CFT for Closed String Tachyon Condensation

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We construct a class of CFTs which describe space-dependent closed string tachyon backgrounds as the IR limit of GLSMs in which the FI-parameter is promoted to a superfield. The entire process of tachyon condensation is described by a single CFT. We apply this construction to several examples, in which the target space is deformed drastically, and the dilaton background may vary, as a tachyon condenses.

§1. Introduction

The understanding of closed string tachyon condensation has been gradually progressing. In recent studies, it has been recognized that there is a class of tachyons whose condensation can be understood rather easily. This is a class in which tachyons are localized in a non-compact space. A typical and well-studied example of such a tachyon is that appearing in string theory on a non-compact and non-supersymmetric orbifold.\(^1\)

The condensations of other types of tachyons have been studied recently in Refs. 2)–4). One property which seems to be common among these tachyons is that they inevitably couple to the dilaton, and the endpoints of their condensations have linear dilaton backgrounds.\(^5,6,2\) Analysis of this phenomenon has been done using a spacetime effective action which is valid when the size of the string is negligible, but the \(\alpha'\)-corrections are not always small, especially when the tachyon mass squared is of order \((\alpha')^{-1}\). There is also an analysis employing string field theory,\(^7\) and one employing worldsheet RG flows.\(^8\)

In this paper, we construct \(N = 2\) SCFTs which describe an \(\alpha'\)-exact background of string theory, in which a tachyon varies along a spatial direction. They are obtained as the IR limit of gauged linear sigma models (GLSMs).\(^9\) A GLSM is used in Ref. 10) to study non-supersymmetric orbifolds, for which GLSM provides control of the RG flow that is assumed to describe the corresponding tachyon condensation. Our construction is, roughly speaking, to promote the RG scale to that of the target space coordinate, and to make the entire tachyon background on-shell. We show that there is such a CFT in which the dilaton gradient varies as the tachyon varies. This can be regarded as an explicit realization of the claim made in Refs. 2) and 8). Throughout this paper, we focus only on the tree-level string theory, and possible problems connecting the strong coupling region will, hopefully, be treated elsewhere.

This paper is organized as follows. In §2, we briefly review,\(^10\) and then we explain how to obtain an on-shell background from the corresponding RG flow. Some applications of our construction to tachyon condensation are discussed in §3. The examples include that of a vanishing target space and the pinching off of a cylinder.
investigated in Ref. 11). Section 4 deals with tachyon condensation in which the dilaton varies among the tachyons. We comment on the properties of our CFT in §5. Section 6 is devoted to discussion. Our conventions for the superfields and a detailed construction of \( N = 2 \) superconformal algebra, which is a review of Refs. 12–14) in a slightly more general case, are given in the appendices.

§2. On-shell tachyon condensation in GLSM

2.1. Tachyon condensation and RG flow

First, let us recall the description of closed string tachyon condensation provided by a GLSM.\(^{10}\) This GLSM is a \( U(1) \) gauge theory in two dimensions with \((2, 2)\) supersymmetry. Suppose that there are \( n \) chiral superfields \( \Phi_i \) whose \( U(1) \) gauge symmetry charges are \( q_i \). The Lagrangian of this GLSM is

\[
L = \frac{1}{2\pi} \int d^4\theta \left( \sum_{i=1}^{n} \bar{\Phi}_i e^{2q_i V} \Phi_i - \frac{1}{2e^2} \bar{\Sigma} \Sigma \right) - \frac{1}{2\pi} \int d^2\bar{\theta} \bar{t} \Sigma + \text{h.c.},
\]

where \( \Sigma = \bar{D}_+ D_- V \) is a twisted chiral superfield constructed from \( V \), and \( t = t_R + it_I \) is a complex parameter. (For conventions, see Appendix A.)

The classical vacua are determined by the D-term potential. The minima of the potential correspond to the solutions of the equation

\[
\sum_{i=1}^{n} q_i |\phi_i|^2 = 2t_R,
\]

where the quantities \( \phi_i \) are the lowest components of \( \Phi_i \). In the \( e \rightarrow \infty \) limit, fluctuations perpendicular to the vacuum manifold given in (2.2) become infinitely heavy, and the GLSM approaches a non-linear sigma model whose target space is the vacuum manifold (2.2).

To see the geometry of (2.2) in more detail, suppose that \( q_1, \ldots, q_l \) are positive, and the rest of the \( q_i \) are negative. Then, if \( t_R \) is negative, all of the quantities \( \phi_{l+1}, \ldots, \phi_n \) cannot vanish simultaneously. These can be regarded as homogeneous coordinates of the weighted projective space \( \mathbb{WCP}^{q_l+1, \ldots, q_n} \). The total space of (2.2) is a non-compact bundle over this \( \mathbb{WCP}^{q_l+1, \ldots, q_n} \), which is topologically equivalent to \( \mathbb{C}^l \times (\mathbb{C}^{n-l} \setminus \{\bar{0}\}) / \sim \), where the identification is defined as follows:

\[
(\phi_1, \ldots, \phi_n) \sim (\lambda^{q_l} \phi_1, \ldots, \lambda^{q_n} \phi_n), \quad \lambda \in \mathbb{C}^*.
\]

If \( t_R \) is positive, then (2.2) determines a similar non-compact bundle over \( \mathbb{WCP}^{q_1, \ldots, q_l} \).

Note that if all \( q_i \) are positive (negative), then there is no solution to (2.2) when \( t_R \) is negative (positive). In these cases, there are no supersymmetric vacua for such \( t_R \) regions.

The GLSM is super-renormalizable, and thus the renormalization procedure is very simple. In fact, all divergences are canceled by the shift of \( t_R \) as

\[
t_R(\Lambda) = t_R + \frac{1}{2} \sum_{i=1}^{n} q_i \log \Lambda,
\]
where $\Lambda$ is a UV cutoff scale. If $\sum q_i > 0$, then $t_R(\Lambda)$ decreases monotonically as the scale $\Lambda$ decreases, and vice versa. Recalling that the shape (even the topology) of the vacuum manifold (in other words, the target space of the corresponding non-linear sigma model) depends on the sign of $t_R$, it is concluded that the target space varies drastically along the RG flow.

To determine what is described by this RG flow, it is convenient to dualize $\Phi_i$, following Ref. 15). The twisted superpotential of the dual theory is

$$\tilde{W} = \Sigma \left[ \frac{1}{2} \sum_{i=1}^{n} q_i Y_i - t \right] + \mu \sum_{i=1}^{n} e^{-Y_i}, \quad (2.5)$$

where the quantity $Y_i$ are the twisted chiral superfields dual to $\Phi_i$, and $\mu$ is a scale parameter related to $\Lambda$ and $t$. The superpotential is absent in the dual theory. Now, $\Sigma$ can be integrated out, and this results in the constraint

$$\sum_{i=1}^{n} q_i Y_i - 2t = 0. \quad (2.6)$$

Let us consider the simple case in which all $q_i$ other than $q_n = -q$ are positive and $\sum_{i=1}^{n} q_i < 0$. Then the twisted superpotential, after integrating out $\Sigma$ is

$$\tilde{W} = \mu \sum_{i=1}^{n-1} e^{-Y_i} + \mu e^{\frac{2t}{q}} \prod_{i=1}^{n-1} e^{-\frac{q}{q} Y_i}. \quad (2.7)$$

Let us define $u_i = e^{-\frac{1}{q} Y_i}$. Then $\tilde{W}$ is

$$\tilde{W} = \mu \sum_{i=1}^{n-1} u_i^q + \mu e^{\frac{2t}{q}} \prod_{i=1}^{n-1} u_i^q. \quad (2.8)$$

In the UV limit ($t_R \to -\infty$), the second term on the RHS vanishes. Naively, it is expected that this limit of the GLSM can be described by an $N = 2$ SCFT which is specified by $\tilde{W}|_{t \to -\infty}$ (up to an orbifolding). The second term on the RHS of (2.8) represents a perturbation by a tachyon background which grows as the energy decreases ($t_R \to +\infty$). In fact, the $U(1)$ charge reveals that $\prod_{i=1}^{n-1} u_i^q$ is relevant, because $\sum_{i=1}^{n-1} q_i < q$ by assumption. Therefore, we naturally expect that the GLSM can describe an RG flow that is induced by a tachyon perturbation. The endpoint of the RG flow is expected to describe a background in which the tachyon condenses, and, hence, the target space is drastically deformed by the tachyon condensation, even at the level of topology. This approach to closed string tachyon condensation has been applied to non-supersymmetric orbifolds. (See also the review paper.)

