We present a proper off-mass-shell extension of the chiral reduction formula (\( \chi_{RF} \)) proposed by Yamagishi and Zahed. This is achieved by rewriting the \( \chi_{RF} \) in a manifestly consistent form with the conventional LSZ reduction formula.

\section{Introduction}

There is no doubt that chiral symmetry is important for a theoretical understanding of the low energy properties of QCD. The spontaneous, and also explicit breaking of this symmetry governs various aspects of hadronic processes and plays a crucial role in the construction of effective models.

On the basis of chiral symmetry, Yamagishi and Zahed developed a general framework for analyzing pion-induced hadronic processes.\(^1\) This framework consists of two master equations for the extended S-matrix, which are based on the gauge covariant divergence equations proposed by Veltman and Bell.\(^2,3\) Each master equation represents transformation properties of the extended S-matrix under the local vector and axial transformations, subject to the following asymptotic conditions for the axial current:

\[
A^a_\mu(x) \to - f_\pi \partial_\mu \pi^a_{in,\text{out}}(x) + \cdots, \quad (x^0 \to \mp \infty)
\]

(1.1)

and

\[
\partial^\mu A^a_\mu(x) \to f_\pi m_\pi^2 \pi^a_{in,\text{out}}(x) + \cdots, \quad (x^0 \to \mp \infty)
\]

(1.2)

One noteworthy result obtained within this framework is that the master equation for the axial transformation provides a new reduction formula, the chiral reduction formula (\( \chi_{RF} \)), for scattering amplitudes involving any number of pions with their physical masses. With this formula, the relevant on-shell Ward identities satisfied by those amplitudes can be derived. These identities are expressed in terms of the Green functions of well-defined current and density operators, and exactly embody the consequences of the broken \( SU(2) \times SU(2) \) chiral symmetry without the need for any model or expansion scheme. A number of investigations based on the \( \chi_{RF} \) have been carried out for hadron reactions in the resonance region\(^4\)–\(^6\) and hadronic matter.\(^7\)–\(^10\) These investigations have demonstrated that the \( \chi_{RF} \) is a powerful tool for clarifying the role of chiral symmetry in those processes in a model-independent way.

To this time, there have been no studies aimed at determining how amplitudes behave in the off-shell region within the framework of Ref. 1). In principle, the \( \chi_{RF} \) can be applied not only to scattering amplitudes but also to internal vertex functions.
and potentials in which the attached pion legs can be off the mass shell. Although
the off-shell amplitudes themselves are artificial, and not directly related to phys-
ical quantities, such amplitudes are required in some theoretical calculations. For
instance, in scattering problems, the on-shell scattering amplitudes are calculated by
using off-shell potentials (in the Lippmann-Schwinger equation, etc.). This situation,
i.e. that the off-shell amplitudes are employed in calculations of observables, is also
found in many-body problems. Thus in attempting to treat these problems on the
basis of the $\chi RF$, it is necessary to make clear the off-shell structure.

However, care must be taken in applying the $\chi RF$ presented in Ref. 1) to off-shell
pions, because this formula was originally constructed for the purpose of treating the
on-shell pions. Indeed, in contrast to on-shell cases, we find that its naive application
is invalid off the mass shell. The purpose of this paper is to carefully examine the
derivation of $\chi RF$ and to formulate a proper method of extending the $\chi RF$ off the
mass shell.

This paper is organized as follows. In §2, we briefly review the theoretical
framework developed in Ref. 1). In §3, we explain in detail the kind of interpolating
pion field chosen within the framework of Ref. 1). Generally, the interpolating pion
field must be defined in order to relate scattering amplitudes to the Green functions
of current and density operators. This is also the case for the $\chi RF$. The problem
inherent in the $\chi RF$ will become clear through this explanation, because the off-shell
behavior of amplitudes is fixed once an interpolating pion field is given. In §4, we
present an off-shell extension of the $\chi RF$ that is consistent with the framework of
Ref. 1). A summary is given in §5.

