A finite-difference technique to incorporate spatial variations in rigidity and planar faults into 3-D models for lithospheric flexure

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SUMMARY
We present a finite-difference formulation for 3-D elastic flexure of the lithosphere, which is solved by a direct-matrix method. To incorporate the effect of spatial variations in rigidity, additional terms for the bi-harmonic 3-D flexure equation have been derived from a variational displacement formulation as used in finite-element methods. Additionally, planar faults are treated as discontinuities. These are implemented by an additional degree of freedom for fault heave, and a coupled continuum equation for zero-differential tilting across the fault. The 3-D finite-difference results have been tested for line loads, point loads and disc loads by analytical solutions, and for spatial variation in effective elastic thickness (EET) by 2-D finite-difference solutions. Fault-related flexure patterns are compared to the 2-D analytical broken-plate model developed by Vening-Meinesz (1950). We subsequently apply the 3-D fault model to investigate fault controlled 3-D basement geometries in Lake Tanganyika (East Africa). We show that our model is capable of predicting 3-D basement geometries, characteristically observed in rifted basins. The modelling results indicate that fault-controlled upper crustal flexure patterns are associated with low values for EET. A comparison with regional scale-model studies, showing a superposition of high EET flexure effects, supports a multilayered rheological control on continental rifting.

Key words: Africa, fault models, finite-difference methods, lithosphere, rifts.

INTRODUCTION
Starting from observations by Vening-Meinesz (1941), it has generally become accepted that on geological timescales, the oceanic lithosphere essentially behaves as a thin elastic plate overlying an inviscid weak asthenosphere, in response to intraplate loads and plate tectonic stresses (Walcott 1970; Watts & Cochran 1974; Caldwell et al. 1976).

2-D flexure studies have shown that the rigidity values in terms of the thickness of the elastic plate (the so-called effective elastic thickness EET) of the oceanic lithosphere increase with age, suggesting a dominant control by cooling of the lithosphere (Caldwell & Turcotte 1979; Watts, Bodine & Steckler 1980; Bodine, Steckler & Watts 1981). This observation is in accordance with rheological models based on the extrapolation of rock-mechanics data (Goetze & Evans 1979; Kirby 1983).

On the other hand, it has been shown by Goetze & Evans (1979), that inelastic processes, occurring for stresses in excess of yield strengths predicted from lithospheric rheology, strongly reduce inferred values for EET. For continental lithosphere underlying sedimentary basins, basement deflections and gravity anomalies also indicate flexural behaviour (Beaumont 1979; Kooi, Cloetingh & Burrus 1992). These flexure studies suggest a strong dependence of EET on thermal- and material-controlled rheology (McNutt, Diament & Kogan 1988; Van der Beek & Cloetingh 1992; Peper & Cloetingh 1992). Deep seismics (Reston 1990) and rheology modelling studies (Van Wees, de Jong & Cloetingh 1992; Cloetingh & Banda 1992), suggest that the continental lithosphere has a layered structure, marked by relatively strong upper crust and subcrustal lithosphere with relatively weak lower crust in between. The strong layers in the lithosphere can, therefore, behave as independent elastic entities. In this perspective, a single-plate elastic analogue can be formulated using different EET values. Kooi et al. (1992), assume a coupling of bending of the layers and, therefore, bending energy is summed in all layers to link it with an EET. On the other hand an EET value can be based on the flexural characteristics of just one of the layers, assuming decoupled bending with zero-bending energy in the other layers (Kusznir, Marsden & Egan 1991).
In continental lithosphere, loading processes are generally accompanied by inelastic internal deformations, such as lithospheric thinning and thickening. Thermodynamic models show that induced deformations strongly depend on rheological layering (Braun & Beaumont 1989; Buck 1991). However, in most basin flexure studies, dynamical effects of thinning and thickening tend to be neglected, by prescribing kinematics. First, a kinematically prescribed deformation of basement and Moho topography is applied which most closely mimics the actual deformation path expected from layered rheology. In a second step a flexural calculation is carried out, in which acting loads correspond to the buoyancy forces related to the kinematically perturbed lithosphere, including effects of erosion and sediment infill. Taking into account dynamical aspects of stretching in the kinematic step, Kooi et al. (1992) showed that the actual flexural response and associated basin shape strongly depend on the dynamics of thinning (e.g. the associated depth of lithospheric necking).

In addition to effects related to bulk thinning and thickening, it is evident that major crustal faults play a crucial role in deformation of continental lithosphere (Wernicke 1981; Lister, Etheridge & Symonds 1986). Seismic data (McGeary & Warner 1986; Jackson 1987; Reston 1990), indicate that crustal-scale normal faults, form at constant and steep angles in the upper crust and sole in the lower crust, where deformation occurs by low-angle shear movements. Shear-stress balance considerations (Mandl 1988) at the interface of low-angle sheared lower crust and the overlying high-angle sheared upper crust, indicate a transition from relatively strong to relatively weak material. Both finite-element models (Melosh 1990) and analogue models (Vendeville et al. 1987; McClay & Ellis 1987) support this feature, highlighting the abrupt change from steeply dipping planar normal faults to low-angle shear zones. Furthermore, analogue models show that the thickness of the weak base (ranging from a certain thickness of silicone to a basal detachment formed by a rubber sheet), has little influence on the pronounced planar shape of faulting. In analogue experiments with a relatively thick weak base (Vendeville & Cobbold 1988), it has been shown that faults can rotate during progressive extension. In these experiments domains in between the faults behave relatively rigidly, rotating together with the faults. The kinematic domino-fault model (Ransome, Emmons & Garrey 1910; Barr 1987), adopts crustal faults in this fashion to explain observed basement tilting observed in sedimentary basins. A characteristic feature of this model is constant basement dip across faults, related to the planar shape of faults and rigid-body rotation in between. Consequently, transition from basement tilting in the faulting domain to non-affected basement outside the basin is not considered. This can be attributed to flexural characteristics of the continental lithosphere.

