Localized and Non-Localized Solutions of $q$-Deformed Oscillators

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Eigenfunctions of the Fock and non-Fock irreducible representations are explicitly derived for the $q$-deformed oscillators of the Macfarlane and Dubna types. While the Fock representations are composed of localized functions, the non-Fock representations consist of nonlocal oscillating functions which are constructed for the first time in this paper. The concept of the $q$-Hermite functions is generalized to include both the Fock and non-Fock types in a unified manner.

§1. Introduction

Studies of the ordinary quantum oscillator have provided very useful illustrations of the general principles and formalism of standard quantum theory. They have also led to basic concepts in quantum field theory, quantum optics and the theory of molecular and crystalline vibrations. For many years, great effort has been made to extend such studies by formulating generalizations and deformations of the ordinary quantum oscillator that play important roles in theoretical physics. There is some hope that the study of such systems will lead to new physical insights and exactly solvable models in quantum mechanics through the discovery and utilization of new symmetries with non-trivial structure.

In this context, particular attention has been focused on studies of $q$-deformed oscillators in the attempt to realize $q$-deformed systems related to quantum groups,$^1$ since the discoveries by Biedenharne$^2$ and Macfarlane.$^3$ Classification in (abstract) representation spaces for $q$-deformed oscillators was first investigated by Kulish,$^4$ within the framework of “a boson oscillator algebra”. He found that a $q$-deformed oscillator possesses two types of irreducible representation spaces which consist of the Fock representation, forming a discrete spectrum, and the non-Fock representation, forming a continuous spectrum. Some authors$^5,6$ have attempted to formulate multi-particle structure and generalizations of $q$-deformed oscillators within the framework of the boson oscillator algebra.

Coordinate representations on the Fock space were derived independently by Macfarlane$^3$ and the Dubna group.$^7-14$ Detailed analyses$^{15-17}$ of the Macfarlane-type oscillator were performed by Shabanov and Rajagopal. Later, we investigated coordinate representations on the Fock space of $q$-deformed oscillators of the Mac-

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farlane type (M-oscillator) and the Dubna type (D-oscillator) in a unified manner.\(^\text{18}\) In the light of the studies of the classical deformed oscillator,\(^\text{19}\) \((c\text{-deformed oscillator})\), we noted the existence of oscillatory solutions of the Bloch type, which can be interpreted as being equivalent to the non-Fock representation presented in Kulish’s paper.\(^\text{4}\)

The main purpose of this paper is to derive “exact coordinate representations” corresponding to the Fock and non-Fock representation spaces. In §2, the formulation of the ordinary oscillator is outlined in terms of the boson oscillator algebra. There exists a kind of gauge transformation, under which the physical states are invariant in the representation space for the boson oscillator algebra. We fix the corresponding gauge degree of freedom to specify a definite physical state. In §3, a boson oscillator algebra of the \(q\)-deformed oscillator is introduced and investigated by using the same procedure as in the case of the boson oscillator algebra of the ordinary oscillator. We find that the \(q\)-deformed oscillator possesses two types of irreducible representation spaces, which consist of the Fock representation and the non-Fock representation. The corresponding eigenfunctions of the M-oscillator and D-oscillator are derived by using the Fourier series expansion in §4. In §5, we investigate the connections between oscillating solutions of the Bloch types\(^\text{19}\) and our new eigenfunctions on the non-Fock space. We extend the unified \(q\)-Hermite function\(^\text{18}\) on the Fock representations to a generalized function including the Hermite function for the non-Fock representation.

§2. Ordinary oscillator in a general setting

The ordinary quantum harmonic oscillator is formulated in terms of the algebra \(A_0\), which is generated by the ladder operators \(\hat{a}\) and \(\hat{a}^\dagger\); that is, \(A_0 = \{\hat{a}, \hat{a}^\dagger\}\). We generalize the algebra \(A_0\) to the boson oscillator algebra \(A\) generated by three independent operators, the ladder operators, \(\hat{a}\) and \(\hat{a}^\dagger\), and the number operator \(\hat{N}\); that is, \(A = \{\hat{a}, \hat{a}^\dagger, \hat{N}\}\). These operators satisfy the algebraic relations:

\[
[\hat{a}, \hat{a}^\dagger] = I, \quad (2.1)
\]

and

\[
[\hat{N}, \hat{a}] = -\hat{a}, \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger. \quad (2.2)
\]