§2. Chiral reduction formula

A fundamental quantity in the theoretical framework developed in Ref. 1) is
the extended S-matrix, $\hat{S} = \hat{S}[v^a, a^a, s, J^a]$, which is a functional of the vector,
alvector, scalar and pseudoscalar external fields. Then, the current and density
operators $\hat{O} = (j^a V_\mu, j^a A_\mu, f^a_\pi \hat{\pi}_a)$ conjugate to the corresponding external
fields $\phi = (v^a, a^a, s, J^a)$ are defined as

$$\hat{O}(x) = -i\delta \hat{S} / \delta \phi(x).$$

(2.1)

Here, $\hat{\pi}_a(x)$ is the pseudoscalar-isovector density operator, and $\hat{\sigma}(x)$ is the scalar-
isoscalar density operator from which the pion decay constant $f_\pi$ has been subtracted.
In the presence of the external fields, the operators $j^a V_\mu, j^a A_\mu$ and $\hat{\pi}_a$ are related to
the ordinary vector and axial currents as

$$V^a_\mu(x) = f_\pi a_{ab}^\mu(x) \hat{\pi}_b(x)$$

(2.2)

and

$$A^a_\mu(x) = f_\pi a_{ab}^\mu(x) - f_\pi \nabla^a_\mu \hat{\pi}_b(x),$$

(2.3)

respectively, where we use the notation $a_{abc}^\mu \equiv \epsilon_{abc} a_{ab}^\mu$ and $\nabla^a_\mu \equiv \delta^a_\mu \partial_\mu + \epsilon^a_\mu$. Owing
to the Bogoliubov causality condition, the $T^*$ product of the operators $\hat{O}$ can be
expressed as

\[ T^n(\hat{O}(x_1) \cdots \hat{O}(x_n)) = (-i)^n \hat{S}^\dagger \delta \frac{\delta}{\delta \phi(x_1)} \cdots \delta \frac{\delta}{\delta \phi(x_n)} \hat{S}. \]  

(2.4)

The essence of the framework presented in Ref. 1) is embodied by two linear master equations for the extended S-matrix,

\[ T^a_V(x) \hat{S} = 0, \]  

(2.5)

\[ T^a_A(x) \hat{S} = 0, \]  

(2.6)

where

\[ T^a_V(x) = \left( \nabla_\mu \frac{\delta}{\delta v_\mu} + a_\mu \frac{\delta}{\delta a_\mu} + J \frac{\delta}{\delta J} \right)^a(x), \]  

(2.7)

\[ T^a_A(x) = \left[ -(\boxdot + m^2_\pi + K) \frac{\delta}{\delta J} + iJ \frac{1}{f_\pi} X_A - \frac{1}{f_\pi} \left( \nabla_\mu a_\mu - \frac{J}{f_\pi} \right) \frac{\delta}{\delta s} \right]^a(x) \]  

(2.8)

and

\[ K^{ab}(x) = \left( \nabla_\mu \nabla_\mu - a_\mu a_\mu + s - \boxdot \right)^{ab}(x), \quad X^a_A(x) = \left( \nabla_\mu \frac{\delta}{\delta a_\mu} + a_\mu \frac{\delta}{\delta v_\mu} \right)^a(x). \]  

(2.9)

In Eqs. (2.7)–(2.9), the isospin indices with respect to which the contraction is taken are suppressed. The operator \( T^a_V(x) \) is the (local) isospin generator, and hence Eq. (2.5) represents the isospin invariance of the extended S-matrix. Equation (2.6) represents the transformation properties of \( \hat{S} \) under axial transformations. By applying the functional derivatives of \( \phi(x) \) to the master equations and using Eqs. (2.1) and (2.4), we straightforwardly obtain the vector and axial Ward identities satisfied by the Green functions of \( \hat{O}(x) \).

A crucial point in the derivation described above is that the asymptotic conditions (1.1) and (1.2) are exactly incorporated in the master equations without the need for any specific model or expansion scheme. Consequently, the resulting Ward identities incorporate not only the spontaneous chiral symmetry breaking, but also its explicit breaking to all orders in the quark masses. This property of the master equations allows us to derive the Ward identities for scattering amplitudes involving pions with their physical masses. This contrasts with the situation in other theoretical frameworks, such as the chiral perturbation theory, in which the pion mass is expressed as a perturbative extrapolation from the chiral symmetric point. The Ward identities are integrated into the form of a reduction formula, called the chiral reduction formula (\( \chi RF \)).

