approximation to the diffraction, integrating across the opening, using the semiclassical Green function at each source point across the slit. This approximation need not concern us if we assume the slits are all small compared to a wavelength, and is quite accurate in any case.\(^3\)

For the case of one slit there is (for a small slit) just one path to each point on the screen and a structureless probability results at the screen. Equation (1·4) has just one term. Classical, semiclassical, and quantum predictions are the same.

The case of two slits is equally familiar, and Eq. (1·4) has two terms. The quantum result oscillates about the classical. Semiclassical and quantum results are the same.

However suppose we have tens or thousands of very small slits. The image at the screen is (practically) the Fourier transform of the slit distribution. Each point of the screen receives \(N\) independent contributions for \(N\) slits. There are two important points. (1) The mere existence of many contributions to each point on the screen does not degrade the accuracy of the semiclassical result. (2) The quantum result does not necessarily merely oscillate about the classical probability. The more terms, the more the semiclassical (and presumably the quantum) result may deviate over large regions from the classical. This makes the whole issue of semiclassical approximations much more interesting than looking for a few oscillations about the classical result.

**Coulomb and Mott scattering**

The semiclassical and the classical probabilities are identical only if there is just one classical path connecting the relevant points. Coulomb scattering in momentum space is an important special case of this. The usual elastic scattering boundary conditions are incoming \(\mathbf{p}\), outgoing \(\mathbf{p}'\), with \(|\mathbf{p}|=|\mathbf{p}'|\). The corresponding Green function is \(G(\mathbf{p}, \mathbf{p}', t)\). The semiclassical version of the Green function has only one
term, since only one trajectory connects any given initial momentum with any given final momentum. Thus the quantum cross section should not oscillate with final momentum. The cross section is exactly the same classically and quantum mechanically, a fact which is well known.\(^4\) For the Hamiltonian

\[ H = \frac{p^2}{2m} + \frac{Ze^2}{r} \]  

(2.1)

the cross section is given by the Rutherford formula

\[ \frac{d\sigma}{d\Omega} = |f(p, p')|^2 = \left( \frac{Ze^2}{4E\sin^2(\theta/2)} \right)^2, \]  

(2.2)

both classically and quantum mechanically.

One should not draw the conclusion however that the existence of only one term in the semiclassical sum means that the semiclassical result is exact (and therefore that quantum and classical results are the same as well, since the one term in Eq. (1.4) squared gives the classical probability). Scattering from a single hard disk has only one classical trajectory connecting different momenta; however, the quantum and semiclassical scattering cross sections are different.

An interesting variant on Rutherford scattering is Mott scattering, which is Coulomb scattering for identical particles. This is a new challenge for the semiclassical approximation, because there will be two terms in the expression for the momentum space cross section: one for the direct process, and one for the exchange process. The direct term is the amplitude already needed for the semiclassical Coulomb scattering. It does not have the correct symmetry for either Bose or Fermi statistics. However, since it is an amplitude we have only to apply the appropriate symmetrization operator to get a semiclassical version of Mott scattering.\(^5\) There are thus two paths, which become experimentally indistinguishable (see Fig. 2), and the cross section is given by

\[ \frac{d\sigma}{d\Omega} = |f(p, p') \pm f(-p, p')|, \]  

(2.3)

where the sum refers to bosons and the difference to fermions. The semiclassical cross section for Mott scattering becomes given by

Fig. 2. Mott scattering trajectories, showing the original (distinguishable particle) trajectories at the top and the symmetrized set at the bottom, revealing that there are just two trajectories leading to the same final momentum, which interfere to give the characteristic oscillations.
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\[
\frac{d\sigma}{d\Omega} = \left( \frac{Ze^2}{4E} \right)^2 \left[ \frac{1}{\sin^4 \Theta/2} + \frac{1}{\cos^4 \Theta/2} \pm \frac{2\cos[2Ze^2/p\ln(\cot \Theta/2)]}{\sin^2 \Theta/2 \cos^2 \Theta/2} \right]
\]

(2.4)

and is identical to the exact quantum cross section (Taylor 1972). This checks the phase of the semiclassical Coulomb scattering amplitude, because in Rutherford scattering the phase is superfluous. Since there are only two terms, the quantum probability just oscillates about the classical. This example illustrates the point that no special exchange forces or procedures are needed to treat particle symmetry in semiclassical methods, except "post-symmetrization" of the distinguishable particle semiclassical result.