The Hamiltonian governing the dynamics of the system is given by

\[
\hat{H} = (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger)/2. \quad (2.3)
\]

There exists a trivial center,

\[
\hat{z} = \hat{N} - \hat{a}^\dagger \hat{a}, \quad (2.4)
\]

which satisfies

\[
[\hat{a}, \hat{z}] = [\hat{a}^\dagger, \hat{z}] = [\hat{N}, \hat{z}] = 0. \quad (2.5)
\]

The commutation relations (2.1)–(2.5) enable us to define simultaneous eigenstates as follows:

\[
\begin{aligned}
\hat{N}|n; z_0\rangle &= n|n; z_0\rangle, \\
\hat{z}|n; z_0\rangle &= z_0 |n; z_0\rangle, \\
\hat{a}^\dagger \hat{a}|n; z_0\rangle &= (n - z_0)|n; z_0\rangle.
\end{aligned} \quad (2.6)
\]
The non-negativity of the norm of the vector \( \hat{a}|n; z_0\rangle \) leads to the necessary condition
\[
\langle n; z_0|\hat{a}^\dagger \hat{a}|n; z_0\rangle = n - z_0 \geq 0.
\]
(2.7)

This condition implies that there exists a lowest eigenvalue of \( \hat{N} \), which is the eigenvalue \( z_0 \) of the center. Therefore, the lowest eigenstate is represented by \( |z_0; z_0\rangle \). Applying \( \hat{a}^\dagger \) to this lowest eigenstate, the normalized basis of the irreducible representation space is constructed as follows:
\[
|n; z_0\rangle = |k + z_0; z_0\rangle = \frac{1}{\sqrt{k!}}(\hat{a}^\dagger)^k|z_0; z_0\rangle. \quad (k \in 0, 1, 2, \cdots)
\]
The spectrum of eigenvalues of \( \hat{N} \) is given by \( \sigma(\hat{N}) = \{z_0, z_0 + 1, z_0 + 2, \cdots\} \) for the eigenvalue \( z_0 \) of \( \hat{z} \).

We introduce the transformation
\[
|n; z_0\rangle \rightarrow |n + \alpha; z_0 + \alpha\rangle.
\]
(2.8)
The states \( |n + \alpha; z_0 + \alpha\rangle \) are also eigenstates of the operators \( \hat{N}, \hat{z} \) and \( \hat{a}^\dagger \hat{a} \), with
\[
\begin{align*}
\hat{N}|n + \alpha; z_0 + \alpha\rangle &= (n + \alpha)|n + \alpha; z_0 + \alpha\rangle, \\
\hat{z}|n + \alpha; z_0 + \alpha\rangle &= (z_0 + \alpha)|n + \alpha; z_0 + \alpha\rangle, \\
\hat{a}^\dagger \hat{a}|n + \alpha; z_0 + \alpha\rangle &= (n - z_0)|n + \alpha; z_0 + \alpha\rangle.
\end{align*}
\]
(2.9)
The eigenvalues of \( \hat{N} \) corresponding to the single eigenvalue \( z_0 + \alpha \) of \( \hat{z} \) form the spectrum \( \sigma(\hat{N}) = \{z_0 + \alpha, z_0 + \alpha + 1, z_0 + \alpha + 2, \cdots\} \). Comparison of Eqs. (2.6) and (2.9) shows that the eigenvalues of the physical operators \( \hat{a}^\dagger \hat{a} \) and the Hamiltonian are invariant under the transformation in Eq. (2.8). This transformation is thus interpreted as a kind of gauge transformation. To specify a definite physical state in the subspace \( \{|n + \alpha; z_0 + \alpha\rangle; \alpha \in R\} \), we fix the parameter \( \alpha \) as \( \alpha = -z_0 \). With this gauge fixing, we express eigenvectors as \( |n_{\text{new}}; 0\rangle = |n - z_0; 0\rangle \), introducing the new quantum number \( n_{\text{new}} = n - z_0 \), where \( n_{\text{new}} = \{0, 1, 2, 3, \cdots\} \). We then have
\[
\begin{align*}
\hat{N}|n_{\text{new}}; 0\rangle &= n_{\text{new}}|n_{\text{new}}; 0\rangle, \\
\hat{z}|n_{\text{new}}; 0\rangle &= 0, \\
\hat{a}^\dagger \hat{a}|n_{\text{new}}; 0\rangle &= n_{\text{new}}|n_{\text{new}}; 0\rangle,
\end{align*}
\]
which correspond to Eq. (2.6) with the eigenvalue \( z_0 = 0 \). The representation subspace spanned by the eigenvectors \( |n_{\text{new}}; 0\rangle \) of the boson oscillator algebra \( \mathcal{A} \) is equivalent to the Fock representation space for the algebra \( \mathcal{A}_0 \).