We now explain the \( \chi RF \) in detail. First, we note that the pseudoscalar-isovector density \( \hat{\pi}^a(x) \) possesses the asymptotic form \( \hat{\pi}^a(x) \rightarrow \pi^a_{\text{in, out}}(x) \) \( (x^0 \rightarrow \mp \infty) \), which follows from the construction of the master equations and the asymptotic conditions (1.1) and (1.2). This fact enables us to identify \( \hat{\pi}^a(x) \) as the (normalized) interpolating pion field. Then, through this identification, the axial Ward identities satisfied
by the Green functions of $\hat{O}(x)$ can be related to scattering amplitudes including pions. The identification is emphasized by formally solving the axial master equation (2.6) for $\delta \hat{S}/\delta J^a$ under the condition $\hat{\pi}^a(x) \rightarrow \pi^a_{\text{in, out}}(x) \ (x^0 \rightarrow \mp \infty)$. The result is\footnote{In Ref. 1), Eq. (2.10) is called the (axial) master equation.}

$$\frac{\delta \hat{S}}{\delta J^a(x)} = i\hat{S}\pi^a_{\text{in}}(x) + i\hat{S} \int d^4y G_R^{ab}(x,y) \left(K\pi_{\text{in}}^b\right)(y)$$

$$+ \int d^4y G_A^{ab}(x,y) \left[-iJ + \frac{1}{f_\pi} \left(\nabla^\mu a_\mu - \frac{J}{f_\pi}\right) \frac{\delta}{\delta s} - \frac{1}{f_\pi} X_A \right]^b(y) \hat{S}$$

$$= i\pi^a_{\text{in}}(x)\hat{S} + i \int d^4y G_A^{ab}(x,y) \left(K\pi_{\text{in}}^b\right)(y) \hat{S}$$

$$+ \int d^4y G_A^{ab}(x,y) \left[-iJ + \frac{1}{f_\pi} \left(\nabla^\mu a_\mu - \frac{J}{f_\pi}\right) \frac{\delta}{\delta s} - \frac{1}{f_\pi} X_A \right]^b(y) \hat{S}. \quad (2.10)$$

Here, $G_R$ and $G_A$ are the retarded and advanced Green functions, respectively, which satisfy

$$(-\Box - m^2 - K)^{ab}(x)G_{R,A}^{bc}(x,y) = \delta^{ac}\delta^{(4)}(x-y). \quad (2.11)$$

Using Eq. (2.10), we can directly derive the commutation relations for the creation and annihilation operators of the pion with the extended S-matrix $\hat{S}$. We find

$$[a^a_{\text{in}}(k), \hat{S}] = R^a(k)\hat{S}, \quad [\hat{S}, a^a_{\text{in}}^\dagger(k)] = R^a(-k)\hat{S}, \quad (2.12)$$

where

$$R^a(k) = \int d^4xe^{ikx} \left[iJ + \frac{1}{f_\pi} X_A - K \frac{\delta}{\delta J} - \frac{1}{f_\pi} \left(\nabla^\mu a_\mu - \frac{J}{f_\pi}\right) \frac{\delta}{\delta s} \right]^a(x). \quad (2.13)$$

Note that we have rewritten the r.h.s. of the commutation relations (2.12) in forms that do not involve the asymptotic pion field. [Compare these with Eqs. (6.2) and (6.3) in Ref. 1.)]

Iterative use of Eq. (2.12) yields the $\chi$RF for scattering amplitudes involving any number of pions with their physical masses,*\footnote{We consider the case in which no two pions have equal momenta.}

$$\langle \alpha; k_1a_1, \cdots, k_ma_m | \hat{S} | \beta; l_1b_1, \cdots, l_nb_n \rangle_{\phi=0} = [R^{a_1}(k_1) \cdots R^{a_m}(k_m)R^{b_1}(-l_1) \cdots R^{b_n}(-l_n)]_{\hat{S}} \langle \alpha | \hat{S} | \beta \rangle_{\phi=0}, \quad (2.14)$$

where $k_i$ ($l_i$) and $a_i$ ($b_i$) are the four-momentum and isospin indices of the outgoing (incoming) pions, respectively, and $\alpha$ and $\beta$ represent the states of other particles. Here, $[\cdots]_{\hat{S}}$ indicates that we take normalized symmetric permutations of the functional derivative operators contained therein; explicitly, we have