Back in 1950, Vening-Meinesz had already quantified the effect of planar faulting with respect to elastic properties of the continental lithosphere. He suggested that a crustal planar fault, cutting through the mechanically strong part of the lithosphere, can be considered to disect the elastic lithospheric plate into a hanging wall and foot wall, which react as two broken plates with opposite endloads and zero-bending moments at the position of the fault. In this model it is assumed that the fault can be treated as a pointwise discontinuity. The 2-D analytical solution (Vening-Meinesz 1950; Heiskanen & Vening-Meinesz 1958; Fig. 1), shows that the broken-plate model satisfies the basic assumption about planar faulting in the domino model, that basin tilting is constant across the fault. In addition, the transition of horizontal basement to basement tilting is explained by flexure. However, buoyancy effects related to crustal thinning or thickening accompanying the faulting and erosion or sediment infill are not included.

Recently, more sophisticated 2-D flexural models have been developed to incorporate fault behaviour (Kuszni, Karner & Egan 1987; Buck 1988; Weissel & Karner 1989; Kuszni et al. 1991). In these models, flexural response related to faulting is calculated in a two-stage fashion. In the first stage, fault-related basement steps are kinematically prescribed. Next, deflections are calculated according to the buoyancy forces which result from the kinematically perturbed lithosphere, taking into account effects of crustal thinning or thickening and erosion or sediment infill. Buck’s (1988) model and Kuszni et al.’s (1991) flexural-cantilever model, assume crustal-scale planar faults. The other models also consider initially listric fault shapes. We assume that listric faults are seldomly created as such. Both finite-element and analogue models support this notion (Mandl 1988; Vendeville & Cobbold 1988), whereas it has been shown by Buck (1988) that observed listric fault shapes can be a result of fault slip related to buoyancy forces, which laterally vary for large displacements. Although we do not fully exclude initial formation of listric faults, we consider here only the effect of planar faulting on flexure patterns. Focusing on the flexural cantilever model (Fig. 2), it is assumed that faults sole in the lower crust. The dynamics of upper crustal fault movements and accompanying lower crustal and mantle-ductile deformation are considered to be characterized by a sinusoidal shape (Kuszni et al. 1991) for $\beta$ values, characterizing the stretching of the lithosphere (McKenzie 1978). In contrast to the broken-plate model for

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**Figure 1.** Broken plate model of Vening-Meinesz (1950) for crustal faulting. The fault is treated as a pointwise discontinuity, half-fault heave corresponds to $w_{\text{max}}$. Analytical solution and flexural parameter $a$ are discussed in text.
A finite-difference technique

Following Kirchoff thin-plate assumptions and a pointwise variational-displacement formulation (Papadopoulos & Taylor 1990; Zienkiewicz & Taylor 1991), a general bending-energy potential \( P \) can be derived for plate bending over a surface domain \( A \):

\[
P = \frac{1}{2} \int \kappa^T Dk \, dA - \int wq \, dA, \tag{1}
\]

where, \( w \) = surface deflection and \( q \) = surface loading, with the in-plane shear strains:

\[
\kappa = \left[ \epsilon_x, \epsilon_y, \gamma_{xy} \right]^T = \left[ \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, \frac{\partial^2 w}{\partial x \partial y} \right]^T.
\]

The flexural rigidity is given by:

\[
D = \frac{EH^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}
\]

where, \( E \) = Young's modulus, \( \nu \) = Poisson's ratio, \( h \) = elastic-plate thickness, with the in-plane bending moments:

\[
D_{e} = [M_x, M_y, M_{xy}]^T.
\]

A finite-element solution of the Kirchoff thin-plate approximation is obtained, by minimizing the bending-energy potential \( P \) with respect to variation in \( w \) over a surface domain \( A \). This corresponds to taking the derivative of \( P \) with respect to \( w \), and to solve this derivative for the 0 value. In the case of finite-difference nodes, discretization of the derivative of this equation is obtained by ignoring the surface weighting as expressed by the surface integral over \( A \). Subsequently the following equation has to be solved:

\[
\frac{\partial}{\partial w} \frac{1}{2} \kappa^T Dk = q. \tag{2}
\]

evaluation of \( Dk \) in (2) leads to:

\[
Dk = D \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \\ (1-\nu)/2 \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x \partial y} \right) \end{bmatrix}
\]

where \( D = Eh^3/12(1-\nu^2) \).

Evaluation of the contribution of plate bending in the energy potential leads to:

\[
\frac{1}{2} \kappa^T Dk = \frac{1}{2} \left[ \frac{\partial^2 w}{\partial x^2} D + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x \partial y} D \left( \frac{1}{2} \frac{\partial^2 w}{\partial x \partial y} \right) \right] + \frac{\partial^2 w}{\partial y^2} D \frac{\partial^2 w}{\partial x \partial y} \left( \frac{1}{2} \frac{\partial^2 w}{\partial x \partial y} \right) \right].
\]