§3. Algebra of the \( q \)-deformed oscillator and irreducible representations

The boson oscillator algebra of the \( q \)-deformed oscillator \( \mathcal{A}_q^{(4)-(6)} \) is generated by the lowering and raising operators \( \hat{A} \) and \( \hat{A}^\dagger \) and the number operator \( \hat{N} \); that is, \( \mathcal{A}_q = \{\hat{A}, \hat{A}^\dagger, \hat{N}\} \). These operators satisfy the algebraic relations
\[
[\hat{A}, \hat{A}^\dagger]_q \equiv q \hat{A} \hat{A}^\dagger - q^{-1} \hat{A}^\dagger \hat{A} = I,
\]
(3.1)
and
\[ [\hat{A}, \hat{N}] = \hat{A}, \quad [\hat{A}^\dagger, \hat{N}] = -\hat{A}^\dagger. \]  

The Hamiltonian of this system is given by
\[ \hat{H}_q = \frac{1}{2}(q\hat{A}\hat{A}^\dagger + q^{-1}\hat{A}^\dagger\hat{A}). \]  

The algebra \( \mathcal{A}_q \) has the non-trivial center
\[ \hat{z} = q^{\hat{N}-1}\left(\frac{q^\hat{N} - q^{-\hat{N}}}{q - q^{-1}} - q^\hat{N}\hat{A}^\dagger\hat{A}\right), \]  

which was first found by Kulish.\(^4\) In the non-deformed limit \( q \to 1 \), the center (3.4) of the \( q \)-deformed oscillator reduces to the center (2.4) of the ordinary oscillator.

The three operators \( \hat{N}, \hat{z} \) and \( \hat{A}^\dagger\hat{A} \) have simultaneous eigenvectors:
\[
\begin{align*}
\hat{N} |n; \Lambda \rangle &= n |n; \Lambda \rangle, \\
\hat{z} |n; \Lambda \rangle &= -q^{-1}(r + \Lambda) |n; \Lambda \rangle, \\
\hat{A}^\dagger\hat{A} |n; \Lambda \rangle &= (r + q^{-2n}\Lambda) |n; \Lambda \rangle.
\end{align*}
\]

Here, we have
\[ r = 1/(q - q^{-1}), \]  

and \( \Lambda \) characterizes the eigenvalue of the center \( \hat{z} \). Equation (3.2) implies
\[ \hat{A} |n; \Lambda \rangle = c_n |n - 1; \Lambda \rangle, \quad \hat{A}^\dagger |n - 1; \Lambda \rangle = c_n^* |n; \Lambda \rangle. \]  

The non-negativity of the norm of the vectors \( \hat{A} |n; \Lambda \rangle \) implies the necessary condition
\[ \langle n; \Lambda |\hat{A}^\dagger\hat{A} |n; \Lambda \rangle = |c_n|^2 = r + q^{-2n}\Lambda \geq 0. \]  

By tuning the phase of \( |n; \Lambda \rangle \), we can keep the coefficient \( c_n \) positive. Therefore, Eq. (3.7) becomes
\[ \hat{A} |n; \Lambda \rangle = \sqrt{r + q^{-2n}\Lambda} |n - 1; \Lambda \rangle \quad \text{and} \quad \hat{A}^\dagger |n - 1; \Lambda \rangle = \sqrt{r + q^{-2n}\Lambda} |n; \Lambda \rangle. \]  

We next introduce the following transformation, which we call the \( \alpha \)-transformation:
\[ |n; \Lambda \rangle \rightarrow |n + \alpha; q^{2\alpha}\Lambda \rangle. \]  