$$[D_1 \cdots D_n]_{\hat{S}} = \frac{1}{n!} \sum_{\text{perms}} D_1 \cdots D_n. \quad (2.15)$$
This operation clearly shows the crossing symmetry in Eq. (2.14). By using Eq. (2.14), together with Eq. (2.4), scattering amplitudes can be expressed in terms of the Green functions of the operators \( \hat{O} = (j_V, j_A, f_\pi \hat{\sigma}, \hat{\pi}) \). The \( \chi \)RF takes the form of a functional derivative, and all constraints that stem from the broken chiral symmetry are contained in \( R^a(k) \). The \( \chi \)RF can be understood as the following replacement of commutators:

\[
[a_{in}^a(k), ] \rightarrow R^a(k), \quad [\ , a_{in}^{a\dagger}(k)] \rightarrow R^a(-k).
\]

(2.16)

For instance, the double commutator is replaced as

\[
[a_{in}^a(k_1), \{ \hat{S}, a_{in}^{b\dagger}(k_2) \}] \rightarrow R^b(-k_2)R^a(k_1)\hat{S}.
\]

(2.17)

§3. Interpolating pion field

In contrast to \( V_\mu^a \) and \( A_\mu^a \), which are identified with physical electroweak currents, the operators \( \hat{O} \) are convention-dependent. The interpolating pion field \( \hat{\pi}^a \) is not uniquely determined: It is only required that the asymptotic condition \( \hat{\pi}^a(x) \rightarrow \pi_{in/out}^a(x) \) \((x^0 \rightarrow \mp \infty)\) be satisfied. Equations (2.2) and (2.3) represent an arbitrary way of dividing \( V_\mu^a \) and \( A_\mu^a \), respectively. For the redefinition of the pion field \( \hat{\pi}^a \rightarrow \hat{\pi}'^a \), with

\[
\hat{\pi}'^a(x) = \hat{\pi}^a(x) + A^a(x),
\]

(3.1)

\( j_{V_\mu}^a \) and \( j_{A_\mu}^a \) also change in such a manner that Eqs. (2.2) and (2.3) remain invariant:\(^1\)

\[
j_{V_\mu}^a(x) = j_{V_\mu}^a(x) - f_\pi a^a_{\mu c} A^c(x),
\]

(3.2)

\[
j_{A_\mu}^a(x) = j_{A_\mu}^a(x) + f_\pi \nabla^a_{\mu c} A^c(x).
\]

(3.3)

The arbitrariness of the operators \( \hat{O} \) is translated into that of the extended S-matrix. The redefined operators \( \hat{O}' = (j_{V}', j_{A}', f_\pi \hat{\sigma}', \hat{\pi}') \) are obtained in terms of an appropriately extended S-matrix \( \hat{S}'[\phi] \), which equals \( \hat{S}[\phi] \) at \( \phi = 0 \) (i.e. \( \hat{S}|_{\phi=0} = \hat{S}'|_{\phi=0} \equiv \hat{S} \)), through functional derivatives as

\[
\hat{O}'(x) = -i \hat{S}' \frac{\delta \hat{S}'}{\delta \phi(x)}.
\]

(3.4)

Therefore, a different way of extending the S-matrix \( \hat{S} \) by introducing the external fields provides a different choice of the current and density operators.

The framework presented in Ref. 1) only takes account of the interpolating pion fields given by the extended S-matrices satisfying the master equations. Because the master equations fix the extended S-matrix up to a phase factor,\(^*)\) the arbitrariness

\(^*)\) A new extended S-matrix \( \hat{S}' \) can be written as \( \hat{S}' = \hat{S} \exp(i f[\phi]) \).\(^1\) Causality implies that the functional \( f[\phi] \) has the form

\[
f[\phi] = \int d^4x P(x),
\]

where \( P(x) \) is a polynomial in \( \phi(x) \) and its derivatives. [A constant term can be eliminated from \( P(x) \) by imposing the normalization \( \langle 0 | \hat{S} | 0 \rangle = 1 \).] Then, for any \( f[\phi] \) satisfying \( T_{V,A} f[\phi] = 0 \), the master equations (2.5) and (2.6) are invariant.
in the choice of $\hat{O}$ still exists. However, as long as the extended S-matrices are introduced in such a manner that the master equations are satisfied, we can apply the $\chi$RF to all of them. The scattering amplitudes obtained from such extended S-matrices then satisfy the Ward identity given by the $\chi$RF, and thus exhibit the same off-shell momentum dependence up to terms that are not fixed by the broken chiral symmetry. (Of course, these amplitudes are identical on shell.) A class of interpolating pion fields that is chosen to be consistent with the master equations ensures the “form invariance” of the Ward identities.