For 3-D modelling of fault deformation, we adopt the broken-plate concept of Vening-Meinesz (1950), since we do not focus on deformation in the fault zone itself. Additionally, the effects of buoyancy forces related to the faulting are incorporated as in the flexural-cantilever model (Kusznir et al. 1991). Results for various fault patterns are discussed in the light of 3-D models for rifted basin formation.
The derivative of this function gives:
\[
\frac{\partial}{\partial w} D \delta \mathbf{x} = \left[ \frac{\partial^2}{\partial x^2} D \delta w + \frac{\partial^2}{\partial y^2} D \delta w + \frac{\delta^2}{\partial x \partial y} D \frac{\delta^2 w}{\partial x \partial y} \right]
\]
\[
= \frac{\partial^2}{\partial y^2} \left( \frac{1 - \nu}{2} D \frac{\delta^2 w}{\partial y^2} + \frac{\partial^2}{\partial x \partial y} D \frac{\delta^2 w}{\partial x \partial y} \right)
\]
Substituting (3) in (2) and assuming that equation (Bodine 1981; Ranalli 1987) for plate bending:
\[
D \frac{\partial^4 w}{\partial x^4} + D \frac{\partial^4 w}{\partial y^4} + 2D \frac{\partial^4 w}{\partial x^2 \partial y^2} = q.
\]
In case of an isostatically compensated lithosphere, downward and upward deflections \((w)\), cause additional loads, which are proportional to \(w\) and the density contrast between the asthenosphere and the material filling a depression \((w < 0)\) or the density contrast between the asthenosphere and eroding material \((w \geq 0)\) (Turcotte & Schubert 1982). Consequently, the following force component \(dq_i\) has to be added to \(q\) in (4):
\[
dq_i = \Delta \rho gw,
\]
where \(g\) is the acceleration of gravity \((9.8 \text{ m s}^{-2})\) and \(\Delta \rho\) is the density contrast between asthenosphere and filling \((w < 0)\) or eroding \((w \geq 0)\) material.

In addition the effect of intra-plate stresses (e.g. Cloetingh & Kooi 1992) can be incorporated using equations given by Turcotte & Schubert (1982) and Bodine (1981):
\[
dq_2 = \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial y^3} + 2 \frac{\partial^3 w}{\partial x \partial y} + \frac{\partial^3 w}{\partial x^2 \partial y},
\]
where in terms of 2-D stress-tensor components:
\[
T_x = \sigma_{xx},
\]
\[
T_y = \sigma_{yy},
\]
\[
T_{xy} = \sigma_{xy}.
\]

Variation of EET values leads to a more complex evaluation of (2), than the simple bi-harmonic eq. (4). If we assume that varying EET is only a function of variation in \(h\) and \(E\), then we derive for (2):
\[
D \frac{\partial^4 w}{\partial x^4} + D \frac{\partial^4 w}{\partial y^4} + 2D \frac{\partial^4 w}{\partial x^2 \partial y^2}
\]
The basic continuum eq. (3) to derive (7), differs significantly from 3-D equations used by Bodine (1981) to derive a 2-D finite-difference formulation for axisymmetric loading problems. Instead of the material matrix \(D\) in eq. (1), Bodine assumed a scalar \(D\), which does not agree with theoretically expected bending moments (Papadopoulos & Taylor 1990). However, in case of line-load problems, considering variations in EET perpendicular to the line load, the scalar \(D\) approximation is correct.

For finite-difference formulations, the deflection-dependent load contributions from eqs (5) and (6) are shifted to the left-hand side of continuum eqs (4) or (7). Using finite-difference approximations (Conte & De Boor 1981), the continuum equation for each node is recast in a linear equation, in which partial differences of \(D\) are directly evaluated and deflection \(w\) needs to be solved. The resulting set of equations is solved by using a Gaussian elimination technique and by setting displacement boundary conditions. Concerning the latter, it is assumed that nodes at the edge of the model have zero deflection and that second derivatives are zero in directions outside the model range. In case \(\Delta \rho\) differs for negative and positive deflections, we adopt a constant \(\Delta \rho\) for matrix factorization and an iterative Newton technique to solve (7), for pseudo-loads corresponding to \(\Delta \rho\), which vary as a function of \(w\).

### NUMERICAL TESTS FOR CONSTANT AND SPATIALLY VARYING EET

To test the numerical performance of the 3-D finite-difference-solution technique for constant EET plate-bending problems, we compared numerical results for a line load and various disc loads, with analytical solutions. As a test for varying EET, we compared results for a line load with 2-D finite-difference results (Bodine 1981). Additionally, for an axisymmetric problem with varying EET, we compared our 3-D results with the erroneous 2-D finite-difference solution of Bodine (1981).

#### Line load (constant EET)

For a line load on an elastic plate with constant EET, an analytical solution for the maximum deflection \(w_0\) is given by Turcotte & Schubert (p. 126, 1982):
\[
w_0 = \frac{F_0 \alpha x^3}{8D}
\]
where
\[
D = \frac{E h^3}{12(1 - \nu^2)}
\]
\[
\alpha = \frac{[4D/\Delta \rho g]^1/4}.
\]