Note that for the choice $\sum q_i > 0$, the operator $\prod_{i=1}^{n-1} u_i^q$ is irrelevant, while for this choice, $t_R \to -\infty$ is the IR limit, and thus the irrelevant perturbation decays as $\Lambda$ decreases, as should be the case. However, in this case, it is not obvious which tachyons would induce this RG flow.

\footnote{The natural variables are the $Y_i$, not the $u_i$ themselves. Therefore, the UV limit is not a product of minimal models.}
2.2. Promoting $t$ to a field

The off-shell analysis based on the RG flow seems to be in contradiction with the c-theorem with respect to generic tachyons that are not localized in a non-compact space. In such a case, for example that of a tachyon localized in a compact space, there is no reason for the c-theorem not to be applicable. It follows that the central charge must decrease along the RG flow induced by the tachyon, and therefore the consistency of string theory comes into question. It may be thought that the endpoint of such a tachyon condensation is a non-critical string theory, which is possible when a non-trivial dilaton background is induced by the tachyon condensation. One mechanism that can accomplish this is discussed in Ref. 6).

Some recent studies 2), 6) and 8) focus on on-shell processes of closed string tachyon condensation. Because they are time-dependent processes, understanding them is usually difficult. However, if it is possible to analyze an on-shell process of tachyon condensation, the endpoint of the condensation can be identified as a state in the original string theory.

In this paper, we show that it is possible to construct an exact CFT background in which a tachyon varies along a spatial direction, not the temporal one. It would be very interesting if this spatial direction could be Wick-rotated, enabling us to study the corresponding time-dependent processes. The construction is a rather straightforward generalization of that in Ref. 10). It is realized by promoting the parameter $t$ to a twisted chiral superfield $Y$; that is, the Lagrangian becomes

$$L = \frac{1}{2\pi} \left[ \int d^4 \theta \left( \sum_{i=1}^{n} \bar{\Phi}_i e^{2q_i V} \Phi_i - \frac{1}{2e^2} \bar{\Sigma} \Sigma - kY Y \right) - \int d^2 \bar{\theta} \ Y \bar{\Sigma} - \int d^2 \bar{\theta} \ \bar{Y} \Sigma \right].$$  

Thus, the shift (2.4) due to the RG flow becomes an additive renormalization of $Y$, and therefore the total Lagrangian remains intact. The dual theory of (2.9) is almost the same as in the previous case, and in particular, the twisted superpotential is (2.8), with $t$ replaced by $Y$. The term $e^{\frac{2}{n} Y} \prod_{i=1}^{n-1} q_i^{q_i}$ in the twisted superpotential can be regarded as a “dressed” operator, which varies along the $Y$-direction. Therefore, the target space of (2.9) should describe an on-shell background in which a tachyon varies along the $Y$-direction. For one regime of $Y$, (2.9) describes a background without a tachyon, and for the other regime, it describes an endpoint of the tachyon condensation.

Note that the GLSM whose Lagrangian is given in (2.9) is the mirror dual of that discussed in Ref. 14). (See also Ref. 17) for its application to tachyon condensation.) One advantage of introducing $Y$ is that when this is done, the whole process of tachyon condensation can be described by one CFT, which is obtained by taking the IR limit of the GLSM. In arguments based on RG flows, it can only be shown that a CFT, which is an IR fixed point of a flow, is a possible endpoint of a tachyon condensation. In our case, a tachyon varies along a spatial direction, with the on-shell condition remaining satisfied, since our model is a CFT that applies everywhere, not only in an asymptotic region. Therefore, the corresponding state with a non-trivial tachyon is connected to the original tachyonic vacuum via a physical process in string theory. It is also interesting that $Y$, along which the tachyon varies, is similar to a
Liouville field. This may be related to our previous works.²,⁸

In addition, \( N = 2 \) superconformal symmetry is realized in our model. The \( N = 2 \) algebra has a \( U(1) \) current which is a combination of \( U(1)_V \) and \( U(1)_A \) symmetries in the GLSM. By regarding the superfields in two dimensions as those obtained from four-dimensional ones through the dimensional reduction, \( U(1)_V \) and \( U(1)_A \) are both found to have their origins in four dimensions: \( U(1)_V \) comes from the R-symmetry, and \( U(1)_A \) is a rotation in the plane which is dimensionally reduced. Since \( U(1)_A \) symmetry has a chiral anomaly in the GLSM (2.1) in the case \( \sum_i q_i \neq 0 \), it is not obvious whether \( N = 2 \) symmetry is realized even in the IR limit, which seems to be generally expected.

When the FI-parameter \( t \) is promoted to \( Y \), the chiral anomaly can be canceled. The chiral anomaly appears for the transformation of fermions expressed by

\[
\psi_\pm \to e^{\mp ia} \psi_\pm,
\]

under which the path-integral measure is not invariant, and the variation of the measure is

\[
\mathcal{D}\psi\mathcal{D}\bar{\psi} \to \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left[ -\frac{iq}{\pi} \int d^2x \, a v_{01} \right],
\]

where \( q \) is the gauge charge of \( \psi_\pm \). All fermions in \( \Phi_i \) give this variation with appropriate charges. All of them can be canceled if the chiral transformation is accompanied by the shift

\[
Y \to Y + ia \sum_{i=1}^{n} q_i,
\]

because the twisted superpotential in terms of component fields includes

\[
\frac{1}{\pi} \int d^2x \, y v_{01},
\]

where \( y = y_R + i y_I \) is the lowest component of \( Y \). Note that the above term indicates that \( Y \) is periodic with period \( 2\pi i \).

In fact, one can construct generators of the \( N = 2 \) algebra acting on a subspace of the full Hilbert space of the CFT, following Refs. 12)–14). The details of the construction of the generators are given in Appendix B. The ambiguity in the choice of the \( U(1) \) current mentioned there can be fixed for the GLSM (2.9). Because the GLSM can be described by a non-linear sigma model in the IR limit, the \( U(1) \) current of the \( N = 2 \) algebra should be just a fermion number current. (The contribution from \( Y \) can include a shift in the \( y_I \)-direction.) Therefore, one should choose \( p_i = 0 \), since the lowest component of \( J^i_2 \) generates a transformation of \( \phi_i \) that may act non-trivially on the target space. As a result, the central charge of this CFT is found to be

\[
c = 3(n - 1) + 3 \left( 1 + k \gamma^2 \right),
\]

as expected from the dimensionality of the target space (plus the linear dilaton background).
In this section, we present explicit examples of GLSMs describing space-dependent tachyon backgrounds.