We explain the situation described above more concretely by considering the $\pi N$ scattering as an example. Using the $\chi$RF (2.14), together with Eqs. (2.1) and (2.4), we obtain the $\pi^a(k) + N(p) \rightarrow \pi^b(k') + N(p')$ scattering amplitude $iT_{\pi N}$ with two off-shell pions (i.e. $k^2, k'^2 \neq m_\pi^2$) as

$$iT_{\pi N} = \left[ R^a(-k)R^b(k')|S\langle N(p')|\hat{S}|N(p)\rangle\right]_{\phi=0} = iT_S + iT_V + iT_{AA}, \quad (3.5)$$

where

$$iT_S = \frac{i}{2f_\pi} \left[ k^2 + k'^2 - m_\pi^2\right] \delta^{ab}\langle N(p')|\hat{\sigma}(0)|N(p)\rangle, \quad (3.6)$$

$$iT_V = \frac{1}{2f_\pi} \int d^4x e^{ikx} \langle N(p')|T^\ast(j^a_{A\mu}(x)j^b_{A\nu}(0))|N(p)\rangle, \quad (3.7)$$

and

$$iT_{AA} = -\frac{1}{f_\pi} \int d^4x e^{ikx} \langle N(p')|T^\ast(j^a_{A\mu}(x)j^b_{A\nu}(0))|N(p)\rangle, \quad (3.8)$$

modulo $2\pi^4\delta^{(4)}(k + p - k' - p')$. Now we redefine the operators as $\hat{O} \rightarrow \hat{O}'$, which follows from the change of the extended S-matrix $\hat{S} \rightarrow \hat{S}'$. Then, if $\hat{S}'$ satisfies the master equations, the $\chi$RF can be applied to it. The amplitude $iT'_{\pi N}$ obtained from $\hat{S}'$ takes the same form as $iT_{\pi N}$, except that the Green functions of $\hat{O}$ are replaced with those of $\hat{O}'$, and thus the overall momentum dependences of $iT_{\pi N}$ and $iT'_{\pi N}$ do not change. The change of the amplitude resulting from the redefinition of the pion field only appears in the Green functions of $\hat{O}$, which are not uniquely determined by the broken chiral symmetry. In addition, the difference between the Green functions of $\hat{O}$ and $\hat{O}'$ is only in some polynomials in the momenta,\(^1\) and this does not alter on-shell quantities that are determined by pole terms.\(^1\)

The interpolating pion fields defined consistently with the master equations can be formally expressed as

$$\hat{n}^a(x) = \hat{n}^a_{\text{PCAC}}(x) + F^a(\phi, \hat{O}), \quad (3.9)$$

where $\hat{n}^a_{\text{PCAC}}(x)$ is the so-called PCAC choice of the pion field $\hat{n}^a_{\text{PCAC}} \equiv \partial^\mu A^a_\mu/(f_\pi m_\pi^2)$, and $F^a(\phi, \hat{O})$ is some function of $\phi$ and $\hat{O}$ satisfying $F^a(\phi, \hat{O})|_{\phi=0} = 0$. The asymptotic one-pion component is entirely included in $\hat{n}^a_{\text{PCAC}}$, and therefore the second

\(^1\)This results from the fact that the phase $f[\phi]$ appearing in $\hat{S}' = \hat{S} \exp(if[\phi])$ consists only of some polynomials in $\phi$ and their derivatives.
term on the r.h.s., \( F^a(\phi, \hat{O}) \), can generate only an off-shell effect. Furthermore, \( F^a(\phi, \hat{O}) \) is the only quantity that depends on the choice of the extended S-matrix, \( F^a(\phi, \hat{O}) \rightarrow F^a(\phi, \hat{O}') \), while \( \hat{\pi}_PCAC^a \) is invariant under the redefinition of the pion field. These facts imply that \( F^a(\phi, \hat{O}) \) influences only the Green functions of \( \hat{O} \) in the Ward identity. The overall kinematical dependence and symmetry structure of the off-shell Ward identity is determined by the first term in Eq. (3.9), i.e. the PCAC pion field \( \hat{\pi}_PCAC^a \). Therefore, the class of interpolating pion fields defined consistently with the master equations and employed in the framework of Ref. 1) gives off-shell amplitudes satisfying Ward identities of the same form as those obtained using the PCAC pion field.