The semiclassical Airy function in momentum space

Starting with the momentum space Schrödinger equation for a particle of mass \(m\) on a linear ramp \(\beta x\), we have

\[
\left( \frac{p^2}{2m} + i\hbar \beta \frac{\partial}{\partial p} \right) \psi(p) = E\psi(p).
\]

(2.5)

The exact solution of this differential equation is:

\[
\psi(p) = \exp\left[ -iE\beta /\hbar + ip^3 /6m\beta \hbar \right].
\]

(2.6)

This is also the semiclassical wave function in momentum space. The prefactor is simply proportional to the square root of the classical probability of having momentum \(p\) and energy \(E\) on the ramp of slope \(\beta\). This probability is independent of \(p\). The phase is the momentum space action integral

\[
\int x(p) dp = pE/\beta + p^3 /6m\beta.
\]

(2.7)

Note there is again only one term in the semiclassical expression. We now move to coordinate space, where it is no longer exact.

\[\text{§ 3. Approximate cases}\]

The semiclassical Airy function in position space

The semiclassical approximation to the Airy function is a well known and very instructive example.

The exact integral representation of the coordinate space eigenfunction is the Fourier transform of \(\psi(p)\),

\[
\psi(x) = \sqrt{\frac{2}{2\pi}} \int_{-\infty}^{\infty} \exp[ipx/\hbar - ipE/\hbar\beta + ip^3 /6m\beta \hbar] dp,
\]

(3.1)

which is an integral form of the Airy function

\[
\psi(x) = (2m\beta \hbar)^{1/3} \sqrt{\frac{2\pi}{\hbar}} \text{Ai} \left[ \left( \frac{2m}{\hbar^2 \beta^2} \right)^{1/3} (\beta x - E) \right].
\]

(3.2)

The semiclassical approximation to the Airy function comes from the stationary
phase evaluation of the integral in Eq. (3·1), which gives (we take $E=0$ without loss of generality) the stationary phase points

$$p = \pm \sqrt{2m\beta x}$$  \hspace{1cm} (3·3)

and the semiclassical result (including Maslov phases)

$$\phi_{sc}(x) = \left(\frac{m\beta}{x\hbar}\right)^{1/4} \exp\left[2i\sqrt{2m\beta x^{3/2}/3\hbar} + i\pi/4\right] + c.c.$$  \hspace{1cm} (3·4)

The stationary phase approximation works very well except when the area enclosed by the intersection of the vertical line representing $x$ in phase space and the parabola representing the equation $p = \pm \sqrt{2m\beta x}$ is less than Planck's constant $\hbar$. (The momentum coordinates of these intersections are the values of the stationary phase points.) This is seen in Fig. 3, where accurate plots are given. The onset of the error is the usual “area $\hbar$ rule” of semiclassical approximations. When it is violated, the phase difference in the exponent between two adjacent stationary phase points falls below $2\pi$, the stationary phase points in the integral Eq. (3·1) have “coalesced”. In discussing accuracy of semiclassical approximations, it is thus clear the area of the “loops” formed by the intersections of two Lagrangian manifolds is so critical.

The Airy function and its semiclassical approximation serve as a useful baseline for similar problems that arise in much more complex situations, e.g., chaos. It has the advantage of having one semiclassical representation where it is exact (momentum space). The folds that inevitably occur in chaotic evolution of classical Lagrangian manifolds are very similar to the single fold found in the Airy function in phase space. The area rule for the accuracy of the semiclassical approximation is crucial to all discussions of the accuracy of semiclassical approximations.

§ 4. Nonlinear dynamics

The semiclassical Van Vleck Green function corresponds to the quantum amplitude (switching to one dimensional notation)

$$\langle x' | x(t) \rangle = \langle x' | e^{-iHt/\hbar} | x \rangle.$$  \hspace{1cm} (4·1)

Under nonlinear time evolution the initially vertical classical manifold corresponding to $|x\rangle$ becomes the folded $|x(t)\rangle$. At first it hardly matters whether the dynamics is chaotic, quasiperiodic, or a mixture of the two (the most common situation). At first
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a few loops or folds form (albeit faster for chaotic dynamics). As long as the loops enclose area $\hbar$ or greater and the loop ends are avoided, we expect the semiclassical Green function to be quite accurate. However for hard chaos the folds grow exponentially in number, and there is no reason for them to avoid any region in coordinate space. Soon, on an exponentially short timescale, there is at least one and perhaps many area-$\hbar$ violating loops contacting the $|x\rangle$ manifold for every position $x$. If we make $\hbar$ smaller, the time for this disaster to occur is longer only by $1/\log \hbar$, since we can reduce the “bad” zones for each caustic by an amount roughly linear in $\hbar$; however, the number of them grows exponentially fast.