The first example is (2.9) with \( n = 1 \). This is the mirror dual of the GLSM studied in Ref. 14), in which it is shown that this GLSM is equivalent (the mirror dual) to both the \( N = 2 \) Liouville theory and the \( SL(2, \mathbb{R})/U(1) \) coset. The latter CFT describes a Euclidean black hole in two dimensions,\(^{18}\) and its geometry is a semi-infinite cigar-like one. The shape of this target space can be seen from the classical vacuum manifold determined by

\[ |\phi|^2 = 2y_R. \tag{3.1} \]

For simplicity, we have chosen \( q = 1 \). For a positive \( y_R \), (3.1) has a solution, and its shape is locally that of a cylinder. For a negative \( y_R \), however, there is no supersymmetric solution. Therefore, the cylinder in the \( y_R > 0 \) region is “capped” at \( y_R = 0 \), resulting in the cigar geometry.

The twisted superpotential of the dual theory is

\[ \tilde{W} = e^{-2Y}. \tag{3.2} \]

This is the \( N = 2 \) Liouville potential. It can also be regarded as a tachyon vertex operator dressed by the Liouville field. If this operator is interpreted as a space-dependent tachyon background, it represents a tachyon that grows as \( Y \to -\infty \). Thus, the original GLSM, which is a non-linear sigma model on the cigar, indicates that the target space “disappears” where the tachyon condenses.

The situation is similar for a general \( n \) case with \( q_i > 0 \). The target space is asymptotically \( WCP_{q_1, \ldots, q_n} \times \mathbb{R} \times S^1 \), which terminates at \( y_R = 0 \). The mirror dual theory has a tachyon background which grows where the target space disappears.

The second example is the GLSM with \( n = 2 \) and \( q_1q_2 < 0 \).

For simplicity, let us assume that \( q_1 = 1 \) and \( q_2 = -2 \). Then, the corresponding twisted superpotential is

\[ \tilde{W} = e^{-Y_1} + e^Y e^{-\frac{1}{2}Y_1}. \tag{3.3} \]

This can be regarded as describing a small tachyon perturbation in the limit \( Y \to -\infty \). Since there is a Liouville potential for \( Y_1 \), the region with large negative \( Y_1 \) is not accessible from the asymptotic \( Y_1 \to +\infty \) region. In terms of the original variable, \( Y_1 \to -\infty \) corresponds to \( \Phi_1 \to 0.\)\(^{15}\)

The vacuum manifold is determined by

\[ |\phi_1|^2 - 2|\phi_2|^2 = 2y_R. \tag{3.4} \]

We fix the gauge by imposing the condition \( \phi_1 \geq 0 \). For the case \( y_R < 0 \), this condition is not appropriate, because \( \phi_1 = 0 \) is a possible solution of (3.4). However, \( \phi_1 = 0 \) is not physically relevant in this case, as mentioned above, and therefore, the gauge fixing condition \( \phi_1 \geq 0 \) is appropriate physically.

For \( y_R < 0 \), the target space is half of a one-sheeted hyperboloid, topologically a cylinder, and its radius, \( |\phi_2| \), decreases as \( \phi_1 \) decreases. Also the minimum value of
the radius decreases as $|y_R|$ decreases. The tachyon vertex $e^{-\frac{1}{2}Y_1}$ in (3.4) is localized near $\phi_1 = 0$ and its amplitude $e^{Y}$ grows with $Y$. By contrast, for $y_R > 0$, the target space is half of a two-sheeted hyperboloid, topologically a plane. This can be understood as the thin part of the one-sheeted hyperboloid being pinched off by tachyon condensation. This is the phenomenon discussed in Ref. 11).

It seems very interesting that the GLSM discussed here is the same as that employed in Ref. 17) to investigate the twisted circle geometry, but it is analyzed here with a different gauge choice, and the $Y$ direction is regarded here as a physical coordinate.

§4. GLSM with superpotential

To this point, we have studied GLSMs without a superpotential and their application to closed string tachyon condensation. In this section, we turn our attention to GLSMs with a superpotential.

One example which we consider is the GLSM (B.1) with $n = 2$, whose superpotential is

$$W = \Phi_1^{n_1} \Phi_2^{n_2},$$

(4.1)

where $n_1, n_2 > 3$ are integers. This superpotential must be compatible with the gauge symmetry, which implies

$$n_1 q_1 + n_2 q_2 = 0.$$

(4.2)

For definiteness, we assume $q_1 > 0 > q_2$ and $q_1 + q_2 < 0$.

The D-term condition is

$$q_1 |\phi_1|^2 + q_2 |\phi_2|^2 = 2 y_R.$$

(4.3)

For $y_R < 0$, $\phi_2$ cannot vanish, while for $y_R > 0$, $\phi_1$ is non-zero. There is also the F-term condition, which implies $\phi_1 = 0$ for $y_R < 0$ and $\phi_2 = 0$ for $y_R > 0$. Summarizing, we have

$$\phi_1 = 0, \quad q_2 |\phi_2|^2 = y_R < 0,$$

(4.4)

$$\phi_2 = 0, \quad q_1 |\phi_1|^2 = y_R > 0.$$

(4.5)

Then the vector superfield becomes massive through the Higgs mechanism. For $y_R < 0$, for example, we have

$$\bar{\Phi}_2 e^{2q_2 V} \Phi_2 = |\Phi_1|^2 + 2q_2 |\Phi_2|^2 V^2 + q_2 |\Phi_2|^2 V^2,$$

(4.6)

where the second term on the RHS cancels the term coming from the twisted superpotential. Here $\Phi_1$ and $Y$ remains massless. The superpotential for $\Phi_1$ is now

$$W = a' \Phi_1^{n_1}.$$

(4.7)

Therefore, around a vacuum with $y_R < 0$, in the low energy limit, the GLSM describes a CFT which consists of the $N = 2$ minimal model specified by (4.7) and
a free field, with a possible linear dilaton background. Similarly, around a vacuum with \( y_R > 0 \), the low energy limit is the product of another \( N = 2 \) minimal model and a free field.

Because there remains a residual gauge symmetry, the minimal model is in fact an orbifold. Consider the \( y_R < 0 \) case. In this case, \( \phi_2 \) has a non-zero vev, and therefore the gauge symmetry is broken to a discrete symmetry whose action on \( \Phi_1 \) is

\[
\Phi_1 \rightarrow e^{2\pi i \frac{q_1}{n_2}} \Phi_1 = e^{-2\pi i \frac{n_2}{n_1}} \Phi_1.
\]

If \( n_1 \) and \( n_2 \) are coprime, the minimal model is orbifolded by \( \mathbb{Z}_{n_1} \); otherwise the orbifold group is a subgroup of \( \mathbb{Z}_{n_1} \).

The dual theory may provide a more detailed description of the model. The mirror dual of a GLSM with a superpotential is discussed in Ref. 15). Applying the technique in Ref. 15) to our case, we obtain the dual superpotential

\[
\tilde{W} = X_1^{q_2} + e^{\frac{2}{n_2} Y} X_1,
\]

in the \( Y \rightarrow -\infty \) limit and

\[
\tilde{W} = X_2^{q_1} + e^{-\frac{2}{n_1} Y} X_2,
\]

in the \( Y \rightarrow +\infty \) limit. (See Appendix C for their derivations.)

The important difference between the cases discussed here and in §2 is that in the present case, the fundamental variables are \( X_1 \) and \( X_2 \), not \( \log X_1 \) and \( \log X_2 \). Therefore, the dual theory is actually a minimal model plus a free field, with an additional term that depends exponentially on \( Y \) and vanishes in the \( Y \rightarrow \pm \infty \) limits.

The minimal model appearing in (4.9) does not differ from (4.7), as the gauge invariance condition (4.2) is solved as

\[
q_1 = mn_2, \quad q_2 = -mn_1,
\]

where \( m \) can be absorbed through a rescaling of the gauge coupling \( e \).

Recall our assumption \( q_1 + q_2 < 0 \). This implies \( n_1 > n_2 \) which means that the central charge coming from the minimal model decreases as a result of the tachyon condensation, as should be the case.