### §4. Extending the \( \chi RF \) off mass shell

The chiral Ward identity for the \( \pi N \) scattering has also been derived by Weinberg, using a different approach based on the chiral algebra satisfied by currents and density operators. In that derivation, the PCAC choice of the pion field, \( \hat{\pi}_PCAC^a \), was employed to relate the \( \pi N \) scattering amplitude to the Ward identity among the Green functions of the currents and density operators. If the prescription (2.14) can be applied to the off-shell pions, the \( \chi RF \) must reproduce Weinberg’s formula, even off the mass shell, because the two approaches use the same definition of the interpolating pion field. However, for \( iT_S \) including the pion-nucleon \( \sigma \)-term, Weinberg’s formula gives

\[
iT_S = -i\frac{k^2 + k'^2 - m^2}{2\pi}\delta^{ab}\langle N(p')|\hat{\sigma}(0)|N(p)\rangle,
\]

instead of Eq. (3.6). This result indicates that a naive use of the \( \chi RF \) (2.14) proposed in Ref. 1) does not correctly relate the off-shell amplitudes to the Ward identities among the Green functions of \( \hat{O} \); more specifically, it does not do this in a manner consistent with the master equations.

To make the \( \chi RF \) applicable to the off-shell pions, first we note that the off-shell extrapolation of scattering amplitudes is generally realized through the LSZ reduction formula. We therefore examine the derivation of \( \chi RF \) starting from the LSZ formalism.

We consider the commutation relations (2.12) on the basis of the LSZ formalism. From the asymptotic behavior of the interpolating pion field, we obtain

\[
[a_{in}^a(k), \hat{S}] = \int d^4x e^{ikx} (\Box + m_\pi^2) \frac{\delta}{\delta J^a(x)} \hat{S},
\]

\[
[\hat{S}, a_{in}^a(k)] = \int d^4x e^{-ikx} (\Box + m_\pi^2) \frac{\delta}{\delta J^a(x)} \hat{S},
\]

where we have used \( a_{out} = \hat{S}^\dagger a_{in} \hat{S} \) and Eq. (2.1). Note that these exact relations are obtained independently of the master equations. Once the commutation relations are rewritten in the form of Eqs. (4.2) and (4.3), the momentum \( k \) can be analytically continued off the mass shell. Using these relations iteratively, we obtain the
conventional LSZ reduction formula,
\[ \langle \alpha; k_1a_1, \ldots, k_ma_m|\hat{S}|\beta; l_1b_1, \ldots, l_nb_n|\phi=0 \]
\[ = \left( \prod_{i=1}^{m} G(k_i; x_i) \frac{\delta}{\delta J^a_i(x_i)} \right) \left( \prod_{j=1}^{n} G(-l_j; y_j) \frac{\delta}{\delta J^b_j(y_j)} \right) \langle \alpha|\hat{S}|\beta|\phi=0 \]
\[ = \left( \prod_{i=1}^{m} iG(k_i; x_i) \right) \left( \prod_{j=1}^{n} iG(-l_j; y_j) \right) \]
\[ \times \langle \alpha|\hat{S}T^a(\hat{\pi}a^1(x_1) \cdot \hat{\pi}a^m(x_m)\hat{\pi}b^1(y_1) \cdot \hat{\pi}b^n(y_n))|\beta \rangle, \quad (4.4) \]
where \( G(k; x) = \int d^4x \exp(ikx)(\Box + m^2) \) and \( \hat{S} = \hat{S}|\phi=0 \), and we have used Eq. (2.4) in the last step. This formula defines the off-shell extrapolation of the scattering amplitude.