Under this transformation, the eigenvalues of \( \hat{N}, \hat{z} \) and \( \hat{A}^\dagger\hat{A} \) become as follows:
\[
\begin{align*}
\hat{N} |n + \alpha; q^{2\alpha}\Lambda \rangle &= (n + \alpha) |n + \alpha; q^{2\alpha}\Lambda \rangle, \\
\hat{z} |n + \alpha; q^{2\alpha}\Lambda \rangle &= -q^{-1}(r + q^{2\alpha}\Lambda) |n + \alpha; q^{2\alpha}\Lambda \rangle, \\
\hat{A}^\dagger\hat{A} |n + \alpha; q^{2\alpha}\Lambda \rangle &= (r + q^{-2n}\Lambda) |n + \alpha; q^{2\alpha}\Lambda \rangle.
\end{align*}
\]  

The eigenvalues of the physical operators \( \hat{A}^\dagger\hat{A} \) and \( \hat{H}_q \) are invariant under the \( \alpha \)-transformation. We interpret the \( \alpha \)-transformation as a kind of gauge transformation in the representation space. To specify a definite physical state in the subspace
\(|n + \alpha; q^{2\alpha}A_0; \alpha \in \mathbb{R}\), we set the parameter \(\alpha\) to a definite value, as in the case of
the ordinary oscillator. The restrictions and classifications for \(\Lambda\) are determined by
the necessary condition given in Eq. (3.8) and are described for the sectors of
\(q > 1\) and \(q < 1\) as follows.

- \(q > 1\) (\(r > 0\)):
  
  The values of \(\Lambda\) are separated into two domains, (i) \(\Lambda = -rq^{2l} < 0\) and
  (ii) \(\Lambda \geq 0\).

  (i) \(\Lambda = -rq^{2l} < 0\): \(\{n = l, l + 1, l + 2, \cdots\, \text{for} \, |n; A\}\}

  By utilizing the \(\alpha\)-transformation, we can fix \(\alpha\) as \(\alpha = -l\) in Eq. (3.11) to
  specify a definite physical state. Then, Eq. (3.11) becomes

  \[
  \begin{align*}
  \hat{N} |n - l; -r\rangle &= (n - l) |n - l; -r\rangle, \\
  \hat{\sigma} |n - l; -r\rangle &= 0, \\
  \hat{A}^\dagger \hat{A} |n - l; -r\rangle &= (1 - q^{-2(n-l)})r |n - l; -r\rangle,
  \end{align*}
  \tag{3.12}
  \]

  where \(\sigma(\hat{N}) = \{0, 1, 2, \cdots\}\). We can express the eigenvectors as \(|n_{\text{new}}; -r\rangle =
  |n - l; -r\rangle\) by introducing the new quantum number \(n_{\text{new}} = n - l\). We thus
  find that the boson algebra of the \(q\)-deformed oscillator \(A_q\) has the following
  irreducible representation space \(\mathcal{F}_0\):

  \[
  \mathcal{F}_0 = \{ |n_{\text{new}}; -r\rangle; \ n_{\text{new}} = 0, 1, 2, \cdots; \ \hat{A} |0; -r\rangle = 0\}. \tag{3.13}
  \]

  The energy spectrum of the Hamiltonian \(\hat{H}_q\) is given by

  \[
  \frac{1}{2} \leq E_{n_{\text{new}}}^{(-r)} = \left(\frac{q + q^{-1}}{2(q - q^{-1})} \right.
  \frac{q^{-2n_{\text{new}} - 1}}{q - q^{-1}} \left.\right) \leq \frac{q + q^{-1}}{2(q - q^{-1})}. \tag{3.14}
  \]

  (ii) \(\Lambda \geq 0\)

  We divide the semi-infinite interval \([0, \infty)\) for \(\Lambda\) as

  \[
  [0, \infty) = \bigcup_{m=-\infty}^{\infty} [rq^{2m}, rq^{2(m+1)}]. \tag{3.15}
  \]

  It is possible to represent \(\Lambda\) in the finite interval \([rq^{2m}, rq^{2m+2})\) by \(\Lambda = q^{2m}A_0\),
  where \(r \leq A_0 < rq^2\) and \(m \in \{\cdots, -1, 0, 1, \cdots\}\). With this division, Eq. (3.11) reads

  \[
  \begin{align*}
  \hat{N} |n + \alpha; q^{2(\alpha+m)}A_0\rangle &= (n + \alpha) |n + \alpha; q^{2(\alpha+m)}A_0\rangle, \\
  \hat{\sigma} |n + \alpha; q^{2(\alpha+m)}A_0\rangle &= -q^{-1}(r + q^{2(\alpha+m)})A_0 |n + \alpha; q^{2(\alpha+m)}A_0\rangle, \\
  \hat{A}^\dagger \hat{A} |n + \alpha; q^{2(\alpha+m)}A_0\rangle &= (r + q^{-2(n-m)}A_0 |n + \alpha; q^{2(\alpha+m)}A_0\rangle.
  \end{align*}
  \tag{3.16}
  \]