Anharmonic oscillator

Before we discuss why things are not usually this bad, we want to distinguish two phenomena: the appearance of many terms in the sum over classical contributions, Eq. (1·4), and the appearance of many $\hbar$ violating loops. First consider a simple one dimensional anharmonic oscillator. Starting at an off center position with $|x\rangle$, a Morse oscillator generates a manifold $|x(t)\rangle$ as shown in Fig. 4. At long times there will be many contributions to the semiclassical $<x'|x(t)\rangle$. Making an extra circuit (one oscillation of the anharmonic oscillator) accumulates more than area $\hbar$ if $|x\rangle$ is far enough off center, so only the usual Airy-like caustics coming from a single part of a whorl appear. Distinct intersections of the manifolds with the $x$-state having different windings around the center are accurate. Note that at long times the intersections between $x$ and $x'(t)$ can be extremely close in phase space. However, this does not mean a violation of the area $\hbar$ rule; since we must find the area enclosed from one intersection to the next. Mere proximity in phase space of the intersections do not indicate failure.

Fig. 4. The time evolution of an initial position manifold in a Morse potential (not shown). All area $\hbar$ violating regions may be avoided for some time in certain regions. An area $\hbar$ violating region is encountered (black) in the position representation, but it is not a problem for a coherent state (gray).

Fig. 5. The coherent states have a uniformizing property which makes them a useful replacement for coordinate space.
of the stationary phase approximation.

The Airy-like caustics are still a problem. However, suppose we examine $\langle a|\psi(t)\rangle$ instead of $\langle x'|\psi(t)\rangle$, where $|a\rangle$ is a coherent state with the position-momentum uncertainty shown in Fig. 4. The coherent states are well known to smooth over caustics, even in their primitive semiclassical representations. The situation shown in Fig. 5, top, is quite safe, but is questionable in Fig. 5, bottom. Returning to the Morse oscillator example Fig. 4, the coherent state is not in trouble and its amplitude is very accurately predicted by semiclassical Green function propagation.

Thus, by using coherent states we can avoid some of the pitfalls of coordinate space. Smooth states are often the physically relevant initial conditions in any case. Since $\langle a|\psi(t)\rangle = \langle \psi|a(t)\rangle^*$ we can put the shoe on the other foot. The uniformizing properties of the coherent states or other smooth states apply whether we propagate them and project onto coordinate space, or propagate the coordinate state and project onto the coherent state basis.

Some results for the semiclassical propagation of an initial Gaussian wave packet are shown in Fig. 6. Note that the semiclassical approximation (which is the Van Vleck Green function applied to the initial Gaussian wave packet) is doing well long after the wave packet has spread, i.e., long after the "Ehrenfest time" after which the wave packet is no longer behaving classically in an obvious way. The spreading of the wave packet and the formation of whorls are intimately connected. The accuracy of the semiclassical wave packet even when the spiraling is advanced is evidence that many terms may exist in the semiclassical sum without degradation of the result.

The manifold corresponding to $|\psi(t)\rangle$ gets increasingly wound up. However for certain $x$ and $x'$ there will be few caustics at first and then only a slow growth of their number as the inner loops squeeze out area and drop below the legal limit. This is true for $x$ near the center of the oscillator and $x'$ starting near an outer turning point. If we look at certain $x$ at long times there can be many hundreds of contributions but almost all will be "legal".

**Revivals**

The spiraling or "whorls" in phase space (which are due to different energies having different periods of motion) causes the net classical probability density to have a relatively smooth distribution (after averaging over local caustics) over the whole bound coordinate range, providing the spiraling is well developed. Classically, no region will suddenly carry most of the probability at the expense of other regions. Quantum mechanically, a relatively abrupt re-localization or "revival" happens from time to time. The event is not necessarily a recurrence in the survival probability,
because the revival need not be at the same position as the initial wavepacket. It is obvious that revivals must happen eventually in all quantum systems with a discrete energy spectrum. Revivals sometimes happen much sooner than one would expect if there are special aspects of the energy eigenvalues (or quasienergies in the case of a periodically driven system such as the cat map\textsuperscript{(11)}. The quantum cat map is known to revive exactly after a short time.\textsuperscript{(12)} It is well known that the wavepacket relocalizes again for the Morse oscillator. Recently, revivals have been seen experimentally in Coulomb wave packets.\textsuperscript{(13)} However, can the revival be correctly predicted semiclassically? The semiclassical wave function is after all based on the delocalized classical trajectories. Might the quantum relocalization signal the demise of semiclassical time propagation?