One can construct an \( N = 2 \) superconformal algebra acting on a subspace of the Hilbert space. (See Appendix B for details.) The presence of such an algebra strongly suggests that the IR limit of the GLSM is actually an \( N = 2 \) SCFT with central charge

\[
c = 3(1 - p_1) + 3(1 - p_2) + (-3) + 3\left(1 + k \gamma^2\right),
\]

where \( p_1 \) and \( p_2 \) are the \( U(1)_V \) charges of \( \Phi_1 \) and \( \Phi_2 \), respectively.

It should be emphasized that the construction of the superconformal algebra does not depend on a particular limit, for example \( Y \rightarrow \pm \infty \). This implies that the IR limit is a CFT describing a background with the following properties. This background has one spatial direction corresponding to \( y_R \). There is a non-trivial
“tachyon field” whose magnitude increases as \( y_R \to -\infty \), and takes the form of a “massive field” whose magnitude decreases as \( y_R \to +\infty \). The existence of such a CFT is very interesting, because it can be regarded as an \( \alpha' \)-exact version of a solution to the equation of motion of classical string theory, which is discussed in Refs. 2) and 5).

Recall that the central charge coming from the minimal model part decreases after tachyon condensation. Because the total central charge must be the same for both the \( y_R \to +\infty \) and the \( y_R \to -\infty \) regions, which are parts of the same theory, the central charge coming from the \( Y \) field must increase after the tachyon condensation. Explicitly, the central charge of \( Y \) for \( y_R \to -\infty \) is

\[
3 + 6Q^2_{-\infty} = c - \frac{3(n_1 - 2)}{n_1},
\]

(4.13)

where \( c \) is given in (4.12), while for \( y_R \to +\infty \), it is

\[
3 + 6Q^2_{+\infty} = c - \frac{3(n_2 - 2)}{n_2}.
\]

(4.14)

This means that the dilaton gradient \( Q_{\pm \infty} \) varies as the tachyon condenses. In fact, this has been observed in the analysis of a solution of a low energy effective theory\(^2\),\(^5\),\(^6\) and also in Ref. 8).

Another example is the GLSM with

\[
W = \Phi_0 G(\Phi_1, \cdots, \Phi_n),
\]

(4.15)

where \( G(x_1, \cdots, x_n) \) is a quasi-homogeneous polynomial satisfying

\[
G(\lambda^{q_1} x_1, \cdots, \lambda^{q_n} x_n) = \lambda^{-q_0} G(x_1, \cdots, x_n),
\]

(4.16)

for \( \lambda \in \mathbb{C}^* \). It is also required that \( G(x_1, \cdots, x_n) \) be regular; that is, \( G = 0 \) and \( \partial_i G = 0 \) have no common solution. The charges are assumed to satisfy the relation

\[
q_1, \cdots, q_n < 0 < q_0, \quad \sum_{i=1}^{n} |q_i| > q_0.
\]

(4.17)

The IR limit in the \( y_R \to -\infty \) region is analogous to the Calabi-Yau phase.\(^9\) More precisely, that region is described by a non-linear sigma model with target space \( M \times \mathbb{R} \) where \( M \) is the hypersurface \( G = 0 \) in \( WCP_{q_1, \cdots, q_n} \). The \( y_R \to +\infty \) region is described by a LG orbifold with superpotential \( W = G(\Phi_1, \cdots, \Phi_n) \). Because an \( N = 2 \) superconformal algebra can be constructed in the previous example, it is reasonable to expect that there is an \( N = 2 \) SCFT which interpolates between the above two theories. This CFT may describe the decay of \( M \) into an LG orbifold via tachyon condensation. This may be an example in which the decay of a compact manifold results in a final state different from “nothing”.

The dual superpotential is, however, not what would be expected. Following Ref. 15), we obtain the dual superpotential

\[
\tilde{W} = \sum_{i=1}^{n} X_i^{q_0} + e^{-\frac{2}{q_0}} \prod_{i=1}^{n} X_i^{\left|q_i\right|}.
\]

(4.18)
This is suitable for the $Y \to +\infty$ limit, but this limit should be described by a different superpotential, as mentioned above. It would be very interesting to obtain the better understanding of this situation.

§5. $U(1)_V$ charges, the central charge and gauge symmetry

We have studied various GLSMs in which $\gamma = \sum_i q_i \neq 0$ and the FI-parameter is promoted to a twisted chiral superfield. These kinds of models have certain properties that are absent in GLSMs with $\gamma = 0$.

It might seem strange that the central charge (B.35) depends on the $U(1)_V$ charges $p_i$ of $\Phi_i$. The definition of $p_i$ is ambiguous, since the $U(1)_V$ current can be modified by adding the gauge current. For this reason, $p'_i = p_i + mq_i$ can also be regarded as $U(1)_V$ charges. However, this modification does not keep the central charge (B.35) fixed, unless $\gamma = 0$. It might be thought that this is related to the fact that the gauge current

$$j_g = \sum_{i=1}^{n} q_i \left[ 2iD_\bar{\phi}_i \phi_i - \bar{\psi}_i \psi_i \right],$$

which acts on the $\bar{Q}_+\text{-closed subspace}$, is not a primary field,

$$T(x)j_g(0) \sim \frac{i \gamma}{(x^-)^3} - \frac{1}{(x^-)^2} j_g(0) - \frac{1}{x^-} \partial_- j_g(0).$$

It is known that the presence of the $(x^-)^{-3}$ term is a signal of the appearance of the mixed anomaly,

$$\partial_\mu j^\mu_g = aR^{(2)},$$

where $R^{(2)}$ is the scalar curvature of the worldsheet. However, contributions to the coefficient $a$ come not only from $\gamma$ but also from the coefficient of $(x^+)^{-3}$ term of the $\bar{T}j_g$ OPE. By exchanging $-$ and $+$, one can similarly construct the $N = 2$ algebra acting on the $\bar{Q}_-\text{-closed subspace}$ of the Hilbert space, and their OPEs are the same as those we have discussed. In particular, the coefficient of $(x^+)^{-3}$ in the $\bar{T}j_g$ OPE is $i\gamma$. As $a$ is proportional to the difference between these coefficients, it has been shown that the mixed anomaly is absent in our GLSM. Note that, of course, the zero mode of $j_g$ commutes with the generators of the $N = 2$ algebra, and therefore the gauge symmetry is preserved, although it is not promoted to an affine symmetry.

In the case $\gamma = 0$, $j_g$ is primary. Moreover, the modified operators

$$T_\beta = T + \frac{i}{4} \beta \partial j_g,$$

$$G_\beta = G,$$

$$\bar{G}_\beta = \bar{G} + i \beta \sum_{i=1}^{n} q_i \partial_\phi (\phi_i \bar{\psi}_i),$$

$$j_\beta = j - \frac{1}{2} \beta j_g.$$
form the \( N = 2 \) algebra with the same central charge. Therefore, the ambiguity in
the choice of the \( U(1)_V \) charges is absent in the \( \gamma = 0 \) case, in the sense that all
choices yield the same central charge. In fact, the dependence of such an ambiguity
in the central charge exists in ungauged chiral model. In Ref. 12), it is shown that
a two-dimensional theory with the Lagrangian

\[
L = \int d^4 \theta \sum_{i=1}^{n} \bar{\Phi}_i \Phi_i + \int d^2 \theta W(\Phi) \tag{5.8}
\]

reduces in the IR limit to an \( N = 2 \) SCFT with central charge

\[
c = 3 \sum_{i=1}^{n} (1 - 2\alpha_i), \tag{5.9}
\]

where the quantities \( \alpha_i \) are determined by

\[
W = \sum_{i=1}^{n} \alpha_i \Phi_i \partial_i W(\Phi). \tag{5.10}
\]

Therefore, if (5.8) possesses \( U(1) \) symmetry, that is, \( W \) also satisfies

\[
0 = \sum_{i=1}^{n} q_i \Phi_i \partial_i W(\Phi), \tag{5.11}
\]

then \( \alpha_i \) can be determined only up to \( q_i \), and the central charge cannot be fixed.