#### Table 1. Constants.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E)</td>
<td>Young’s modulus</td>
<td>(7.1 \times 10^10 \text{N} \cdot \text{m}^{-2})</td>
</tr>
<tr>
<td>(\nu)</td>
<td>Poisson’s ratio</td>
<td>0.25</td>
</tr>
<tr>
<td>(g)</td>
<td>gravity</td>
<td>9.8 m s(^{-2})</td>
</tr>
<tr>
<td>(\rho_c)</td>
<td>crustal density</td>
<td>2800 kg m(^{-3})</td>
</tr>
<tr>
<td>(\rho_m)</td>
<td>mantle and asthenosphere density</td>
<td>3300 kg m(^{-3})</td>
</tr>
</tbody>
</table>
Table 2. Defined parameters.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>h</td>
<td>effective elastic thickness (EET)</td>
<td>m</td>
</tr>
<tr>
<td>D</td>
<td>flexural rigidity, equal to ( \frac{Eh^3}{12(1-\nu)} )</td>
<td>Nm</td>
</tr>
<tr>
<td>( \Delta \rho )</td>
<td>density contrast</td>
<td>kg m(^{-3} )</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>flexural parameter, equal to ( \frac{4D/\Delta \rho g}{\gamma} )</td>
<td>m(^{1} )</td>
</tr>
<tr>
<td>( w )</td>
<td>deflection</td>
<td>m</td>
</tr>
<tr>
<td>q</td>
<td>load</td>
<td>Nm(^{-2} )</td>
</tr>
<tr>
<td>( F_l )</td>
<td>lineload</td>
<td>Nm(^{-2} )</td>
</tr>
<tr>
<td>Q</td>
<td>pointload</td>
<td>N</td>
</tr>
<tr>
<td>( P_0 )</td>
<td>discload</td>
<td>N</td>
</tr>
<tr>
<td>r</td>
<td>radius of discload</td>
<td>m</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>finite extension</td>
<td>m</td>
</tr>
<tr>
<td>( \theta )</td>
<td>fault node spacing</td>
<td>m</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>fault dip</td>
<td>radians</td>
</tr>
<tr>
<td>( \Delta \gamma )</td>
<td>radius of spherecap function</td>
<td>m</td>
</tr>
<tr>
<td>( \Delta x )</td>
<td>gridspace (x)</td>
<td>m</td>
</tr>
<tr>
<td>( \Delta y )</td>
<td>gridspace (y)</td>
<td>m</td>
</tr>
<tr>
<td>( w_{\text{mean}} )</td>
<td>mean deflection (split node)</td>
<td>m</td>
</tr>
<tr>
<td>( w_{\text{half}} )</td>
<td>half fault heave (split node)</td>
<td>m</td>
</tr>
<tr>
<td>( \beta )</td>
<td>stretching factor</td>
<td></td>
</tr>
</tbody>
</table>

where \( \alpha \) is the flexural parameter and \( F_l \) is the line load (N m\(^{-1} \)).

The deflection at a point \( x \) relative to the position of the line load is:

\[
w(x) = w_0 e^{-x/\alpha} \left[ \cos \left( \frac{x}{\alpha} \right) + \sin \left( \frac{x}{\alpha} \right) \right].
\]

We test the finite-difference solution for a line load of \(-1 \times 10^{12} \text{N m}^{-1}\) in a 50 x 50 grid (spacing \( \Delta x = \Delta y = 20 \text{ km} \)). The boundaries of the mesh are constrained in deflection, \( w = 0 \). Additionally, at these boundaries, both \( d^2w/dx^2 \) and \( d^2w/dy^2 \) are set to 0. The line load is distributed over 30 nodes in the \( Y \) direction. The surface loading \( q \) for these 30 nodes, which is equivalent to the line load, is given by:

\[
\int_{-0.5\Delta x}^{0.5\Delta x} q \, dx = F_l.
\]

This implies \( q = F_l / 20 \times 10^3 = -5 \times 10^7 \text{N m}^{-2}\). For the elastic thickness of the plate a value of \( h = 20 \text{ km} \) is adopted, for Poisson's ratio a value of \( \nu = 0.25 \) and for Young's Modulus \( E = 7 \times 10^{10} \text{N m}^{-2} \). In this case the flexural rigidity is equal to \( D = 4.98 \times 10^{22} \text{N m} \). For the density contrast for both positive and negative deflections, a \( \Delta \rho = 3.3 \times 10^1 \text{kg m}^{-3} \) is adopted, corresponding to asthenospheric density. Intrinsically, a zero density is assumed for depression infill, and elevations are not eroded. The corresponding flexural parameter is equal to \( \alpha = 49.8 \text{ km} \).

Figure 3(a) shows the calculated plate-deflection contours under the line load. Fig. 3(b) shows an E-W profile from the centre of the domain to the east, scaled to the flexural parameter in \( X \) and to the analytical solution of \( w_0 \) in \( Y \). The analytical solution for the \( w_0 \) gives a value of \(-310 \text{ m} \), the actual maximum deflection is \(-322 \text{ m} \). The shape and amount of flexure compares well with the analytical solution as is evident from comparing the numerical and analytical results.

Point load and disc loads (constant EET)

For disc loads with various radii \( a \), analytical solutions of the deflection at a radius \( r \) from the centre of the load are given by Brotchie & Silvester (1969). These require evaluations of
Kelvin–Bessel functions:

\[ w(r) = \frac{P_0}{\Delta \rho g} \left[ \frac{a}{\alpha} \left( \frac{r}{\alpha} \right) \text{be} \left( \frac{r}{\alpha} \right) - \frac{a}{\alpha} \text{ke} \left( \frac{r}{\alpha} \right) \text{be} \left( \frac{r}{\alpha} \right) + 1 \right] \]

\[ r < a \]

\[ w(r) = \frac{P_0}{\Delta \rho g} \left[ \frac{a}{\alpha} \left( \frac{r}{\alpha} \right) \text{be} \left( \frac{r}{\alpha} \right) - \frac{a}{\alpha} \text{ke} \left( \frac{r}{\alpha} \right) \text{i} \left( \frac{r}{\alpha} \right) \right] \]

\[ r \geq a \]

where

\[ a = \left[ \frac{D}{\Delta \rho g} \right]^{1/4} \]

\[ D = \frac{E h^3}{12(1 - v^2)} \]

\[ P_0 = \text{disc load (N m}^{-2} \) \]

\[ a = \text{radius of disc load.} \]

