Note on Massless Bosonic States in Two-Dimensional Field Theories

Hidenori Fukaya,1,* Masashi Hayakawa,2,** Issaku Kanamori,1,*** Hiroshi Suzuki1,† and Tomohisa Takimi1,3,††

1 Theoretical Physics Laboratory, RIKEN, Wako 351-0198, Japan
2 Department of Physics, Nagoya University, Nagoya 464-8602, Japan
3 Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

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In a wide class of $G_L \times G_R$ invariant two-dimensional super-renormalizable field theories, the parity-odd part of the two-point function of global currents is completely determined by a fermion one-loop diagram. For any non-trivial fermion content, the two-point function possesses a massless pole which corresponds to massless bosonic physical states. As an application, we show that two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theory without a superpotential possesses $U(1)_L \times U(1)_R$ symmetry and contains one massless bosonic state per fixed spatial momentum. The $\mathcal{N} = (4, 4)$ supersymmetric pure Yang-Mills theory possesses $SU(2)_L \times SU(2)_R$ symmetry, and there exist at least three massless bosonic states.

§1. Introduction

In this paper, we show that in a wide class of $G_L \times G_R$ invariant two-dimensional super-renormalizable field theories, the parity-odd part of the two-point function of global currents can be determined to all orders in perturbation theory. The two-point function possesses a massless pole for any non-trivial fermion content, and this fact provides a simple criterion to determine the existence of massless bosonic physical states without solving the dynamics. Our argument is based on anomalous Ward-Takahashi (WT) identities and is somewhat similar to that for the ’t Hooft anomaly matching condition.1–3) In fact, applying the anomaly matching argument to the systems we consider (assuming that the anomalous behavior of the two-point function does not receive higher-order radiative corrections), one would arrive at a similar conclusion concerning massless states. (See, for example, Refs. 4)–6).) The point of this paper is, however, to show that in the two-dimensional systems we consider, an elementary argument suffices to obtain an explicit form of the two-point function to all orders in perturbation theory. In particular, our argument is applicable even to systems in which the left and right moving modes are not

* E-mail: hfukaya@riken.jp
** E-mail: hayakawa@eken.phys.nagoya-u.ac.jp
*** E-mail: kanamori-i@riken.jp
† E-mail: hsuzuki@riken.jp
†† E-mail: ttakimi@riken.jp
decoupled due to the Yukawa interaction and the scalar potential.\textsuperscript{a}) This point is crucial for application to two-dimensional extended supersymmetric gauge theory, which is our main concern. As an illustration, we show that all solvable super-renormalizable theories in which the existence of massless bosonic states is known are covered by the argument. (For earlier studies on these two-dimensional field theories, see Ref. 7.) We then consider two-dimensional supersymmetric gauge theories and observe that the $\mathcal{N} = (2, 2)$ supersymmetric gauge theory without a superpotential and the $\mathcal{N} = (4, 4)$ supersymmetric pure Yang-Mills theory contain massless bosonic states. We can also determine the explicit form of the two-point function of global currents in these theories. Combined with supersymmetric WT identities, these findings should be useful in examining recently developed lattice formulations of these two-dimensional supersymmetric gauge theories.\textsuperscript{8)–20)}

§2. All-order proof of the massless bosonic state

We consider two-dimensional field theories of the form\textsuperscript{2)}

$$
\mathcal{L} = -\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} + \bar{\psi} i \gamma^\mu D_\mu \psi + \frac{1}{2} D_\mu \sigma D^\mu \sigma + \frac{1}{2} D_\mu \pi D^\mu \pi - V(\sigma, \pi) - Y(\psi, \bar{\psi}, \sigma, \pi),
$$

(2.1)

where $F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + g f_{ABC} A^B_\mu A^C_\nu$ are the field strengths of the gauge fields $A^A_\mu$ (with $f_{ABC}$ are the structure constants of the gauge group), and $D_\mu = \partial_\mu - ig A^A_\mu t^A$ is the gauge covariant derivative. The representation $t^A$ may differ for each field, and the fermion belongs to the representation $r$ of the gauge group. The potential energy, $V$, and the Yukawa interaction, $Y$, contain scalar fields $\sigma$ and $\pi$ (but not their derivatives), and $Y$ is bi-linear in the fermion fields. We assume that the system possesses parity invariance with the assignment that $\sigma$ is a scalar and $\pi$ is a pseudo-scalar. Power counting in two dimensions, we find that all coupling constants contained in $D_\mu$, $V$ and $Y$ are dimensional and thus the system is super-renormalizable.