To understand the origin of this ambiguity, consider the following superpotential:

\[
W_{\xi,m} = \Phi_1^{m_1} \Phi_2^{m_2} + \xi \Phi_1^{m_1} \Phi_2^{m_2}. \tag{5.12}
\]

For this superpotential with \( \xi \neq 0 \), (5.10) uniquely determines \( \alpha_i \), unless \( n_1 m_2 = n_2 m_1 \). Then, the central charge depends on \( m_i \), but not on \( \xi \). One can take the
\( \xi \to 0 \) limit while keeping the central charge fixed, and thus various CFTs with
different central charges become degenerate at \( W = \Phi_1^{m_1} \Phi_2^{m_2} \), which we discussed in
the previous section. Therefore, the IR limit is not uniquely determined for such a
“degenerate” superpotential, and the \( U(1)_V \) charges \( \alpha_i \) (or \( p_i \) in our notation) are
not determined uniquely.

As mentioned above, any GLSM with a superpotential may have such an ambi-
guity, since the superpotential must preserve the gauge symmetry. It would be very
interesting to determine how to remove this ambiguity.

\section{6. Discussion}

We have investigated various GLSMs in which the FI-parameter is promoted
to a twisted chiral superfield \( Y \) and their application to closed string tachyon con-
densation. The presence of \( Y \) enables us to cancel a possible chiral anomaly, which
results in the possibility of having an \( N = 2 \) SCFT in the IR limit. It is interesting
that any RG flow described by a GLSM with an FI-parameter can be used to
construct an \( N = 2 \) SCFT. Because it is conformal, it can be used for a background of string theory, which is an \( \alpha' \)-exact solution of the equations of motion at the tree level. From the mirror description, at least some of them can be regarded as describing a background in which a tachyon grows along the \( y_R \)-direction. Therefore, the GLSM facilitates the construction of an \( \alpha' \)-exact description of on-shell tachyon condensation from the corresponding worldsheet RG flow. Using such GLSMs, we have described tachyon condensation in which a part of the target space disappears or the topology of the target space changes. In other cases, we have shown that the dilaton gradient varies as a tachyon condenses when the condensation decreases the central charge of a part of the system. The latter phenomenon has been observed in Refs. 2), 5) and 6) at the level of the spacetime effective theory, and in this paper we have shown that this is indeed the case also at the level of \( \alpha' \)-exact solutions. It is also interesting that the RG scale (the FI-parameter in the GLSM) is related to the field \( Y \) with regard to obtaining a consistent background for tachyon condensation, and the \( Y \) field actually is similar to the Liouville field. This can be understood as an explicit realization of the description of tachyon condensation proposed in Ref. 8).

It may seem curious that the explicit expression for the energy-momentum tensor (B.39) contains a simple linear dilaton background for \( y \), although we claimed that the dilaton gradient varies along \( y_R \). In fact, our claim is that a non-trivial dilaton appears after some degrees of freedom, becoming massive, are integrated out. Note that the contribution to the central charge from the vector superfield \( V \) is \(-3\), which is the right amount to cancel the contribution from a chiral superfield with zero \( U(1)_V \) charge. Therefore, the Higgs mechanism and the successive integrating out of massive fields does not change the total central charge if the \( U(1)_V \) charges of the fields are suitably chosen. For example, if one chooses \( p_1 = \frac{2}{n_1} \) and \( p_2 = 0 \) for the case (4.1), then in the \( y_R \to -\infty \) region, where \( \Phi_2 \) becomes massive, the contributions to the central charge from \( \Phi_2 \) and \( V \) cancel. Thus we can simply ignore these fields, and therefore the dilaton gradient is that which appears in the energy-momentum tensor (B.39). Then, what happens in the \( y_R \to +\infty \) region? As mentioned in §5, we cannot redefine the \( U(1)_V \) charges as \( p_1 = 0 \) and \( p_2 = \frac{2}{n_2} \) by using the gauge symmetry. Instead, we can regard the \( N = 2 \) algebra for \((p_1, p_2) = (\frac{2}{n_1}, 0)\) as the algebra for \((p_1, p_2) = (0, \frac{2}{n_2})\) “twisted” by the gauge current. For example, we have

\[
T_{p_1=\frac{2}{n_1}, p_2=0} = T_{p_1=0, p_2=\frac{2}{n_2}} + \frac{i}{4} \beta \partial - j_g,
\]

where \( \beta = \frac{2}{n_1 q_1} = -\frac{2}{n_2 q_2} \). The central charge coming from \( \Phi_1 \) through \( T_{p_1=0, p_2=\frac{2}{n_2}} \) is canceled by \( V \), but \( j_g \) also contributes to the central charge, due to the presence of the \((x^v)^{-3}\) term, and this part can be absorbed into the Liouville part by a non-trivial field redefinition, resulting in a dilaton gradient different from that in (B.39).

It is worth emphasizing again that for any RG flow described by a GLSM one can construct an \( N = 2 \) SCFT in which the RG scale is replaced with a spatial coordinate. Therefore, in principle, we have obtained various CFTs describing tachyon condensation of various kinds. Because it has on-shell backgrounds, it can indeed be realized in string theory. Several examples were discussed in this paper. It is expected that a
CFT for Closed String Tachyon Condensation

systematic study of these CFTs and the corresponding tachyon condensation would provide a deeper understanding of closed string tachyon condensation. In particular, it would help understanding which kinds of tachyons result in the disappearance of the target spaces and which do not. Note that, as mentioned at the end of in §4, not all CFTs obtained from the GLSM describe target space dynamics induced by tachyon condensation. It is very important to clarify this issue.

The relation between two-dimensional black holes[^18] and tachyon condensation, mentioned in §3, seems to be interesting. Naively, by Wick-rotating in the \( y_I \)-direction (not the \( y_R \)-direction), one might obtain the Lorentzian black hole background, and the region where the tachyon condenses is behind the horizon. Therefore, it is tempting to guess that inhomogeneous tachyon condensation would result in the formation of a black hole. Then, it might be possible to understand homogeneous tachyon condensation as a phenomenon in which the whole target space falls inside the horizon. Another interesting point of this relation is that a two-dimensional black hole has a matrix model description[^19]. It could be very interesting if such a matrix model technique could be used to analyze closed string tachyon condensation non-perturbatively. A non-perturbative analysis of closed string tachyon condensation is very important because a typical endpoint of tachyon condensation is strongly coupled. A relation between tachyon condensation and two-dimensional black holes was mentioned in Ref. 20).