For a point load (\( a = 0 \)) these reduce to

\[ w(r) = \frac{Q a^2}{2 \pi D} \text{i}(r) \]

where

\[ Q = \text{point load (N).} \]

Similar to the line-load analysis, the deflections can be scaled to a maximum deflection (normally for \( a = 0 \)) and to some flexural parameter (normally \( \alpha \)). The maximum deflection for \( a = 0 \) is given by eq. (12) as:

\[ w_0 = \frac{Q a^2}{2 \pi D} \text{i}(0) = \frac{Q a^2}{8D}. \]

As a test for a point load, we adopt a similar grid and similar plate properties as in the line-load test. In this case, however, we used a grid spacing of 10 km. Owing to a different definition, the flexural parameter now is equal to \( \alpha = 35.2 \text{ km}. \) For the point-load test, \( q = \frac{P_0}{\pi} = -4 \times 10^7 \text{ N} \) is applied at the central node of the grid. The equivalent point load \( Q \) is assumed to be related to \( q \) by:

\[ q \Delta x \Delta y = Q, \]

which results in \( Q = -4 \times 10^{17} \text{ N}. \) Substitution of this value in (6) gives a value of \(-1246 \text{ m} \) for the deflection of the central node, whereas from the 2-D finite-difference calculations a value of \(-1286 \text{ m} \) is obtained (Fig. 4). The deflection distribution determined from the finite-difference calculations compares well with the analytical solution.

For tests of disc loads, we assign a number of grid-nodes within a specific radius \( a \), and a load \( q_{\text{disc}} \). This load times the number of nodes \( N \) assigned to it, is defined to be equal to \( q \), which was adopted for the point-load model. Therefore, \( q_{\text{disc}} = q / N \). The resulting deflections are scaled by the analytical solution for the point-load model and are depicted in Fig. 4. The corresponding analytical solutions are obtained by substituting \( P_0 = -4 \times 10^7 / (\pi a^2) \) in eq. (11). It is evident that calculated deflection distributions show only small errors relative to the analytical curves.

**Line load and point load (varying EET)**

For tests with varying EET, no analytical solutions are available. However, 3-D results can be tested with 2-D finite-difference formulations (Bodine 1981).
As a test for a line load, we subjected a plate with varying EET to a load of \(-1 \times 10^{12} \text{ N m}^{-1}\), equal to the constant EET line-load test. Grid geometry and boundary conditions are equal to the previous point-load model. For the line load, we set \(q = -1 \times 10^6 \text{ N m}^{-2}\) for 30 nodes aligned in the \(Y\) direction in the central part of the grid. We adopt similar plate properties as in the previous models, except for the flexural parameter \(D\) which now varies as a function of the effective elastic thickness \(h\). The EET is chosen to vary between 10 and 20 km as indicated in Fig. 5(a).

The resulting plate bending (Figs 5b and c), deviates significantly from the constant EET results, and corresponds well to the 2-D finite-difference solution. To check the sensitivity of the results to the choice of the coordinate system, we can repeat the experiment by rotating the line load and EET distribution 45°. The 3-D finite-difference solution yields similar results.

For an axisymmetric point-load problem with varying EET, results can be compared as well to a finite-difference solution of Bodine (1981). However, in this case results are expected to be different due to an error in Bodine's derivation. To illustrate the discrepancy between our 3-D finite-difference solution and Bodine's axisymmetric solution, we compared both for a point load subjected to a plate with similar grid geometry, properties and boundary conditions as in the previous model. In this case the EET varies in a radial symmetric pattern from 20 to 10 km (Fig. 6a) and the magnitude of the point load of \(-4 \times 10^{17} \text{ N}\), applied to the centre of the mesh, corresponds to the previous constant EET point-load test. Results depicted in Fig. 6(b) clearly illustrate a deviation from constant EET solution in Fig. 4(b). Our deflection values for varying EET differ significantly (up to 10 per cent) from Bodine's finite-difference solution (Fig. 6c). The close agreement between our results and Bodine's solution for a constant EET (Fig. 6c), prove that the discrepancy for varying EET is mainly related to an erroneous derivation of Bodine (1981).

### 3-D IMPLEMENTATION OF PLANAR FAULTS

For 3-D calculations, faults are treated as linewise discontinuities, following the 2-D concepts of Vening-Meinesz (1950) and Heiskanen & Vening-Meinesz (1958). Nodes on the fault are characterized by two degrees of freedom (Fig. 7), one for mean deflection (\(w_{\text{mean}}\)) and one for half the heave (\(w_{\text{half}}\)). The deflection in the hanging wall at the fault edge is equal to \(w_{\text{mean}} - w_{\text{half}}\), whereas for the foot wall this equals \(w_{\text{mean}} + w_{\text{half}}\). Partial differences which include fault nodes are evaluated using either \(w_{\text{mean}} - w_{\text{half}}\) or \(w_{\text{mean}} + w_{\text{half}}\), depending on hanging wall or footwall positions of the nodes outside the fault respectively. If both...
Figure 6. 3-D finite-difference solution for an elastic plate with axisymmetric variation of EET, subjected to a point load of $-4 \times 10^{17} \text{N m}^{-1}$ (pertinent parameters discussed in text). (a) Variation of EET from 10 to 20 km (contour interval 500 m). (b) Deflections from solution for varying EET (contour interval 25 m). (c) Varying EET solution compared to 2-D finite-difference solution (Bodine 1981) and 2- and 3-D finite-difference and analytical solutions for constant EET (20 km).

Figure 7. Linewise discontinuities in finite difference mesh to represent faults. See text for explanation.