  To specify a definite physical state, we fix \(\alpha\) as \(\alpha = -m\) for each \(m\). We can
  represent eigenvectors as \(|n_{\text{new}}; A_0\rangle = |n - m; A_0\rangle\) in terms of the new quantum
number \( n_{\text{new}} = n - m \). We thereby find that the irreducible representation space \( \mathcal{F}_1 \) of the algebra \( \mathcal{A}_q \) is
\[
\mathcal{F}_1 = \{ |n_{\text{new}}; A_0\rangle; \ n_{\text{new}} = \cdots, -2, -1, 0, 1, 2, \cdots; \ r \leq A_0 < rq^2 \}, \quad (3.17)
\]
where the eigenvalues of \( ˆz \) are given by
\[
ˆz |n_{\text{new}}; A_0\rangle = -q^{-1}(r + A_0)|n; A_0\rangle. \quad (3.18)
\]
Also, the energy eigenvalues of \( ˆH_q \) are given by
\[
\frac{q + q^{-1}}{2(q - q^{-1})} \leq E_{n_{\text{new}}}^{(A_0)} = \left( \frac{q + q^{-1}}{2(q - q^{-1})} + q^{-2n_{\text{new}} - 1}A_0 \right). \quad (3.19)
\]

- \( q < 1 \) \( (r < 0) \):
The non-negativity condition \((3.8)\) gives the restriction \( A = -rq^{2l} > 0 \) \( (n - l \in \mathbb{N} \cup \{0\}, l \in \mathbb{Z}) \). Similarly to the case of \( q > 1 \), we fix \( \alpha \) as \( \alpha = -l \) to specify the physical state in Eq. \((3.11)\). We can represent eigenvectors as \( |n_{\text{new}}; -r\rangle = |n - l; -r\rangle \) in terms of \( n_{\text{new}} = n - l \). We then find that the boson algebra of the \( q \)-deformed oscillator \( \mathcal{A}_q \) has the following irreducible representation space \( \tilde{\mathcal{F}}_0 \):
\[
\tilde{\mathcal{F}}_0 = \{ |n_{\text{new}}; -r\rangle; \ n_{\text{new}} = 0, 1, 2, \cdots; \ ˆA |0; -r\rangle = 0 \}. \quad (3.20)
\]
The energy spectrum of \( ˆH_q \) is given by
\[
\frac{1}{2} \leq E_{n_{\text{new}}}^{(-r)} = \left( \frac{q + q^{-1}}{2(q - q^{-1})} - \frac{q^{-2n_{\text{new}} - 1}}{q - q^{-1}} \right). \quad (3.21)
\]

From the above considerations, it is shown that there exist a Fock representation space \( \mathcal{F}_0 \) and a non-Fock representation space \( \mathcal{F}_1 \) for \( q > 1 \), while only the Fock representation space \( \tilde{\mathcal{F}}_0 \) exists for \( q < 1 \). The two spaces \( \mathcal{F}_0 \) and \( \tilde{\mathcal{F}}_0 \) possess the discrete energy spectra \((3.14)\) and \((3.21)\), and \( \mathcal{F}_1 \), which is characterized by a vanishing eigenvalue of the center \( ˆz \), possesses the continuous spectrum \((3.19)\). For simplicity, in the remainder of the paper, we write \( n_{\text{new}} \) and \( A_0 \) as \( n \) and \( A \).