To this point, we have studied static, space-dependent tachyon background. To study the time evolution of tachyon condensation, we should perform the Wick rotation of the space-like Liouville direction into the time-like one. There may be another way to extract a time-dependent process from our CFT. To elucidate this point, we consider the spacetime effective action employed in Refs. 2) and 5). To obtain a space-dependent solution, we consider the ansatz

\[
d s^2 = e^{A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \quad \Phi = \Phi(y), \quad T = T(y).
\] (6.2)

Then, the equations of motion reduce to the following:

\[
\frac{1}{2} (T')^2 - V(T) = 2 (\Phi')^2,
\] (6.3)

\[
\Phi'' = \frac{1}{2} (T')^2,
\] (6.4)

\[
A = \text{const.}
\] (6.5)

At a local minimum below zero, the tachyon \( T \) can stop rolling, and the dilaton is linear in the \( y \)-direction. Consider geodesic motion in this background in the Einstein frame. The geodesic equation is

\[
\frac{d^2 y}{dt^2} = -\frac{1}{2} \phi' \left[ 1 - \left( \frac{dy}{dt} \right)^2 \right],
\] (6.6)

where \( \phi = -\frac{4}{D - 1} \Phi \) and \( t = x^0 \). This indicates that all test particles are accelerated in the positive \( y \)-direction, and their speeds approach the speed of light. The monotonicity of the geodesics seems to be related to the monotonicity of the RG flow.
behind the tachyon condensation. It would be interesting to investigate whether the target space which an observer moving along the geodesic sees is the Penrose limit of the above solution. It is known that the Penrose limit of any solution in string theory and M-theory preserves at least 16 supersymmetries, and hence, if this is the case, it may allow us to understand the properties of the endpoint of the condensation, although the corresponding background is strongly coupled, and the metric becomes singular.

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Appendix A

Conventions

Our conventions for superfields are obtained through the dimensional reduction of four-dimensional ones to two dimensions. More specifically, we follow the conventions in Ref. 21), reducing the 1- and 2-directions and renaming $x^3$ as $x^1$. The super-derivatives are

$$D_\pm = \frac{\partial}{\partial \theta^\pm} - 2i\bar{\theta}^\pm \partial_\pm, \quad \bar{D}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} + 2i\theta^\pm \partial_\pm,$$

(A.1)

where $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$.

The chiral superfield $\Phi$ in the component fields is

$$\Phi = \phi - i\theta - \bar{\theta} - \partial_\phi - i\theta^+ \bar{\theta}^+ \partial_\phi + \theta^+ \bar{\theta}^+ \bar{\theta}^- \partial_+ \partial_- \phi + 2\theta^+ \theta^- F + \sqrt{2}(\theta^- \psi_+ + \theta^+ \psi_-) - i\sqrt{2}\theta^+ \theta^- (\bar{\theta}^- \partial_- \psi_+ - \bar{\theta}^+ \partial_+ \psi_-).$$

(A.2)

The vector superfield $V$ in the Wess-Zumino gauge is

$$V = \theta^+ \bar{\theta}^+ v_+ + \theta^- \bar{\theta}^- v_- - \theta^- \bar{\theta}^+ \sigma - \theta^+ \bar{\theta}^- \bar{\sigma} - 2\theta^+ \theta^- \bar{\theta}^+ \bar{\theta}^- D - 2i\theta^+ \theta^- (\bar{\theta}^- \bar{\lambda}_- + \bar{\theta}^+ \bar{\lambda}_+) + 2i\bar{\theta}^+ \bar{\theta}^- (\theta^- \lambda_- + \theta^+ \lambda_+).$$

(A.3)

From this, we obtain the following expression for $\Sigma$:

$$\Sigma = \bar{D}_+ D_- V$$

$$= \sigma + 2i\theta^+ \bar{\lambda}_+ - 2i\bar{\theta}^- \lambda_- + 2\theta^+ \bar{\theta}^- (D - iv_0)$$

$$- i\theta^+ \bar{\theta}^+ \partial_+ \sigma + i\theta^- \bar{\theta}^- \partial_- \sigma - \theta^+ \bar{\theta}^+ \bar{\theta}^- \partial_+ \partial_- \sigma$$

$$- 2\theta^+ \theta^- \bar{\theta}^- \partial_- \bar{\lambda}_+ - 2\theta^+ \bar{\theta}^- \theta^+ \partial_+ \lambda_. $$

(A.4)

To define twisted superfields, it is convenient to introduce the involution $I$, given by

$$I(\theta^-) = -\bar{\theta}^-, \quad I(\bar{\theta}^-) = -\theta^-,$$

(A.5)
which preserves the ordering of the Grassmann coordinates. The involution $I$ exchanges $D_-$ and $\bar{D}_-$, while keeping $D_+$ and $\bar{D}_+$ fixed, and thus $I$ exchanges chiral superfields and twisted chiral superfields. The component expansion of the twisted superfield $Y$ is therefore

$$Y = y + i\theta^-\bar{\theta}^- \partial_- y - i\theta^+\bar{\theta}^+ \partial_+ y - \theta^+\bar{\theta}^- \partial_+ \partial_- y - 2\theta^+\bar{\theta}^- F_Y + \sqrt{2}(-\bar{\theta}^- \chi^- + \theta^+ \bar{\chi}^-) + i\sqrt{2}\theta^+\bar{\theta}^- (-\theta^- \partial_- \bar{\chi}^- - \bar{\theta}^- \partial_+ \chi^-).$$

The kinetic term of $Y$ is obtained as the usual kinetic term of the chiral superfield $I(Y)$, that is,

$$\int d^4\theta I(\bar{Y}) I(Y) = - \int d^4\theta YY.$$  \hfill (A.7)

The Grassmann integration measure is defined as $d^2\bar{\theta} = d\bar{\theta}^- d\theta^+$, and therefore the twisted superpotential can be rewritten as a superpotential, for example,

$$\int d^2\bar{\theta} Y \Sigma = - \int d^2\theta I(Y)^I(\Sigma).$$  \hfill (A.8)

### Appendix B

#### N=2 Algebra from the GLSM

The Lagrangian of the GLSM discussed in this appendix is

$$L = \frac{1}{2\pi} \int d^4\theta \left[ \sum_{i=1}^{n} \bar{\Phi}_i e^{2q_i V} \Phi_i - \frac{1}{2e^2} \bar{\Sigma} \Sigma - k\bar{Y} Y \right] + \frac{1}{2\pi} \left[ \int d^2\theta a W(\Phi) - \int d^2\bar{\theta} Y \Sigma + \text{h.c.} \right],$$  \hfill (B.1)

where $a$ is a constant. The equations of motion for the superfields are

$$\bar{D}_+ \bar{D}_- (\bar{\Phi}_i e^{2q_i V}) = -2a \partial_i W(\Phi),$$  \hfill (B.2)

$$\frac{1}{2e^2} (D_+ \bar{D}_- \Sigma + \bar{D}_+ D_- \Sigma) = \sum_{i=1}^{n} 2q_i \bar{\Phi}_i e^{2q_i V} \Phi_i - 2(\bar{Y} + Y),$$  \hfill (B.3)

$$k\bar{D}_+ D_- \bar{Y} = -2\Sigma.$$  \hfill (B.4)

Our goal is to construct generators of the $N=2$ superconformal algebra acting on a subspace of the full Hilbert space which preserves $\bar{Q}_+$. Such generators must be $\bar{Q}_+$-closed, that is, an operator $O$ satisfying $\{\bar{Q}_+, O\} = 0$. Because $\bar{Q}_+ = \bar{D}_+ - 2i\theta^+ (\partial_0 + \partial_1)$ on the superspace, the lowest component of a $\bar{D}_+$-closed superfield is $\bar{Q}_+$-closed. Therefore, one way to obtain the $N=2$ generators is to construct a $\bar{D}_+$-closed superfield whose lowest component is the $U(1)$ current of the $N=2$ algebra. The other generators can be obtained as higher components of the same superfield.