Formulations, this results in the following equation for a node on a fault in the Y direction and constant mesh spacing $\Delta x$ (Fig. 7):

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{\Delta x^2} \left( w_{+1} - (w_{\text{mean}} + w_{\text{half}}) - ([w_{\text{mean}} - w_{\text{half}}] - w_{-1}) \right)$$

$$= \frac{1}{\Delta x^2} (w_{+1} - 2w_{\text{mean}} + w_{-1}) = 0.$$  (15)

The fault nodes are aligned in the Y direction of the coordinate system. For arbitrary aligned faults, finite-difference formulations are obtained by local coordinate transformations. The components of eq. (15) are scaled by $D/\Delta y^2$, where $\Delta y$ is the mesh spacing in the Y direction, in order to obtain equal weighting compared to the flexure equations. Furthermore, in the finite-difference component matrix, the flexure equation is connected to $w_{\text{half}}$ and the constant tilt eq. (15) to $w_{\text{mean}}$, to prevent zero-diagonal entries. We follow the assumption that deformation of upper lithosphere faults sole in the lower crust and that fault displacement in itself does not affect Moho topography (Kusznir et al. 1991). Consequently, for this part of the faulting process, the lithosphere can be treated as an elastic plate floating on the lower crust. In 2-D modelling studies (e.g. Kusznir et al. 1991), it is a normal procedure to fix fault
A finite-difference technique

and ductile levels of the lithosphere, lead to a smooth sinusoidal $\beta$ distribution of Moho deformation relative to zero surface level (Fig. 2). The $\beta$ distribution is determined by balancing the bulk lithospheric extension or shortening to the upper lithosphere fault deformation:

$$L = \int_{x_1}^{x_2} [\beta(x) - 1] \, dx.$$  \hspace{1cm} (17)

The buoyancy loads arising from the kinematic perturbations of the Moho are applied to the faulted basement topography. In contrast to what is suggested by Kusznir et al. (1991), the $\beta$ distribution does not actually represent the bulk extension or shortening of the crust, since the reference surface for the $\beta$ distribution is not the actual topography, but the zero level. Eq. (17) still holds, since the integration of the fault-related surface deflections over $[x_1, x_2]$, in the absence of loads, yields a zero value.

In the following, we use eqs (16) and (17) to calculate the 3-D response of the lithosphere to tectonic loads. It is assumed that extension or shortening due to displacement at a fault node, leads to a smooth 2-D radial symmetric $\beta$ distribution, describing deformation of the Moho with respect to the zero level. The 2-D $\beta$ function is balanced to the fault-related extension or shortening in a way similar to (17):

$$L \Delta f = \int_0^r [\beta(r) - 1] \, dr,$$  \hspace{1cm} (18)

where $L$ is determined as in 2-D, $\Delta f$ is the fault-node spacing, $r$ is the distance from the fault node and $r_i$ is the radius of the domain of the $\beta$ distribution. For the $\beta(r)$ distribution, a spherical-cap function is assumed (Fig. 9):

$$\beta(r) = 1 + C(V[(r_0^2 - r^2) - d]),$$  \hspace{1cm} (19)

where for $r_0$, $d$ and $h$, a spherical cap angle of $45^\circ$ is assumed [$r_0 = r_i/cos(45^\circ)$, $d = sin(45^\circ)r_i$, $h = r_0 - d$]. The volume of the spherical cap of radius $r_0$ and height $h$ is given by:

$$V_{\text{sphere cap}} = \frac{1}{3} \pi h^2 (3r_0 - h).$$  \hspace{1cm} (20)

Subsequently, using eq. (18), the scale factor $C$ is determined by:

$$L \Delta f = CV_{\text{sphere cap}}.$$  \hspace{1cm} (21)

As a consequence of fault displacements, the upper part of the lithosphere is extended or shortened, depending on fault dip and fault displacement (Fig. 8). The finite amount of extension or shortening in 2-D is given by $L$:

$$L = \frac{2w_{\text{half}}}{\tan(\Theta)}$$  \hspace{1cm} (16)

where $\Theta$ is fault dip, and $w_{\text{half}}$ is half the fault heave. Clearly, positive fault heaves lead to extension, whereas negative values result in shortening. Adopting concepts of Kusznir et al. (1991), we assume that fault extension or shortening in the upper crust is compensated by ductile deformation in the lower crust and subcrustal lithosphere (Fig. 2). In their 2-D model it is assumed that the summation of different types of deformation in the brittle
To obtain a \( \beta \) distribution for the whole fault, the individual contributions from fault nodes, as determined by eq. (19), are multiplied together. Moho deflections, determined by this \( \beta \) distribution are used to calculate buoyancy forces and added to \( q \) in eqs (4) or (7):

\[
\Delta q_v(x) = - (\rho_m - \rho_c) g h_{\text{crust}}
\]

where

\[
\Delta q_v(x) = \text{buoyancy load (N m}^{-2}\text{)}
\]

\( \rho_m \) = mantle density

\( \rho_c \) = crustal density

\( h_{\text{crust}} \) = crustal thickness (m).

Buoyancy forces related to associated thermal perturbations are not taken into account.

**NUMERICAL FAULT TESTS AND RELEVANCE TO MODELS FOR RIFTED BASINS**

To test the numerical performance of our fault model by incorporating a single fault over 30 nodes in the \( Y \) direction of a 50 \( \times \) 50 grid, for which plate properties and boundary conditions are equal to the first line-load model. The fault heaves of the central 26 nodes are set to 10 km. It is assumed that deflections are compensated by lower crustal flow, therefore \( \Delta \rho = 2.8 \times 10^{15} \text{ kg m}^{-3} \). The 2-D analytical solution is given by Vening-Meinesz (1950):

\[
w(x) = \pm w_{\max} e^{-0.701(x/\alpha)} \cos \left( \frac{0.701x}{\alpha} \right)
\]

where

\( x \) = position relative to fault

\( w_{\max} \) = half-fault heave

\( \alpha = \left[ D/\Delta \rho \right]^{1/4} \).