We assume that the lagrangian density $\mathcal{L}$ in Eq. (2.1) possesses $G_L \times G_R$ global flavor symmetry. That is, $\mathcal{L}$ is invariant under the transformations

$$
\psi_L(x) \rightarrow \exp\{i \theta^a_L T^a\} \psi_L(x), \quad \psi_R(x) \rightarrow \psi_R(x),
$$

(2.2)

and

$$
\psi_R(x) \rightarrow \exp\{i \theta^a_R T^a\} \psi_R(x), \quad \psi_L(x) \rightarrow \psi_L(x),
$$

(2.3)

where $\theta^a_L$ and $\theta^a_R$ are independent global parameters, and $T^a$ are hermitian generators of a certain compact Lie group $G$ in the representation $R$, if supplemented with suitable transformations of the scalar fields $\sigma$ and $\pi$. In the following discussion,

\textsuperscript{a}) For such a system, application of the standard technique of conformal field theory is not straightforward.

\textsuperscript{2)} Throughout the paper, we use the following notational conventions. The Greek indices $\mu$, $\nu$, $\ldots$ run from 0 to 1. The flat metric is of lorentzian signature, $g_{\mu\nu} = \text{diag}(1, -1)$. We have $\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$, $\bar{\psi} = \psi^\dagger \gamma_0$, $\gamma_5 = \gamma^0 \gamma^1$ and $\mathcal{P}_\pm = (1 \pm \gamma_5)/2$. The chiralities are defined by $\psi_{R,L} = \mathcal{P}_\pm \psi$ and $\bar{\psi}_{R,L} = \bar{\psi} \mathcal{P}_\mp$. The anti-symmetric tensor $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ is defined by $\epsilon^{01} = +1$. 
it is very important that the generators $T^a$ in Eqs. (2.2) and (2.3) are the same. More precisely, they are in the same representation $\hat{R}$ of the group $G$, and we use the terminology $G_L \times G_R$ in this restricted sense. This property depends on the structure of the Yukawa interaction $Y$ and the scalar potential $V$ whose explicit forms we do not specify.

For the quantization of the system (2.1), we need gauge fixing and introduction of Faddeev-Popov (FP) ghosts. Although we do not explicitly write down these additional parts, we assume that the gauge fixing condition respects the Lorentz invariance and global symmetries of $\mathcal{L}$.

Now, corresponding to the global symmetry $G_L \times G_R$ of Eqs. (2.2) and (2.3), there exist the Noether currents $J^a_{L,\mu}(x)$ and $J^a_{R,\mu}(x)$, respectively. We assume that the above symmetry is strictly global, i.e., that there are no gauge fields which couple to the Noether currents. We also assume that the current conservation of these Noether currents does not suffer from the anomaly in the usual sense. Specifically, we assume that the divergence of the diagrams in Fig. 1 identically vanishes.

For the following treatment, it is convenient to define linear combinations of $J^a_{L,R,\mu}$ as

$$J^a_\mu(x) = J^a_{R,\mu} + J^a_{L,\mu}, \quad J^a_{5\mu}(x) = J^a_{R,\mu} - J^a_{L,\mu},$$

(2.4)

taking the forms

$$J^a_\mu(x) = \bar{\psi}\gamma_\mu T^a \psi(x) + \cdots, \quad J^a_{5\mu}(x) = \bar{\psi}\gamma_\mu \gamma_5 T^a \psi(x) + \cdots,$$

(2.5)

where omitted terms are possible contributions from scalar fields.\(^*\) We now consider the two-point function of the global currents $\langle 0| T^* J^a_{5\mu}(x) J^b_{\nu}(y) |0 \rangle$. We show below

\(^*\) If there is no Yukawa interaction (i.e., $Y = 0$) as is the case in two-dimensional massless QCD, there are no omitted terms in Eq. (2.5), and one may require the condition $\epsilon_{\mu\nu} J^{5\nu} = -J^a_{5\mu}$ on current operators, as suggested by the identity $\epsilon_{\mu\nu} \gamma^\nu = -\gamma_\mu \gamma_5$. Then $J^a_{L} + J^a_{R} = J^a_{R} - J^a_{L} = 0$, and the current conservation condition $\partial_\mu J^a_{\mu L,R} = 0$ implies that the combinations $J^{a-} \equiv J^a_{L} - J^a_{R}$ and $J^{a+} \equiv J^a_{L} + J^a_{R}$ are chiral in the sense that $(\partial_0 - \partial_1) J^{a-} = 0$ and $(\partial_0 + \partial_1) J^{a+} = 0$ (i.e., $J^{a-}$ and $J^{a+}$ are left- and right-moving, respectively). The current algebra is thus decomposed into
that, under the assumptions placed on the two-dimensional model (2.1), this two-point function receives a contribution only from a fermion one-loop diagram and can be determined to all orders in perturbation theory.