#### B.1. Classical analysis

There are various $U(1)$ symmetries in (B.1). Let us first consider a $U(1)$ symmetry acting on $\Phi$, suppressing the subscripts. It is generated by $\partial_- \phi \phi$, $\bar{\phi}_- \partial \phi$, $\bar{\psi}_- \psi_-$,
where $\partial_- = \frac{1}{2}(\partial_0 - \partial_1)$, and their right-moving partners. Their gauge-invariant and supersymmetric extensions are as follows:

\[
D_-\{e^{-2qV}D_-(e^{2qV}\Phi)\}e^{2qV}\Phi = 4i(\partial_-\phi - iv_-\phi)\phi + \cdots, \quad \text{(B.5)}
\]
\[
\Phi e^{2qV}D_-\{e^{-2qV}D_-(e^{2qV}\Phi)\} = 4i\tilde{\phi}(\partial_-\phi + iv_-\phi) + \cdots, \quad \text{(B.6)}
\]
\[
D_-(\bar{\phi}e^{2qV})e^{-2qV}D_-(e^{2qV}\Phi) = 2\bar{\psi} - \psi + \cdots. \quad \text{(B.7)}
\]

It can be shown that the second superfield cannot contribute to a $\bar{D}_+$-closed superfield, even with the equations of motion. For this reason, below we discuss a linear combination of the first and the third ones. Let us define

\[
J^i_1 = \bar{D}_-(\bar{\Phi}e^{2q_iV})e^{-2q_iV}D_-(e^{2q_iV}\Phi_i), \quad \text{(B.8)}
\]
\[
J^i_2 = D_-\bar{D}_-(\bar{\Phi}_i e^{2q_iV})e^{2q_iV}\Phi_i. \quad \text{(B.9)}
\]

Note that $J^i_1 + J^i_2$ is equal to (B.5). Next, define

\[
J_\Phi = \sum_{i=1}^{n}(\alpha_i J^i_1 + \beta_i J^i_2). \quad \text{(B.10)}
\]

It can be shown that

\[
\bar{D}_+J_\Phi = -\Sigma D_-\left[\sum_{i=1}^{n}2\alpha_i q_i\Phi_i e^{2q_iV}\Phi_i\right] - 4a\sum_{i=1}^{n}\left[\alpha_i q_i\Phi_i \partial_i W(\Phi)\right] D_-V
\]
\[
+ a\sum_{i=1}^{n}\left[+2\beta_i D_-(\Phi_i \partial_i W(\Phi)) - 2\alpha_i \partial_i W(\Phi)D_-\Phi_i\right]. \quad \text{(B.11)}
\]

Now suppose that $a \neq 0$. Then the second term on the RHS vanishes if all $\alpha_i$ are equal. Let us consider this case and write $\alpha_i = \alpha$. For a generic superpotential, this is the unique choice, as in this case there exists only the gauge symmetry. Then, the third term vanishes if the relation

\[
\sum_{i=1}^{n}2\beta_i \Phi_i \partial_i W(\Phi) = 2\partial_i W(\Phi). \quad \text{(B.12)}
\]

holds. If $\alpha = 0$, then the solution is generically $\beta_i = q_i$. Otherwise, let $\beta_i = \frac{a}{2}p_i$. Then we have

\[
\sum_{i=1}^{n}p_i \Phi_i \partial_i W(\Phi) = 2W(\Phi). \quad \text{(B.13)}
\]

Therefore, the quantities $p_i$ are the $U(1)_V$ charges of $\Phi_i$.

Next, we obtain

\[
\bar{D}_+J_\Phi = -\alpha\Sigma D_-\left[\sum_{i=1}^{n}2q_i\Phi_i e^{2q_iV}\Phi_i\right] \quad \text{(B.14)}
\]
provided that either $a = 0$ or

$$\alpha_i = \alpha \quad (i = 1, \ldots, n), \quad \beta_i = \begin{cases} q_i, & (\alpha = 0), \\ \frac{\alpha}{2} p_i, & (\alpha \neq 0), \end{cases}$$

(B.15)

where the $p_i$ satisfy (B.13). Note that $J_{\Phi}$ with $\alpha = 0$ is the current for the global gauge symmetry, which is not appropriate for our purpose. In what follows, we assume a non-zero $\alpha$ and normalize $J_{\Phi}$ so that $\alpha = 1$. Note also that the $p_i$ are not determined uniquely by (B.13), due to the global gauge symmetry.

Using the equations of motion, one can show that

$$J_c = \sum_{i=1}^{n} \left[ J_i^1 + \frac{p_i}{2} J_i^2 \right] - \frac{1}{2} e^2 \Sigma \bar{D}_+ D_- \bar{\Sigma} - kD_- \bar{Y} \bar{D}_- Y$$

(B.16)

is $\bar{D}_+$-closed.

B.2. Chiral anomaly

At the quantum level, the current $J_c$ may be anomalous. The chiral anomaly actually appears from

$$\mathcal{D}_- \bar{\phi}_i \phi_i, \quad \bar{\psi}_i, - \psi_i, -$$

(B.17)

whose definitions are

$$\mathcal{D}_- \bar{\phi}_i \phi_i(0) := \lim_{x \to 0} \left[ \mathcal{D}_- \bar{\phi}_i(x) e^{-iq_i \int_0^x d\xi v_\mu(\xi) \phi_i(0)} + \frac{1}{2x} \right]$$

$$= : \mathcal{D}_- \bar{\phi}_i(0) \phi_i(0) : + \frac{i}{2} q_i v_-(0) + \frac{i}{2} q_i \lim_{x \to 0} \frac{x^+}{x^-} v_+(0).$$

(B.18)

$$\bar{\psi}_i, - \psi_i, -(0) := \lim_{x \to 0} \left[ \bar{\psi}_i, -(x) e^{-iq_i \int_0^x d\xi v_\mu(\xi) \psi_i, -(0)} + \frac{i}{x} \right]$$

$$= : \bar{\psi}_i, -(0) \psi_i, -(0) : - q_i v_-(0) - q_i \lim_{x \to 0} \frac{x^+}{x^-} v_+(0).$$

(B.19)

Note that for the singular part of the OPE for $\phi_i$, the gauge-covariant one,

$$\bar{\phi}_i(x) \phi_i(0) \sim - \frac{1}{2} e^{iq_i \int_0^x d\xi v_\mu(\xi)} \log(x^- x^+),$$

(B.20)

should be used. Although the RHS goes to $- \frac{1}{2} \log(x^- x^+)$ in the $x \to 0$ limit, terms of the form $x^n \log x$ cannot be ignored, since their derivatives may be singular in this limit. This should be the same for fermions, but in these cases, the exponential term is always irrelevant.

According to Ref. 14), the following hold:

$$\bar{D}_+ J_i^1 = -q_i \bar{D}_- \Sigma,$$

(B.21)

$$\bar{D}_+ J_i^2 = 0.$$  

(B.22)

Therefore, we have

$$\bar{D}_+ J_c = -\gamma \bar{D}_- \Sigma, \quad \gamma = \sum_{i=1}^{n} q_i.$$  

(B.23)
Note that the current \( J_c \) is \( \bar{D}_+ \)-closed even at the quantum level if \( \gamma = 0 \), which is relevant for a GLSM in the case of a supersymmetric background of string theory.

As the chiral anomaly can be canceled due to the presence of \( Y \), \( J_c \) can be modified by using \( Y \) to recover the quantum \( \bar{D}_+ \)-closed property. Now, consider a superfield \( \bar{D}_- D_- \bar{Y} - D_- \bar{D}_- Y \) whose lowest component generates a shift of \( y_I \). This satisfies

\[
\bar{D}_+ \left[ \bar{D}_- D_- \bar{Y} - D_- \bar{D}_- Y \right] = \frac{2}{k} \bar{D}_- \Sigma. \tag{B.24}
\]

This equation is valid at the quantum level. Therefore, the current

\[
J = J_c + \frac{k\gamma}{2} (\bar{D}_- D_- \bar{Y} - D_- \bar{D}_- Y) \tag{B.25}
\]

is exactly \( \bar{D}_+ \)-closed.