The calculated deflections, in the absence of loads, are depicted in Fig. 10(a). Comparison with the broken-plate analogue of eq. (23), shows that the finite-difference results closely agree with this analytical solution (Fig. 10b).

To demonstrate the effect of buoyancy forces for crustal thinning and sediment infill, related to this fault setting, we adopt an extensional fault dip of 60° and a number of radii \( r_i \) for the \( \beta \) distribution describing the Moho deflection, according to the previous section. An example of a \( \beta \) distribution, is depicted in Fig. 11(a). The effects of buoyancy forces, adopting a crustal thickness of 30 km, are shown in Fig. 11(b). It is evident that the thinning of the lithosphere and radius of \( \beta \) distributions have a moderate control on the resulting deflections. In addition, if we consider the effects of infill of depressions (\( w < 0 \)) with water (\( \rho = 1.0 \times 10^3 \text{ kg m}^{-3} \)) or sediments (\( \rho = 2.3 \times 10^3 \text{ kg m}^{-3} \)), instead of air, it is clear these have a far more pronounced influence on basin shape.

**Models for rifted basins**

Basement topography inferred from seismic-reflection profiles reveals that rifted basins are characterized by block structures, intersected by steeply dipping faults, with spacing in the order of 10 km (Rosendahl 1987). Generally basins form at a larger scale than these individual rift blocks. At their deep sides, the basins are bounded by so-called border faults (Versfelt & Rosendahl 1987), which, relative to the other rift block faults, show markedly greater fault displacements. The displacements vary along strike of the border faults and in map view they have a slightly or pronounced arcuate shape (Rosendahl 1987). In this sense, the most elementary basin geometry is a half graben, in which only one side of the basin is bounded by a border fault. More complex basin geometries are the result of linkage of a number of border faults and associated half grabens. 2-D modelling studies (Marsden et al. 1990), suggest that during basin formation, the upper crustal rift blocks behave as an elastic entity, floating on a lower crustal and/or subcrustal ductile substratum, and are dissected by the planarrift block faults. Owing to the planar (domino) style of faulting, basement-titting patterns are indicative for the actual flexural response of the upper crust in rifted basins. In addition, as pointed out by Versfelt & Rosendahl (1989), rift-block faults, outside the border faults, play a passive role, and merely accommodate deformation. Consequently, curvatures of basement topography in between border faults are expected to reflect the flexural characteristics of upper crust in the wavelength domain. Evident basin features in this respect are convex curvatures of basement topography, which are commonly referred to as Low Relief Accommodation Zones or Interference Accommodation Zones (Rosendahl 1987; Versfelt & Rosendahl 1989).

In view of the previous concepts, we analyse the northern part of Lake Tanganyika (Fig. 12a), which forms part of the north–south-trending Western Rift System in east Africa, formed mainly under E-W extensional-stress conditions, with minor components of oblique slip (Delvaux et al. 1992; Ring, Betzler & Delvaux 1992). We choose this area, because it represents a number of typical features of 3-D basin architecture (Rosendahl 1987). For a 3-D model (Fig. 12b), we adopt a 60 \( \times \) 80 grid (spacing \( \Delta x = \Delta y = 5 \text{ km} \)) and similar boundary conditions for the edge of the model as in the previous models. In the central part of the model we included a number of line-wise discontinuities, representing main boundary faults. Displacements along the minor rift-block faults are neglected. Throw along the boundary faults have been constrained from water depth and sediment thickness derived from TWTT seismic data (Morley 1988; Rosendahl, Kilembe & Kaczmarick 1992; Fig. 12a), adopting a low mean seismic velocity of 2.75 km s\(^{-1}\) for sediments given by Morley (1988). In the
Figure 10. Single-fault 3-D finite-difference solution for fault heave of 10 km (see Section 5 for discussion of parameters). (a) Deflections in the absence of erosion and sedimentary infill, and neglecting loads related to thinning or thickening of the lithosphere (contour interval 250 m). At the position of the fault, \( w_{\text{max}} \) is adopted for the deflection value. (b) Comparison of finite-difference results in (a) to analytical solution of Vening-Meinesz (1950). Deflections are scaled to half the fault heave = 5 km. Horizontal coordinates \( x \) are scaled to the flexural parameter \( \alpha \), in this case equal to 35.2 km.
Figure 11. Configuration with single fault. 3-D finite-difference solution for fault heave of 10 km, (see text for discussion of parameters) taking into account effects of lithospheric deformation and sediment infill. (a) $\beta$ distribution corresponding to fault dip $\Theta = 60^\circ$ and radius $r = 100$ km, used for $r_1$ in the spherical cap function. (b) E-W sections of finite-difference solutions, taking into account the thinning of the lithosphere for various radii $r$, and the effects of infill of depressions with water ($\rho = 1.0 \times 10^3$ kg m$^{-3}$) or sediment ($\rho = 2.3 \times 10^3$ kg m$^{-3}$).
LAKE TANGANYIKA (NORTH)

Figure 12. Basin configuration and geometry of 3-D flexure model for northern Lake Tanganyika. (a) Structure of the northern half of Lake Tanganyika, showing the basic architecture of half-graben and accommodation zones (after Morley 1988; Rosendahl et al. 1992). Bold solid lines mark main border faults. Top-sediment and top-basement reflectors inferred from seismic sections are depicted. Interference accommodation zones are indicated by INAZ. (b) 3-D finite-difference fault-model geometry (300 × 400 km, grid spacing Δx = Δy = 5 km) for the northern half of Lake Tanganyika, corresponding to basin architecture depicted in (a). Bold solid lines denote main boundary faults.