It is remarkable that the localization is accurately reproduced semiclassically even though it is a "pure quantum effect". This fact has been demonstrated by Suarez Barnes, Tomsovic and Nauenberg\textsuperscript{(10)} for the Coulomb potential. This means the terms in the semiclassical sum in the universal formula Eq. (1.4) destructively interfere over broad regions of coordinate space at particular times, a peculiar and beautiful behavior. Recall the many-slit example presented earlier. The classical probability is uniformly spread, but the quantum amplitude, summed over all the contributions, interferes destructively almost everywhere.

§ 5. Proliferation of loops and the log time

We now face the problem of counting how fast the area $\hbar$ violating loops will proliferate in the troublesome chaotic limit.\textsuperscript{(15)} A useful model is a Smale horseshoe construction (for an area preserving map). Starting with a line segment representing a $p$-state, we suppose that time evolution is chaotic and leads to exponential stretching of the segment (Fig. 7). Since the phase space (we suppose) is finite in area, the line grows thinner as it collapses perpendicular to its axis. It necessarily folds to stay in bounds, and clearly the folds increase exponentially in number with time. In real life the folds will appear everywhere essentially randomly, so that indeed the coordinate space wave function will be ruined on a timescale only logarithmically shorter as $\hbar \to 0$ (see Fig. 8).

Fig. 7. The Smale-like horseshoe construction for an area preserving map.

Fig. 8. Proliferation of caustics—both in chaotic and integrable regions.
Fraction of good manifold

However there is another very useful way to view the situation. Suppose we consider points along the initial line segment. Each point maps into a unique point in a successive iteration, and either that point is in a caustic region (shown black) or it is not. If it is, we erase it from the copy of the initial segment (shown in each case on the right in Fig. 7). A Cantor set is being created in this way. The “good” remaining regions still disappear exponentially fast, since, approximately, a constant fraction of the remaining good region is removed at each iteration. The fraction $F_p$ of initial $p_0$ remaining good constitutes a reasonable figure of merit, because all “good” regions translate into pieces of the evolving Lagrangian manifold which yield accurate amplitudes. To get the expression for the fraction of remaining good region, we consider the evolution of the $p$-like (horizontal) Lagrangian manifold. At a fixed time, a finite number of folds will have developed. As $h \to 0$, $F_p \to 1$.

First we need to understand the $\hbar$ dependence of a single fold. To second order in $p_0$, near the fold, we have $x' = y + \alpha p_0 + \beta p_0^2$, $y' = \alpha' p_0$. The line corresponding to $|p_0\rangle$ is now rotated by the transformation, and cuts the fold at an angle. If the area enclosed is $\hbar$, the range of $p_0$ corresponding to “bad” parts of the fold are easily shown to go as $\hbar^{1/3}$, at least for sufficiently small $\hbar$, where the folds are isolated and the quadratic expansion in $p_0$ holds. This dependence has been checked for the standard map.

The initial $p$-like Lagrangian manifold is first folded once to a $U$-shape, lying on its side, making one caustic in coordinate space. After compression by a factor of 2 and stretching by the same factor, the line segment is folded again, making a total of three caustics. The $n^{th}$ cycle yields $2^{n+1}-1$ total caustics, but the stretching by the factor $2^n$ means that each successive caustic spans a smaller range of initial manifold, by a factor of $1/2^n$. A constant fraction of the initial manifold therefore lands in caustics at each step, whether it has previously been part of another caustic or not. Thus the fraction removed is not confined just to the remaining “good” regions of the initial manifold, but is applied apparently at random to the whole of the initial manifold. The differential equation describing this removal procedure is

$$\frac{-dF_p}{dt} = a\lambda h^{1/3} F_p; \quad \implies F_p(t) = \exp(-a\lambda h^{1/3}t),$$

(5.1)

where $\lambda$ is the Lyapunov exponent and $a$ a system dependent proportionality constant. Thus “good” initial manifold disappears exponentially fast in time, but the dependence on $\hbar$ is much more optimistic than the log time: we can increase the half time for the removal of “good” initial manifold (i.e. the time when half of the Lagrangian manifold is contributing to errors) as $h^{-1/3}$, much better than $1/\log h$.