The most general form of the two-point function that is consistent with Lorentz covariance and parity invariance is given by

\[
FT \langle 0 | T^* J_5^a(x) J_5^b(y) | 0 \rangle = -\frac{i}{2\pi} \left\{ \frac{1}{p^2} F^{ab}(p^2) (p_\mu \epsilon_{\nu\rho} p^\rho + p_\nu \epsilon_{\mu\rho} p^\rho) + G^{ab}(p^2) \epsilon_{\mu\nu} \right\},
\]

(2.6)

where “FT” denotes the Fourier transformation with \( \int d^2 x e^{ip(x-y)} \). The divergence of the two-point function yields

\[
FT \langle 0 | T^* \partial_\mu J_5^a(x) J_5^b(y) | 0 \rangle = -\frac{1}{2\pi} \left\{ F^{ab}(p^2) - G^{ab}(p^2) \right\} \epsilon_{\nu\rho} p^\rho,
\]

\[
FT \langle 0 | T^* J_5^a(x) \partial_\nu J_5^b(y) | 0 \rangle = \frac{1}{2\pi} \left\{ F^{ab}(p^2) + G^{ab}(p^2) \right\} \epsilon_{\mu\rho} p^\rho.
\]

(2.7)

Next we note that naive WT identities based on the \( G_L \times G_R \) invariance would imply that the quantities in Eq. (2.7) are equal to \( i f_{abc} \langle 0 | J_5^c(0) | 0 \rangle \) and \( -i f_{abc} \langle 0 | J_5^c(0) | 0 \rangle \), respectively (where \( f_{abc} \) are the structure constants of the group \( G \)), which vanish due to Lorentz invariance of the vacuum. This implies that \( F^{ab} = G^{ab} = 0 \) and Eq. (2.6) identically vanishes. Thus, the two-point function (2.6) can be non-zero only as a result of an anomalous breaking of the WT identities. This “anomaly” (which is exactly the central extension of the current algebra in two dimensions) can arise only from potentially UV divergent diagrams, because the naive WT identities safely apply to UV convergent diagrams. The point here is that under the above assumptions placed on the model (2.1), it turns out that only a fermion one-loop diagram contributes to Eq. (2.7). Thus, to all orders in perturbation theory, the coefficients \( F^{ab} \) and \( G^{ab} \) in Eq. (2.6) can be completely determined.

Since our model (2.1) is super-renormalizable, among all Feynman diagrams that contain two global currents, only a few are UV divergent. Let us enumerate such UV divergent diagrams. The first such class comprises diagrams such as those shown in Fig. 3; they diverge at a loop which contains only a single current. This class, however, does not contribute to Eq. (2.7), because of our assumption of current conservation for the Feynman diagrams appearing in Fig. 1. Therefore the diagrams that contribute to the non-conservation of current described by Eq. (2.7) are those in which the UV divergence arises from a loop which contains both currents \( J_5^a \) and \( J_5^b \).

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\( \star \) Note the identity \( p_\mu \epsilon_{\nu\rho} p^\rho - p_\nu \epsilon_{\mu\rho} p^\rho = -p^2 \epsilon_{\mu\nu} \).
$J^b_{\mu}$. We note, however, that $J^a_{5\mu}$ contains an odd number of $\pi$ fields and $J^b_{\nu}$ contains an even number of $\pi$ fields, due to the parity invariance. Therefore, there is no scalar one-loop diagram that contains both currents, because the scalar propagator connects only $\sigma-\sigma$ and $\pi-\pi$ pairs. In this way, we conclude that the unique diagram which contributes to Eq. (2.7) is the fermion one-loop diagram of Fig. 4.

Applying a standard argument\(^{21}\) to the diagram in Fig. 4, we have

$$
\text{FT}\langle 0|T^* \bar{\psi}\gamma^\mu \gamma^5 T^a \psi(x) T^* \bar{\psi}\gamma^\nu T^b \psi(y)|0\rangle = -\frac{i}{2\pi} (\text{dim } r) T(R) \delta_{ab} \left\{ \frac{1}{p^2} (p_\mu \epsilon_{\nu \rho} p^\rho + p_\nu \epsilon_{\mu \rho} p^\rho) + L(p^2) \epsilon_{\mu \nu} \right\}, \tag{2.8}
$$

where $\text{dim } r$ is the dimension of the gauge-group representation $r$, and $T(R)$ is the second Casimir of the “flavor-group” representation $R$, $\text{tr}\{T^a T^b\} = T(R) \delta_{ab}$, which is positive-definite [i.e., $T(R) > 0$] for any non-trivial representation $R$ of the compact group $G$. In deriving this expression, we have assumed that the regularization respects covariance under Lorentz and global $G$ transformations. The scalar function $L(p^2)$ depends on the regularization applied to the diagram. For example, the Pauli-Villars regularization respects the covariance and gives $L(p^2) = -1$. As we see below, however, the existence of a massless pole is independent of the precise form of $L(p^2)$, and thus of the regularization. The current divergence is therefore given by

$$
\text{FT}\langle 0|T^* \partial_\mu \{ \bar{\psi}\gamma^\mu \gamma^5 T^a \psi(x) \} \bar{\psi}\gamma^\nu T^b \psi(y)|0\rangle = -\frac{1}{2\pi} (\text{dim } r) T(R) \delta_{ab} \{1 - L(p^2)\} \epsilon_{\nu \rho} p^\rho,
$$

$$
\text{FT}\langle 0|T^* \bar{\psi}\gamma^\mu \gamma^5 T^a \psi(x) \partial_\nu \{ \bar{\psi}\gamma^\nu T^b \psi(y)\}|0\rangle = \frac{1}{2\pi} (\text{dim } r) T(R) \delta_{ab} \{1 + L(p^2)\} \epsilon_{\mu \rho} p^\rho. \tag{2.9}
$$