Migration procedure effects of moderate dip of reflectors have been neglected. Additional amounts of throw along the main boundary faults are reflected by present-day topographic elevations of foot-wall blocks bounding Lake Tanganyika, which range up to 1000 m or more above lake level. In the absence of data on erosion of foot-wall blocks, precise estimates of this component cannot be made.

To incorporate effects related to crustal thinning, we adopt fault angles of 60°, and assumed a crustal thickness of 40 km, with values that agree with data published by Ebinger, Karner & Weissel (1991). For the β distribution, according to eq. (19), we assume a radius r1 of 50 km. It is assumed that depressions (w < 0) are filled with low-density material (ρ = 2.0 × 103 kg m−3), which accounts for water and sediment infill, whereas no erosion takes place of basement culminations (w ≥ 0).

Starting from minimum-throw estimates given by seismic data we increased throw to fit migrated basement depths for four seismic sections (Figs 12b and 13). Fig. 13 shows flexural-response results for different EET values for a fixed-throw distribution which agrees closely to observed basement depths. The β distribution corresponding to these throw values is depicted in Fig. 14(a). The 3-D flexural response for an EET of 5 km is shown in Figs 14(b) and (c). Calculated deflection patterns in Fig. 13 show that only for EET = 3 km, an interference accommodation zone is predicted in profile 206, marked by a convex shape in basement tilting. For larger values of EET, the flexural wavelength is too large to predict a convex shape. Also for the more southern lines, the lateral variations of basement tilting best fit to EET = 3 km. From a comparison of amplitudes of calculated deflection with observed ones, the best fitting EET values tend to be higher: 5 km for profile 210 and 7 km for profile 212. These values differ from the EET values derived from the flexural wavelength. We note that observed and modelled deflection amplitudes do not agree with the small-deflection approximation (w ≪ h) for a thin elastic plate. Therefore, it is likely that progressive plate bending involved large amounts of plastic deformation, caused by stresses exceeding yield strength of the lithosphere. Since the distribution of plastic deformation is controlled by the elastic flexural wavelength prior to yielding, it is likely that the present-day basement deflection patterns correspond to a low EET value (3–5 km) for upper crustal deformation. Deviation from elastic behaviour during progressive plastic deformation may have led to the observed difference between low EET values fitting flexural wavelength and higher values fitting amplitude.

The EET = 5 km results (Fig. 14) show that the predicted
basin shape corresponds relatively well to the complete set of seismic sections (Fig. 12a) in the wavelength domain. However, our model fails to predict present-day submersion of central horst block in seismic sections 224, 214 and 216. This strong discrepancy can probably be explained by the fact that the model is not taking into account erosional processes and displacements along adjacent minor faults (Fig. 12a).

The modelling results indicate that the EET for the Western African Rift System in the northern part of Lake Tanganyika is characterized by a low value of about 3–5 km. This value is much less than EET values of about 21 km obtained by Ebinger et al. (1991) for the same area. Their modelling study focused mainly on fitting gravity data on the flanks outside the Tanganyika Basin, neglecting dynamical aspects of planar faulting and associated basement tilting.

Figure 13. Sections of 3-D flexure models of northern Lake Tanganyika for three values of EET (3, 5 and 7 km) and a fixed-throw distribution, which agrees closely to observed basement depths in seismic sections 206, 210, 212 and 218. Locations of sections are indicated in Fig. 12(b).
patterns within the Basin. We, therefore, propose that our EET value of 3–5 km accounts only for upper crustal basement faulting, whereas Ebinger et al. (1991) high EET estimates are probably related to flexure of strong deep crustal or subcrustal lithosphere on a regional scale. The implied decoupled behaviour of both elastic cores is most likely facilitated by a relatively ductile lower crustal level.

**DISCUSSION AND CONCLUSION**

We have adopted 3-D continuum equations for Kirchoff thin-plate flexure for finite-difference formulations, including varying EET and fault-like discontinuities. As a result of the advent of fast and large memory computers, these finite-difference formulations can be successfully used in the analysis of basin-scale flexure characteristics. Typical model runs take less than 10 min on a Sparc workstation. Model results for varying EET and faults closely agree with analytical and 2-D finite-difference solutions. In addition, we have shown for the Lake Tanganyika area that our 3-D fault model is well capable to predict basement geometries, typically observed in rifted basins. Adopting planar faults, our results suggest that upper crustal flexure is marked by low values for EET, in the order of 3–5 km. From a large number of flexure studies (Buck 1988; Stein, King & Rundle 1988; Kusznir & Ziegler 1992), it has become clear that low EET values, explaining basement tilting patterns due to planar faulting, are not only restricted to the Lake Tanganyika rift setting. Visual inspection of 3-D basement tilting patterns in other areas, such as the Baikal Rift (Hutchinson et al. 1992) and the Tucano Rift Basin in Brazil (Karner, Egan & Weissel 1992), also reveals low EET values for upper crustal flexural behaviour. In the latter basin, Karner et al. (1992) showed that regional basin subsidence and gravity anomalies are characteristic for a high value for EET (in the order of 30 km). In this case, the superposition of low EET and high EET effects on basin evolution strongly supports a multilayer, partly decoupled flexural model for rifting. This opposes the idea that flexural behaviour of the continental lithosphere is fully characterized by a one-layer model (Kusznir et al. 1991), in which higher EET values are only laterally reduced to low EET values, due to significant amounts of plastic bending.

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