Thus even though the coordinate space Green function quickly degrades, it is still useful long after the degradation for propagating smooth states, provided that they lie in the “good” regions of phase space. From the phase space, Lagrangian manifold analysis it is quite clear that the accuracy of semiclassical amplitudes is very non-uniform. It would be possible to find inaccurate regions almost immediately, while other zones are well behaved far past the log time. Very often, smooth state correla-
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Fig. 9. Upper: part of the $q_0=0$ manifold unfolded in the angle variable after several iterations; Left: fragment of the initial $q$-manifold after removing "bad" segments for times 1, 2, 3 and 4; Right: detail of the upper diagram showing the coherent state and the semiclassically inaccurate region in dark.

tion functions are physically the desired quantity; other times, as when determining energy eigenvalues for example (by Fourier transform of a correlation function\textsuperscript{16}), one has a choice of states. This makes the present results of far reaching consequence for applications.

**Standard map**

For a view of a more conventional system, Fig. 9 shows the shape of a segment of the $x_0=0$ manifold near $p_0=0$ after several iterations of the standard map,\textsuperscript{17} except that we have unfolded the map by not applying the $2\pi$ modulo condition in angle. This has the advantage of simplifying the structure locally in phase space; the overlap with a localized state is now obtained by replicating that state periodically, as is shown by the shaded disk. We consider the Green function first, i.e. $\langle x|x(t)\rangle$. The figure has regions blackened out that violate the area $h$ rule for a particular value of $h$. Some of the blackened areas are standard textbook caustics; others are thin "tendrils".\textsuperscript{18} However, no matter how thin a tendril has become nor how many folds upon folds have been generated by the dynamics, the sole criterion for accuracy of a given pair of coalescing stationary phase points in the integration variable $p_0$ is their distance as measured by the phase accumulation of the exponent between the points in question; this accumulation should be $2\pi$ or greater, which translates into the area $h$ in the phase plane. Thus, even though the points marked C, D in Fig. 9 are not separated by $2\pi$, and are thus generating inaccuracies if $|x|$ should happen to cut through those regions, the contribution from A and B encloses area greater than $h$ and is accurate even though the distance between A and B is minuscule on the scale of $h^{1/2}$. Note that subsequent evolution will fold the Lagrangian manifold further but cannot reduce the area enclosed between A and B (which have been chosen to lie on a stable manifold and will move exponentially closer together). The Poincaré-Cartan theorem of dynamics also guarantees that the area preservation and phase difference between A and B hold even if the phase plane is a surface of section.

The area rule leads to the remarkable conclusion that stronger chaos may actually help the semiclassical accuracy by producing larger folds with more area enclosed. This leads to the worry that the generic "soft" chaos dynamics would be
problematical, since it tends to generate many small loops in phase space. However, using the cellular dynamics approach, we have had gratifying success with mixed dynamics.\textsuperscript{14} This is good, but now there is only speculation why things work so well. Nonetheless, we present some of the speculations below.

§ 6. Soft chaos and quantum smoothing

The most common type of system found in nature, i.e. "soft chaos" or mixed dynamical systems, now have the most worries associated with them. The slow mixing associated with the boundaries between the regular and the chaotic regimes in phase space results in small oscillations of the manifolds which profoundly violate the $\hbar$ rule. The horseshoe arguments do not apply, since the loops do not have sufficient area. Yet, numerical trials have been encouraging.\textsuperscript{14} One aspect of these trials is that the fine details of the oscillation were not followed, due mainly to their complexity and size. This means that some sort of average was taken in the mixed regimes, an average which is not very well defined yet. We next give arguments how this averaging may take place. One thing is quite certain: one does not want to find all the tiny caustics and apply Maslov conditions, etc. The reason is that the machinery of stationary phase assumes the area $\hbar$ rule is satisfied; this is far from the case if small homoclinic and heteroclinic oscillations develop in the phase space.

\textit{Kicked quartic oscillator}

A good model to study in this regard is the periodically kicked quartic oscillator.

\begin{equation}
\begin{align*}
\dot{p}_{n+1} &= p_n - V'(x_n)\tau, \\
x_{n+1} &= x_n + p_{n+1}\tau,
\end{align*}
\end{equation}

where $V(x) = x^4 - x^2$. With small timesteps $\tau$, the dynamics reduces to integrable motion on a one dimensional double well oscillator. The model retains the character of unstable motion near the barrier top at all values of the timestep $\tau$. By using a Verlet integration (the $p_{n+1}$ appears in the equation for $x_{n+1}$, not $p_n$) the map 6.1 is area preserving.