We thus conclude that Eq. (2.7) is, to all orders in the perturbation theory, given by the expressions in Eq. (2.9). Explicitly, we have $F^{ab}(p^2) = (\text{dim } r) T(R) \delta_{ab}$ and $G^{ab}(p^2) = (\text{dim } r) T(R) \delta_{ab} L(p^2)$, and hence we have

$$
\text{FT}\langle 0|T^* J^a_{5\mu}(x) J^b_{\nu}(y)|0\rangle = -\frac{i}{2\pi} (\text{dim } r) T(R) \delta_{ab} \left\{ \frac{1}{p^2} (p_\mu \epsilon_{\nu \rho} p^\rho + p_\nu \epsilon_{\mu \rho} p^\rho) + L(p^2) \epsilon_{\mu \nu} \right\} \tag{2.10}
$$

as the expression which holds to all orders in perturbation theory. It is surprising that the two-point function is completely determined by the one-loop diagram and receives no higher-order radiative corrections; this is a consequence of the (anomalous) WT identities. In other words, we showed that the level of the Kac-Moody algebra does not receive higher-order corrections in the present super-renormalizable theories.
One can also explicitly confirm that there are no radiative corrections to the two-point function, for example at the two-loop order.

From the expression (2.10), we observe that there exists a massless pole which corresponds to massless bosonic states (since $J^{a\mu}_5$ and $J^{b\nu}_5$ are bosonic operators).\textsuperscript{a)} This pole cannot be eliminated by the regularization ambiguity $L(p^2)$, because the first term in Eq. (2.10) is symmetric under the exchange $\mu \leftrightarrow \nu$, while the second is anti-symmetric. Recall also that $T(R) > 0$ for any (non-trivial) representation $R$. Thus, the residue is always non-zero for any (non-trivial) fermion content.

We note that the existence of massless bosonic states does not contradict Coleman’s theorem,\textsuperscript{22)} which rules out the spontaneous breaking of bosonic symmetry in two dimensions. The spontaneous symmetry breaking implies that a massless bosonic state, the Nambu-Goldstone boson, appears in the intermediate state in a channel between a conserved current and a scalar field. It can be shown that,\textsuperscript{22)} in two dimensions, such a massless intermediate state is inconsistent with the normalizability and positivity of physical states.\textsuperscript{**}

This argument, however, does not place any restriction on possible massless states appearing in a channel between two conserved currents, such as Eq. (2.10). In fact, the massless pole in Eq. (2.10) has nothing to do with the spontaneous symmetry breaking. Rather, it corresponds to (as we have already noted) the central extension of the current algebra. These points can be explicitly seen by applying the Bjorken-Johnson-Low (BJL) prescription\textsuperscript{23)},\textsuperscript{24)} to Eq. (2.10). Assuming that the function $L(p^2)$ is a polynomial in $p_0$, we have (see Appendix A)

$$\langle 0| [J^{a0}_5(x), J^{b\nu}_5(y)]|0\rangle \delta(x^0 - y^0) = -\frac{i}{\pi} (\dim r) T(R) \delta_{ab} \delta^0 \delta(x^0 - y^0) \frac{d}{dx^1} \delta(x^1 - y^1).$$

Integrated both sides over the spatial coordinate $x^1$, we see that the vacuum expectation value of the commutator between the axial charge $Q^a_5 = \int dx^1 J^{a0}_5(x)$ and the vector current $J^{b\nu}_5(y)$ vanishes. This clearly demonstrates that the existence of the massless pole does not imply the spontaneous symmetry breaking, and also that the $G$-algebra, $[Q^a_5, Q^b_5] = i f_{abc} Q^c_5$, where $Q^b_5 = \int dx^1 J^{b0}(x)$ is the vector charge, does not suffer from the ($c$-number) anomaly. It is interesting that in the present two-dimensional models, the situation is the opposite of that described in the Nambu-Goldstone theorem: The $G_L \times G_R$ symmetry remains exact, and there arise massless bosonic states.

\textsuperscript{a)} Here we assume that the vacuum is a bosonic state.