We stress that owing to Eqs. (2.8) and (2.13), the identity
\[ \int d^4xe^{+ikx(\Box + m^2)} \frac{\delta}{\delta J^a(x)} = R^a(k) - T^a_A(k) \quad (4.5) \]
exists among functional derivative operators, where \( T^a_A(k) \) is the Fourier transformation of \( T^a_A(x) \). Thus, we immediately find the following replacements for the commutators:
\[ [a^a_m(k), ] \rightarrow R^a(k) - T^a_A(k), \quad (4.6) \]
\[ [ , a^a_m(k)] \rightarrow R^a(-k) - T^a_A(-k). \quad (4.7) \]
With these replacements, the scattering amplitude including the \( n \) incoming and \( m \) outgoing (on-shell or off-shell) pions becomes
\[ \langle \alpha; k_1a_1, \ldots, k_ma_m|\hat{S}|\beta; l_1b_1, \ldots, l_nb_n|\phi=0 \]
\[ = [(R - T_A)^a_1(k_1) \cdots (R - T_A)^a_m(k_m) \]
\[ \times (R - T_A)^b_1(-l_1) \cdots (R - T_A)^b_n(-l_n)]_{S}(\alpha|\hat{S}|\beta)|\phi=0 \]
\[ = [R^{a_1}(k_1) \cdots R^{a_m}(k_m)R^{b_1}(-l_1) \cdots R^{b_n}(-l_n)]_{S}(\alpha|\hat{S}|\beta)|\phi=0 \]
\[ + (\text{terms including } T_A). \quad (4.8) \]
This is just another expression of the LSZ reduction formula. Note here that we have not yet imposed any constraints from the broken chiral symmetry.

The \( \chi RF \) in our case is obtained as a combination of Eq. (4.8) and the master equation (2.6), which includes any effects from the broken chiral symmetry. To see our \( \chi RF \) concretely, we consider the case \( n + m = 2 \), i.e. that in which the sum of the incoming and outgoing pions is 2. In this case, Eq. (4.8) gives
\[ [(R - T_A)^a(R - T_A)^b)]_{S}(\alpha|\hat{S}|\beta)|\phi=0 \]
\[ = [R^aR^b]_{S}(\alpha|\hat{S}|\beta)|\phi=0 + \left( [T^a_AR^b_A]_{S} - [T^a_AR^b_A]_{S} - [T^a_AR^b_A]_{S} \right) (\alpha|\hat{S}|\beta)|\phi=0, \quad (4.9) \]
where we have suppressed the momentum indices. Note again that this is simply the LSZ reduction formula expressed in terms of \( R \) and \( T_A \). Next, we move all \( T_A \) to the
right of all $R$, using the relation $T_A R = [T_A, R] + R T_A$, and then we use the master equation (2.6). As a result of this operation, Eq. (4.9) becomes

\[
[(R - T_A)\alpha(R - T_A)\beta]_S \langle \alpha | \hat{S} | \beta \rangle |_{\phi = 0} = [R^a R^b]_S \langle \alpha | \hat{S} | \beta \rangle |_{\phi = 0} - \frac{1}{2} \left( [T_A^a, R^b] + [T_A^b, R^a] \right) \langle \alpha | \hat{S} | \beta \rangle |_{\phi = 0}.
\] (4.10)

This is the desired $\chi RF$ for the case $n + m = 2$. The above procedure is applied similarly for all values of $n$ and $m$. The $\chi RF$ obtained in this manner is manifestly consistent with the LSZ reduction formula, and there is no ambiguity in its derivation.

We note the presence of “(terms including $T_A$)” appearing in Eq. (4.8). In the following, we show that (i) such additional terms vanish if all pions are on shell, and therefore in that case our $\chi RF$ properly reduces to that proposed in Ref. 1), and that (ii) those terms are essential for deriving the correct identities for the off-shell amplitudes within the framework of Ref. 1).

The assertion (i) can be readily demonstrated by considering the explicit expression of the commutation relation between $T_A$ and $R$,

\[
[T_A^a(k), R^b(k')] = -\epsilon^{abc} \frac{1}{f_\pi^2} \int d^4xe^{i(k+k')x} T_V^c(x) - i\delta^{ab}(-k'^2 + m_\pi^2)(2\pi)^4\hat{\delta}(4)(k + k') - \delta^{ab}(-k'^2 + m_\pi^2) \frac{1}{f_\pi^2} \int d^4xe^{i(k+k')x} \delta \delta S(x). \] (4.11)

As found from Eqs. (2.6) and (4.11), “(terms including $T_A$)” in Eq. (4.8) is always proportional to $-k^2 + m_\pi^2$, where $k$ is the four momentum of some external pion. [By virtue of the symmetric permutation (2.15), the contribution from the first term in Eq. (4.11) involving the antisymmetric tensor $\epsilon^{abc}$ vanishes.] As a result, our $\chi RF$, i.e. Eq. (4.8) together with Eq. (2.6), reduces to the $\chi RF$ proposed in Ref. 1) if all pions are on shell.