For small timestep the homoclinic oscillations are very tiny. They violate the area-$\hbar$ rule, and stationary phase is not justified. What to do with them? Certainly we do not want to find the caustics carefully and add a Maslov phase shift for each, because the area enclosed is not sufficient. No phase shift of $\pi/2$ could have developed. Because the area is so small in the loops, the trajectories on either side of a loop have nearly the same phase. The right thing to do is a kind of averaging which is discussed next.

\textit{Quantum smoothing}

The quantum mechanics smoothes over the oscillations in a way we can quantify. The situation can be simulated by considering the Lagrangian manifold defined by

\begin{equation}
p(x) = p_0 + \hbar\beta\lambda x^2/2\sin(\beta\pi x^3).
\end{equation}
The function 6.2 (see Fig. 10) has oscillations of increasing amplitude and frequency, which however have the same area \( \int p(x)dx = \lambda \) between successive zeros. This mimicks some aspects of the homoclinic oscillations characteristic of nonintegrable motion near a periodic orbit. The momentum \( p(x) \) spans a large range as \( x \) increases, but we now show that for \( \lambda < 1 \) the manifold's semiclassical behavior can be understood by replacing it with three smooth manifolds. The rules for the replacement are simple to derive. The action function \( S(x) \) is the integral of \( p(x) \):

\[
S(x) = \int_0^x p(x)dx = p_0x + \hbar \frac{\lambda}{2} \cos(\beta \pi x^3). \tag{6·3}
\]

The "wavefunction"

\[
\psi(x) = A(x)\exp[iS(x)/\hbar] \tag{6·4}
\]

can be approximated by

\[
\psi(x) \approx A(x)\exp[ip_0x/\hbar]\left(1 + i\frac{\lambda}{4}\cos(\beta \pi x^3)\right) = A(x)\exp[ip_0x/\hbar]\left[1 + \frac{i}{4}\exp[i\beta \pi x^3] + \frac{i}{4}\exp[-i\beta \pi x^3]\right] \tag{6·5}
\]

if \( \lambda < 1 \). For small \( \lambda \) this expression is nearly identical with Eq. (6·4) with the original action 6.3, because \( \cos(\beta \pi x^3) \) is bounded by \( \pm 1 \). However semiclassically it has the interpretation of the sum of three smooth classical manifolds, i.e. \( p = p_0, p = p_0 + 3\beta \pi x^2, \) and \( p = p_0 - 3\beta \pi x^2 \). They have the weight \( A(x), \lambda A(x), \lambda A(x) \), respectively. The situation is depicted in Fig. 11.

Almost every discussion of the relation of classical and quantum mechanics for chaotic systems alludes to quantum smoothing, but here we have seen explicitly one way this smoothing comes about. For all reasonable semiclassical purposes the three smooth classical manifolds accurately replace the rapid oscillations of the original manifold. Note too that depending on the parameters the outlying manifolds \( p = p_0 + 3\beta \pi x^2, \) and \( p = p_0 - 3\beta \pi x^2 \) lie far beyond the limits of the original distribution. This may have very interesting consequences.

As the area of the loops in \( p(x) \) exceeds \( \hbar \), they can no longer be replaced by a few smoother manifolds. This happens in the present example when the parameter...
§ 7. Conclusion

We have given perspectives on the role of multiple contributions to a semiclassical amplitude. The most important conclusion is that sheer numbers of terms do no harm (except to make the calculation more cumbersome); harm is done when the areas of enclosed loops fall below $\hbar$. Even so, the harm is much more benign when coherent state representations are used, which smooth out the caustics. The troubling "log time" was seen to be a paper tiger, as had been pointed out earlier by us.\textsuperscript{15,16} The improvement to semiclassical approximations is power law in $\hbar$, not logarithmic. We noted that mixed dynamical systems apparently give good results with cellular dynamics, which is an implementation of the Van Vleck Green function, but we do not fully understand why the results are so good. Finally, one hint about the good results for mixed systems comes from the way in which quantum mechanics smoothes over classical detail, and a new explicit example of this smoothing was given.

References

5) J. M. Rost and E. J. Heller, Semiclassical Scattering of Charged Particles, submitted for publication.
17) The standard map is a 2-d mapping of the cylinder onto itself defined by the following eqs.: $p_{n+1} = p_n + K \sin(x_n)$, $x_{n+1} = x_n + p_{n+1}$, (mod $2\pi$). The parameter $K$ determines the chaoticity of the mapping.