\textsuperscript{**} An example demonstrating this statement is provided by the free massless scalar field $\phi(x)$ in two dimensions. In this system, the shift symmetry $\phi(x) \rightarrow \phi(x) + \epsilon$ of the action is always spontaneously broken, because the vacuum expectation value of $i[Q, \phi(x)] = 1$, where the charge $Q = \int dx^1 j^{0}(x)$ is defined in terms of the conserved current $j_{\mu}(x) = \partial_{\mu} \phi(x)$, is unity. Correspondingly, in the intermediate state between $j_{\mu}$ and $\phi$, there appears the massless Nambu-Goldstone boson that is simply $\phi$ itself. However, the state created by $\phi(x)$ from the vacuum is not normalizable, because $\delta(p^2)$ has no well-defined Fourier transform in two dimensions (due to the infrared divergence). Spontaneous symmetry breaking does occur in this theory, but this theory itself is ill-defined.
Equation (2-10) implies that there exists a state $|X\rangle$ such that
\[ (X|J^a_{\mu}(0)|0) \neq 0. \] (2-14)

Assuming covariance under global $G$ transformations, we see that the massless state belongs to a non-trivial multiplet of $G$ [unless $G = U(1)$] and that the right-hand side of the above equation can be written in terms of invariant tensors of $G$. The simplest possibility would be that $|X^b\rangle$ transforms as a conjugate of the current $J^b_{\mu}$ under $G$ transformations, and the right-hand side is the invariant tensor $\delta_{ab}$. Thus the minimum number of massless bosonic states per fixed spatial momentum $p_1$ is given by the index $a$, the dimension of the group $G$.

Moreover, the bosonic state $|X\rangle$ is physical in the sense that it can be chosen to contain no unphysical modes, such as FP ghosts and longitudinal modes of gauge fields. This can be seen most clearly by using the form of the completeness relation in the present gauge system, $1 = P^{(0)} + \{Q_B, R\}$. Here $P^{(0)}$ is the projection operator to the Hilbert space $\mathcal{H}_{\text{phys}}$ that does not contain any unphysical modes and $Q_B$ is the BRST charge.$^{25}$ The second term in the completeness relation, where $R$ is a certain operator, is the projection operator to states which contain at least one unphysical mode. Since the global currents $J^a_{\partial\mu}$ and $J^b_{\nu}$ are gauge invariant and commute with the BRST charge, the second term in the completeness relation does not contribute when inserted into the two-point function (2-10). This shows that the state $|X\rangle$ is an element of $\mathcal{H}_{\text{phys}}^{**}$

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$^{*1}$ More specifically, when $Y = 0$, Eq. (2-10) shows that there exists a massless asymptotic field $\tilde{J}^a_{\mu}(x)$ that can be expanded in the creation and annihilation operators as
\[ J^a_{\mu}(x) \rightarrow Z^{1/2} \tilde{J}^a_{\mu}(x) = Z^{1/2} \int \frac{dk_1}{\sqrt{2\pi k_0}} \left\{-ik_\mu a^a(k_1)e^{-ikx} + ik_\mu a^a(k_1)e^{ikx} \right\} \quad \text{for } x^0 \rightarrow +\infty, \] (2-12)

where $k_0 = |k_1|$ and $Z = (1/\pi)(\text{dim}r)T(R)$. When acting on the vacuum $|0\rangle$, the asymptotic field $\tilde{J}^a_{\mu}(x)$ and the creation operator $a^a(k_1)$ define normalizable states. Note that this is in sharp contrast to the case of a massless asymptotic scalar field $\tilde{\phi}(x) = \int \frac{dk_1}{\sqrt{2\pi k_0}} \{a(k_1)e^{-ikx} + a^\dagger(k_1)e^{ikx}\}$, for which the state $\tilde{\phi}(x)|0\rangle$ is not normalizable due to the infrared divergence caused by the factor $\sqrt{k_0}$ in the denominator. In the case considered presently, therefore, the LSZ reduction formula provides well-defined transition amplitudes when applied to multi-point functions of current operators. For example, from a fermion one-loop diagram which contains three vector currents, we have
\[ \int d^2x e^{ipx} \int d^2y e^{iqy} \langle k, c|T^a J^a_{\mu}(x) J^b_{\nu}(y)|0\rangle \]
\[ = \frac{-2\pi i}{\sqrt{2\pi k_0}} Z^{-1/2}(\text{dim}r)T(R) f^{abc} \delta^2(p + q + k) \]
\[ \times \frac{1}{p^2 - q^2} \left\{ (p_\mu - q_\mu) k_\nu + (p_\nu - q_\nu) k_\mu + g_{\mu\nu}(p^2 - q^2) + 2k_\mu k_\nu \frac{p^2 + q^2}{p^2 - q^2} \right\} \ln \left\{ \frac{p^2}{q^2} \right\} - 4k_\mu k_\nu, \] (2-13)

where $\langle k, c \rangle \equiv (0|a^\dagger(k_1)|$. This expression of a well-defined transition amplitude, which holds at zeroth order in gauge and Yukawa couplings, illustrates that these massless asymptotic states pose no problem in defining the S matrix through the LSZ formula.