To verify the assertion (ii), we return to the investigation of the $\pi N$ scattering. With the one-nucleon states for $\langle \alpha |$ and $| \beta \rangle$, using Eqs. (2.1) and (4.11), it can be shown that Eq. (4.10) becomes

\[
[(R - T_A)\alpha(-k)(R - T_A)\beta(k')]_S \langle N(p') | \hat{S} | N(p) \rangle |_{\phi = 0} = [R^a(-k)R^b(k')]_S \langle N(p') | \hat{S} | N(p) \rangle |_{\phi = 0} - i\delta^{ab} \frac{1}{f_\pi} \left( \frac{k'^2}{2} - m_\pi^2 \right) \langle N(p') | \hat{\sigma}(0) | N(p) \rangle,
\] (4.12)

modulo $(2\pi)^4\hat{\delta}(4)(k + p - k' - p')$, where we drop the non-scattering parts arising from the second term on the r.h.s. of Eq. (4.11). Now we obtain the additional contribution to Eq. (3.5) arising from “(terms including $T_A$)” . Our result is identical with Weinberg’s formula for the $\pi N$ scattering, even off the mass shell, as should be the case. We can thus conclude that our $\chi RF$, being manifestly consistent with the LSZ reduction formula, is a proper off-shell extension of the on-shell $\chi RF$ presented in Ref. 1).
Before closing this section, several comments are in order:

- A naive application of the replacement of the single commutators expressed in Eq. (2.16) to multi-commutator cases results in the nontrivial replacement of the functional derivative operator \( \int d^4x \exp(ikx)(\square + m^2)\langle \delta/\delta J^a \rangle \rightarrow R^a(k) \).
  Such replacement is consistent with the LSZ reduction formula for on-shell amplitudes, but it is incompatible off the mass shell, because \( R \) and \( T_A \) do not commute, and thus “(terms including \( T_A \))” does not vanish. This is the reason that the \( \chi \)RF proposed in Ref. 1) cannot be naively applied to off-shell pions.

- Practically, the \( \chi \)RF formulated in Ref. 1) gives the same off-shell result as ours for single pion emission and absorption processes [see Eqs. (6.4) and (6.5) in Ref. 1)]. According to our results, however, it seems to be a rather special case, because in this case, “(terms including \( T_A \))” survives in neither the on-shell nor off-shell case:
  \[ \langle \alpha|a^{a}_{in}(k), \hat{S}||\beta \rangle = (R - T_A)^a(k)\langle \alpha|\hat{S}||\beta \rangle = R^a(k)\langle \alpha|\hat{S}||\beta \rangle. \]

- In principle, the \( \chi \)RF presented in this paper is applicable for any value of (on-shell or off-shell) pion momenta. This is due to the fact that the master equations are exact formulas essentially equivalent to the current conservation law or Noether’s theorem. The nature of specific points, such as the chiral limit and the soft pion limit, is relevant only if extra assumptions are introduced, for example, the possibility of a loop expansion around those points.\(^1\)

\section{Summary}

We have investigated the off-shell structure of the \( \chi \)RF because of its importance in actual calculations and in the pursuit of a deeper understanding of the theoretical framework developed in Ref. 1). We have seen that the on-shell \( \chi \)RF proposed in Ref. 1) cannot be naively applied to off-shell pions. We then found its proper extension off the mass shell. This was achieved by reconstructing the \( \chi \)RF in a form manifestly consistent with the conventional LSZ reduction formula.

The analysis employed in this work can be readily applied to the three flavor case studied in Refs. 5) and 9).

\section{Acknowledgements}

The author would like to thank Dr. Masaki Arima for discussions and his helpful advice in completing the manuscript. The author would also like to thank Dr. Toru Sato for valuable comments. This work was supported by Research Fellowships of the Japan Society for the Promotion of Science (JSPS) for Young Scientists.

\section{References}