B.3. \( \mathcal{N} = 2 \) algebra

The lowest component of the current \( J \) may play the role of the \( U(1) \) current \( j \) of the \( \mathcal{N} = 2 \) algebra. It is obvious that \( D_- J, \bar{D}_- J \) and \( [D_-, \bar{D}_-] J \) are also \( \bar{D}_+ \)-closed, and in fact, their lowest components are the supercharges \( G \) and \( \bar{G} \) and the energy-momentum tensor \( T \), respectively. To confirm this point, we must calculate their OPE. This is a difficult task in general, but in our case the situation is quite simple. Because the interaction terms only contribute to non-singular terms of the OPE, one can calculate the OPE of any operators by using (gauge-covariant) free propagators.

It can be shown that

\[
j = -\frac{1}{2} J \big|_1, \quad G = \frac{1}{2\sqrt{2}} D_- J \big|_1, \quad \bar{G} = \frac{1}{2\sqrt{2}} \bar{D}_- J \big|_1, \quad T = \frac{1}{16} [D_-, \bar{D}_-] J \big|_1 \tag{B.26}
\]

form the \( \mathcal{N} = 2 \) algebra,\(^*\).\(^{14}\)

\[
T(x)T(0) \sim \frac{\gamma}{(x^-)^4} - \frac{2}{(x^-)^2} T(0) - \frac{1}{x^-} \partial_- T(0), \tag{B.27}
\]

\[
T(x)G(0) \sim -\frac{3}{(x^-)^2} G(0) - \frac{1}{x^-} \partial_- G(0), \tag{B.28}
\]

\[
T(x)\bar{G}(0) \sim -\frac{3}{(x^-)^2} \bar{G}(0) - \frac{1}{x^-} \partial_- \bar{G}(0), \tag{B.29}
\]

\[
G(x)G(0) \sim \frac{2}{3\sqrt{3}} \frac{i c}{(x^-)^3} - \frac{2}{(x^-)^2} j(0) - \frac{i}{x^-} (2T(0) - \partial_- j(0)), \tag{B.30}
\]

\[
T(x)j(0) \sim -\frac{1}{(x^-)^2} j(0) - \frac{1}{x^-} \partial_- j(0), \tag{B.31}
\]

\[
j(x)G(0) \sim -\frac{i}{x^-} G(0), \tag{B.32}
\]

\(^*\) This algebra reduces to a well-known one after the Wick rotation and a suitable rescaling of the operators related to the conformal transformation from a cylinder to a plane.
\begin{align}
  j(x)\bar{G}(0) &\sim \frac{i}{x} \bar{G}(0), \quad \text{(B.33)} \\
  j(x)j(0) &\sim -\frac{\xi}{(x^-)^2}, \quad \text{(B.34)}
\end{align}

with the central charge

\begin{equation}
  c = \sum_{i=1}^{n} 3(1 - p_i) + (-3) + 3 \left(1 + k\gamma^2\right), \quad \text{(B.35)}
\end{equation}

coming from $\Phi_i$, $V$ and $Y$, respectively.

The explicit form of the generators are as follows:

\begin{align}
  j &= \sum_{i=1}^{n} \left[ -\bar{\psi}_i \psi_i - \frac{p_i}{2} (2iD\bar{\phi}_i\phi_i - \bar{\psi}_i\psi_i) \right] + \frac{i}{e^2} \sigma \partial \bar{\sigma} + k\bar{\chi}\chi - ik\gamma \partial (\bar{y} - y), \quad \text{(B.36)} \\
  G &= \sum_{i=1}^{n} 2iD\bar{\phi}_i\psi_i - \frac{\sqrt{2}}{e^2} \sigma \partial \bar{\lambda} + 2ik\bar{\chi} \partial y + ik\gamma \partial \bar{\chi}, \quad \text{(B.37)} \\
  \bar{G} &= \sum_{i=1}^{n} \left[ i(p_i - 2)D\phi_i \bar{\psi}_i + ip_i \phi_i D\bar{\psi}_i \right] + \frac{\sqrt{2}}{e^2} \lambda \partial \bar{\sigma} - 2ik\bar{\chi} \partial \bar{y} - ik\gamma \partial \chi, \quad \text{(B.38)} \\
  T &= \sum_{i=1}^{n} \left[ 2D\bar{\phi}_i D\phi_i + \frac{i}{2} (\bar{\psi}_i D\psi_i - D\bar{\psi}_i\psi_i) + \frac{i}{4} p_i \partial (2iD\bar{\phi}_i\phi_i - \bar{\psi}_i\psi_i) \right] \\
  &\quad + \frac{1}{2e^2} (\partial \sigma \partial \sigma - \sigma \partial^2 \bar{\sigma}) + \frac{i}{e^2} \lambda \partial \bar{\lambda} + 2k\partial \bar{y} \partial y + \frac{1}{2} k\gamma \partial^2 (\bar{y} + y) + \frac{i}{2} k(\bar{\chi} \partial \chi - \partial \bar{\chi} \chi), \quad \text{(B.39)}
\end{align}

where the subscript “−” is omitted.

**Appendix C**

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**Dual Superpotential**

Consider the GLSM given in (B.1) with $n = 2$ and the superpotential (4.1). The dual superpotential $\tilde{W}$ for this GLSM can be read off of the BPS mass formula,\(^{15}\)

\begin{equation}
  \Pi \propto \int d\Sigma dY dy_1 dy_2 \Sigma e^{-\tilde{W}_0}, \quad \text{(C.1)}
\end{equation}

where

\begin{equation}
  \tilde{W}_0 = \frac{1}{2} \Sigma (q_1 Y_1 + q_2 Y_2 - 2Y) + e^{-Y_1} + e^{-Y_2} \quad \text{(C.2)}
\end{equation}

is the dual superpotential for the GLSM without $W$. Note that the derivation of (C.1) does not depend on the details of $W$, and thus the mass formula is applicable in our case. Equation (C.1) can be rewritten as

\begin{equation}
  \Pi \propto \int dY dy_1 e^{\frac{q_1}{q_2} Y_1 - \frac{2}{q_2} Y} \exp\left[ -e^{-Y_1} - e^{\frac{q_1}{q_2} Y_1 - \frac{2}{q_2} Y} \right]. \quad \text{(C.3)}
\end{equation}
In our case, the FI parameter is promoted to a field $Y$, and hence it is possible to carry out a field-dependent shift of $Y$. In this way, we obtain the BPS mass formula to be

$$
\Pi \propto \int dY dY_1 e^{-\frac{1}{|q_2|}Y_1 + \frac{2}{|q_2|}Y} \exp\left[-e^{-Y_1} - e^{-\frac{1}{|q_2|}Y_1 + \frac{2}{|q_2|}Y}\right],
$$

where $X_1 = e^{-\frac{1}{|q_2|}Y_1}$. (See §4 for the choice of the charges $q_1$ and $q_2$.) This suggests that the dual superpotential is

$$
\tilde{W} = X_1^{q_2} + e^{\frac{2}{|q_2|}Y} X_1 - \frac{2}{|q_2|}Y.
$$

The term linear in $Y$ does not contribute to any physical phenomena, and therefore this term can be ignored. Note that the possible background charge of $Y$ appears in the normalization of the kinetic term, not in the superpotential. In the $\text{Re}(Y) \to -\infty$ limit, the CFT specified by this superpotential is a minimal model plus a free field with a possible linear dilaton background. By exchanging the roles of $Y_1$ and $Y_2$, we then obtain

$$
\tilde{W} = X_2^{q_1} + e^{-\frac{2}{q_1}Y} X_2 + \frac{2}{q_1}Y,
$$

which is used to describe the CFT in the $\text{Re}(Y) \to +\infty$ limit.

References

10) C. Vafa, hep-th/0111051.