$^{**}$ Since there is ambiguity in adding BRST exact states to $|X\rangle$, it is more precise to say that $|X\rangle$ can be chosen as an element of $\mathcal{H}_{\text{phys}}$. 
We must be careful, however, in the interpretation of the massless bosonic states. We have intentionally used the term “state” instead of “particle”, because in two dimensions, even multi-particle intermediate states can produce a massless pole. This point becomes clear if we consider a system of free massless fermions (see the next section). It is thus not obvious whether the massless pole we observed corresponds to a single massless boson or is due to (say) two massless fermions. In two dimensions, it is not clear which interpretation is appropriate, as the bosonization of fermions is possible. Nevertheless, Eq. (2.10) provides non-trivial information regarding the low-energy spectrum of the model. In particular, if we assume the validity of Eq. (2.10) in a non-perturbative level (like the anomaly matching condition), it constrains the possible patterns of low-energy spectra with possible assignment of the $G$-representation.

As a final remark, we explain why $G_L \times G_R$ symmetry should not be gauged for our argument to be applied. In other words, our argument is not applied to the two-point function of gauge currents. The reason is that we use the fact that the $G_L \times G_R$ symmetry is chiral (i.e., it involves $\gamma_5$). Gauging $G_L \times G_R$ means that we introduce two distinct gauge fields, one coupled to $J^\mu_{L,a}$ and the other coupled to $J^\mu_{R,a}$; the resulting system is a chiral gauge theory. In two dimensions, however, such a chiral gauge theory is always anomalous, and it would be meaningless to consider such a theory from the outset. The best thing we can do is to gauge the vector subgroup $G_V$ of $G_L \times G_R$. In such a system, however, there are an infinite number of potentially UV diverging diagrams (chains of fermion one-loop diagrams connected by gauge propagators) which contribute to the two-point function of currents. This infinite set of diagrams can produce a massive pole, as is well known for the two-point function of currents in the single-flavor Schwinger model. It is thus seen that the assumption that $G_L \times G_R$ is not gauged is important for our argument.

In the next section, we first illustrate the power of our argument by considering exactly solvable models. Then we consider its application to non-solvable models, two-dimensional supersymmetric gauge theories.

§3. Applications

3.1. Solvable models

The simplest example to which the above argument can be applied is a system of $n$ free massless Dirac fermions. The global symmetry in this system is $U(n)_L \times U(n)_R$, and hence we expect the existence of at least $n^2$ massless bosonic states (per fixed spatial momentum). As is well known, this system can equivalently be expressed by $n$ free massless real scalar fields—the so-called abelian bosonization.\textsuperscript{26-28} The existence of massless bosonic states itself is thus manifest in this picture of bosonized theory.

In the abelian bosonization, among the global currents $J^\mu_{L,a}$ and $J^\mu_{R,a}$, those associated with the Cartan sub-algebra of $U(n)$ are simply given by derivatives of the scalar fields, and the appearance of a massless pole in the corresponding two-point functions is obvious. Other currents, associated with off-diagonal generators of
SU(n), are non-local and non-polynomial functions of the scalar fields, and therefore the appearance of a massless pole is less obvious. This asymmetric treatment of currents is necessary for the abelian bosonization, and a symmetric treatment is possible through non-abelian bosonization,\(^{29}\) which uses \(n^2\) real scalar fields. In any case, the two-point functions are given by Eq. (2.10), which indicates (at least) \(n^2\) massless states.

This example of a free fermion illustrates subtlety in the interpretation of the massless pole in Eq. (2.10) in terms of particles. In the original theory of fermions, which corresponds precisely to the diagram appearing in Fig. 4, a pair of massless fermions produces the massless pole. In the bosonized theory, the pole is produced by the exchange of massless bosons. These pictures are equivalent.

The first non-trivial example is two-dimensional QED with \(n\) massless electrons (where \(n > 1\), i.e., the massless Schwinger model\(^{30,31}\) with many flavors.\(^{32}\) In this case, the global symmetry that is not gauged and does not suffer from the anomaly is SU\((n)_L \times SU(n)_R\), and thus we expect at least \(n^2 - 1\) massless bosonic states. As emphasized in Ref. 33), the global currents \(J^a_{\mu,L}\) and \(J^a_{\mu,R}\), associated with the SU\((n)_L \times SU(n)_R\) symmetry can be expressed in terms of \(n - 1\) free massless real scalar fields. The existence of massless bosonic states itself is manifest in this bosonized theory, and the expression (2.10), which indicates (at least) \(n^2 - 1\) massless states, is reproduced. It is interesting that the single-flavor Schwinger model\(^{30,31}\) has no (physical) massless bosonic states. The would-be global symmetry \(U(1)_V \times U(1)_A\) in this model either is gauged or suffers from the anomaly; hence our argument does not apply.

A somewhat different type of solvable non-gauge model is the Kogut-Sinclair model,\(^{34,35}\) in which the Yukawa interaction takes the form

\[
Y = \overline{\psi} e^{i\gamma_5 \pi} \psi, \tag{3.1}
\]

and the global \(U(1)_L \times U(1)_R\) symmetry is realized by Eq. (2.2) with \(\pi \rightarrow \pi + \theta_L\) and Eq. (2.3) with \(\pi \rightarrow \pi - \theta_R\). (There is no scalar potential \(V\) in this model.) The abbreviated term of \(J^a_{5\mu}\) in Eq. (2.5) is linear in the pseudo-scalar \(\pi\). Again, according to the previous argument, we conjecture the existence of one massless bosonic state. Indeed, here, this is the case, as seen from the exact solution.\(^{34,35}\) What is more interesting about this model is that one can observe saturation of the residue of the massless pole by a massless boson and the absence of fermions with a non-zero \(U(1)_A\) charge.\(^{34,35}\) This picture provides a very non-trivial realization of Eq. (2.10), because the elementary fermion in the original lagrangian has a non-zero \(U(1)_A\) charge.

The two-dimensional \(SU(N)\) gauge theory with \(n\) massless fermions (i.e., two-dimensional massless QCD) is not exactly solvable. Nevertheless, it can be shown that this model contains \(n^2\) massless bosonic states\(^{4,36}\) for finite \(N\), as well as for \(N \rightarrow \infty.\(^{37}\)\) (See also Refs. 38) and 39). This is in accord with the above argument, because the global symmetry of this model is \(U(n)_L \times U(n)_R\). We note that our counting of the minimal number of massless bosonic states is independent of the gauge-group representation of fermions.
3.2. **Two-dimensional supersymmetric gauge theories**

A particularly interesting application is provided by supersymmetric gauge theories in two dimensions, which are not exactly solvable. We are thus interested in their spectra, especially at low energies. As the parity invariant two-dimensional supersymmetric gauge theories, we have $\mathcal{N} = (1,1)$, $(2,2)$, $(4,4)$ and $(8,8)$ models. In the $\mathcal{N} = (1,1)$ supersymmetric gauge theory, the gaugino is a single Majorana fermion, and consequently the action generically does not possess any continuous symmetry of the $G_L \times G_R$ type. Here we do not attempt to carry out a general classification of the $\mathcal{N} = (1,1)$ models. Rather, as a special case, we first consider the $\mathcal{N} = (2,2)$ supersymmetric $SU(N)$ pure Yang-Mills (YM) theory.

In the $\mathcal{N} = (2,2)$ $SU(N)$ pure YM theory, the Yukawa interaction and the scalar potential take the forms

$$
Y = g \text{tr} \left\{ \overline{\psi} [\sigma + i\gamma_5 \pi, \psi] \right\}, \quad V = -g^2 \text{tr} \left\{ [\sigma, \pi]^2 \right\},
$$

where all fields belong to the adjoint representation of the gauge group $SU(N)$, and the commutator and the trace are taken with respect to this gauge-group representation. From these, we see that the model possesses $U(1)_L \times U(1)_R$ symmetry and that, applying the above argument, there must exist at least one massless bosonic state (per fixed spatial momentum). These conclusions are valid even in the case that there exist matter multiplets, as long as there is no superpotential. The two-dimensional $\mathcal{N} = (2,2)$ supersymmetric gauge theory can be obtained from the four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory by dimensional reduction. The rotational $SO(2)$ invariance in the reduced dimensions becomes $U(1)_A$ in two dimensions, and thus the two-dimensional $\mathcal{N} = (2,2)$ models always possess the $U(1)_A$ symmetry with $U(1)_A$ charges (i.e., the representation) fixed by the underlying rotational invariance. The $U(1)_A$ charges come to be unity for all fermions. Then, the $U(1)_V$ charges for each fermion are fixed (to unity), because they must be identical to the $U(1)_A$ charges for our argument to apply. It turns out that any superpotential is inconsistent with this assignment of the $U(1)_V$ charges.

If the gauge group contains the $U(1)$ factor, like the $\mathcal{N} = (2,2)$ supersymmetric massless QED, we need “anomaly cancellation” for the current conservation of global currents $J_{L,\mu}$ and $J_{R,\mu}$, as explained in Fig. 1. This requires that, for each $U(1)$ factor, the sum of the $U(1)$ charges $Q_i$ of the fermion over all matter multiplets vanishes, i.e., $\sum_i Q_i = 0$. If this condition is satisfied (and there is no superpotential), there must exist at least one massless bosonic state in this system.\(^1\)

If we consider the $\mathcal{N} = (4,4)$ supersymmetric pure YM theory, which is a special case of the $\mathcal{N} = (2,2)$ supersymmetric gauge theory without a superpotential, the symmetry is enhanced to $SU(2)_L \times SU(2)_R$. In this case, the Yukawa interaction and the scalar potential of the $\mathcal{N} = (4,4)$ pure YM theory are given by

$$
Y = g \text{tr} \left\{ \overline{\psi} [\sigma + i\gamma_5 \pi^i, \psi^i] \right\},
$$

\(^1\) The possible presence of the Fayet-Iliopoulos D-term does not change the conclusion, because after the integration over auxiliary fields, the lagrangian density takes the form of Eq. (2.1). However, we cannot include the $\theta$-term in the present argument, because it breaks the parity invariance.
\[ V = -g^2 \text{tr} \left\{ [\sigma, \pi_i]^2 + \left( \frac{1}{2} \epsilon_{ijk} [\pi_j, \pi_k] \right)^2 \right\}, \] (3.3)

where the pseudo-scalars \( \pi_i \) \((i = 1, 2, 3)\) and fermions \( \psi \) and \( \bar{\psi} \) are the triplet and doublets of the flavor \( SU(2) \), respectively, and \( \tau^i \) denotes the Pauli matrices. The theory thus possesses \( SU(2)_L \times SU(2)_R \) global symmetry, which is a realization of the \( SO(4) \) part of the \( R \) symmetry of this model. According to the above argument, therefore, there must be at least three massless bosonic states (per fixed spatial momentum).

As soon as we couple matter multiplets to the \( \mathcal{N} = (4,4) \) pure YM theory, however, all chiral symmetries are broken, because the \( \mathcal{N} = (4,4) \) supersymmetry requires that we introduce a particular form of the superpotential [in terms of the \( \mathcal{N} = (2,2) \) theory]. As already noted, then, the lagrangian has no \( G_L \times G_R \) symmetry. Since the \( \mathcal{N} = (8,8) \) gauge theory is a particular case of the \( \mathcal{N} = (4,4) \) gauge theory with matter multiplets, the \( \mathcal{N} = (8,8) \) theory also possess no \( G_L \times G_R \) symmetry, and again our argument does not apply.

Thus we have observed that the \( \mathcal{N} = (2,2) \) models without a superpotential and the \( \mathcal{N} = (4,4) \) pure YM theory contain massless bosonic states. The fact that the \( \mathcal{N} = (2,2) \) and \( \mathcal{N} = (4,4) \) pure YM theories have no mass gap was noted in Ref. 6) on the basis of the ’t Hooft anomaly matching condition.

In supersymmetric theories, if supersymmetry is not spontaneously broken, a massless bosonic state implies a massless fermionic state. For example, in the \( \mathcal{N} = (2,2) \) pure YM theory, Eq. (2.10) and a supersymmetric WT identity show that there exists a massless fermionic state which is created by the action of the supercurrent on the vacuum.\(^\ast\) This fact provides a further constraint on possible low-energy spectra and should be very useful in the examination of recently developed lattice formulations of two-dimensional supersymmetric gauge theories.\(\text{8)–20)\) The details of such a study will be reported elsewhere.\(\text{40)\)

\section*{§4. Conclusion}

In summary, using an elementary argument, we showed that in a wide class of \( G_L \times G_R \) invariant two-dimensional super-renormalizable field theories, the parity-odd part of the two-point function of global currents can be determined to all orders in perturbation theory. For any non-trivial fermion content, the two-point function possesses a massless pole, and this fact provides a simple criterion for the existence of massless bosonic states in the theory. As a particular application, we considered two-dimensional supersymmetric gauge theories.

\(^\ast\) This occurs, however, in a way that does not directly imply spontaneous breaking of supersymmetry.
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Appendix A

BJL Prescription and the Derivation of Eq. (2.11)

The BJL prescription\(^{23,24}\) enables one to extract the \(T\) product from the corresponding \(T^\ast\) product. (See also Ref. 41) for an analysis of a related problem.) The prescription (for two-point functions) is based on the following two requirements: (1) The possible difference between the \(T\) product and the \(T^\ast\) product arises only at equal times. (2) The \(p_0 \rightarrow \infty\) limit of the Fourier transform of the \(T\) product vanishes.\(^{23}\) From these, the Fourier transform of the \(T\) product is obtained from that of the corresponding \(T^\ast\) product by subtracting a polynomial of \(p_0\), so that the \(p_0 \rightarrow \infty\) limit vanishes.

For example, from Eq. (2.10), we have

\[
\text{FT} \langle 0| T J^{\mu}_5(x) J^\nu(y) |0 \rangle = -\frac{i}{2\pi} (\dim r) T(R) \delta_{ab} \frac{1}{p_0} (p_\mu \epsilon_{\nu \rho} p^\rho + p_\nu \epsilon_{\mu \rho} p^\rho - \delta^0_{\mu} \epsilon_{\nu 0} - \delta^0_{\nu} \epsilon_{\mu 0}). \tag{A.1}
\]

Similarly, from the divergence of Eq. (2.10), we have

\[
\text{FT} \langle 0| T \partial_\mu J^{\mu}_5(x) J^\nu(y) |0 \rangle = 0. \tag{A.2}
\]

Now, the divergence of Eq. (A.1) yields

\[
\text{FT} \partial_\mu \langle 0| T J^{\mu}_5(x) J^\nu(y) |0 \rangle = -\frac{1}{\pi} (\dim r) T(R) \delta_{ab} \delta^0_\nu p_1, \tag{A.3}
\]

which is, using Eq. (A.2), precisely the commutator \(\text{FT} \langle 0| [J^{\mu}_5(x), J^\nu(y)] |0 \rangle \delta(x^0 - y^0)\), and then we have Eq. (2.11).

References

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