\section*{§4. \( q \)-deformed oscillators in the coordinate representation}

We begin by introducing the coordinate representation for Eq. \((3.1)\) as
\[
[A(x), A(x)^\dagger]_q = qA(x)A(x)^\dagger - q^{-1}A(x)^\dagger A(x) = 1, \quad (4.1)
\]
where the deformation parameter \( q \) is given by
\[
q = \exp(s^2 + t^2 + 3st), \quad (4.2)
\]
in terms of the real parameters \( s \) and \( t \). Following the method developed in Refs. 18 and 19), we postulate that the ladder operators \( A(x) \) and \( A(x)^\dagger \) have the separable forms
\[
A(x) = \frac{f(x)}{g(x)} \exp[-ih(x)] D(s, t) \frac{1}{f(x)g(x)} \quad (4.3)
\]

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and

\[ A^\dagger(x) = -\frac{1}{f(x)g(x)}D(s, t) \frac{f(x)}{g(x)} \exp[ih(x)], \quad (4.4) \]

where \( D(s, t) \) is defined by the difference operator

\[ D(s, t) = \frac{i}{(s-t)} \left[ \exp(-is\frac{d}{dx}) - \exp(-it\frac{d}{dx}) \right]. \quad (4.5) \]

Here, \( f(x), g(x) \) and \( h(x) \) are functions, called part-functions, that take real values for \( x \in \mathbb{R} \). These functions are determined by the conditions that the ladder operators satisfy the \( q \)-mutator relation given in Eq. (4.1) and reduce to those of the ordinary harmonic oscillators in the limit \( q \to 1 \). By definition, the part-functions are determined up to arbitrary functions that commute with the difference operator \( D(s, t) \), and the sign of the function \( g(x) \) also is indeterminate.

For the relation given in Eq. (4.1) to hold, the parameters \( s \) and \( t \) must satisfy either the conditions

\[ s \neq 0 \text{ and } t = 0 \quad (s = 0 \text{ and } t \neq 0) \quad (4.6) \]

or the condition

\[ s + t = 0. \quad (4.7) \]

The former and latter conditions correspond to M-oscillator and D-oscillator, respectively. The part-functions \( f(x), g(x) \) and \( h(x) \) of M-oscillator and D-oscillator can be expressed in a unified manner, by using parametric representations, as

\[ f(x) = \sum_{m=-\infty}^{\infty} \eta(sm - ix) \times \sum_{k=-\infty}^{\infty} b_k \exp \left[ -\frac{1}{2} \left( x - \frac{2k\pi}{s-t} \right)^2 \right], \quad (4.8) \]

and

\[ g(x) = \left( \frac{q - q^{-1}}{\ln q} \right)^{\frac{1}{4}} \sqrt{\cos \left( \frac{2st}{s-t}x \right)}, \quad (4.9) \]

and

\[ h(x) = -2(s+t)x + a_0, \quad (4.10) \]

where \( \eta(x) \) is an arbitrary function, and \( b_k \) and \( a_0 \) are arbitrary constants.\(^1\)

4.1. The \( x \)-representation of M-oscillator

In this subsection, the eigenvalue problem for M-oscillators is investigated. The deformation parameter of M-oscillator is given by \( q = e^{s^2} \geq 1 \). As shown in §3, there are two irreducible representation spaces, \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \), for \( \Lambda = -r \) and \( r \leq \Lambda < rq^2 \), respectively. We derive eigenfunctions corresponding to these two irreducible abstract representations.

\(^{1}\) Note that the global structure of \( q \)-deformed systems is determined by the selection of \( \eta(x) \) and \( b_k \).\(^{19}\)
From Eq. (3.5), we obtain the equation
\[ A(x)\psi_n^{(A)}(x) = (r + q^{-2n}A)\psi_n^{(A)}(x). \] (4.11)

Assuming the form
\[ \psi_n^{(A)}(x) = f(x)g(x)\xi_n^{(A)}(x) \]
for the eigenfunctions, we obtain the following equation for the function \( \xi_n^{(A)}(x) \):
\[ e^{s^2 + 2isx}\xi_n^{(A)}(x - 2is) - (e^{s^2 + 2isx} + 1)\xi_n^{(A)}(x - is) - q^{-2n}\frac{A}{r}\xi_n^{(A)}(x) = 0. \] (4.12)

Substitution of the Fourier expansion
\[ \xi_n^{(A)}(x) = \sum_{m=-\infty}^{\infty} d_{m,q}^{(n,A)} e^{-2imsx} \]
into Eq. (4.12) yields the recurrence relation
\[ e^{s^2 - 3(m+1)s^2}(e^{(m+1)s^2} - e^{-(m+1)s^2})d_{m+1,q}^{(n,A)} = -\left(e^{-2ms^2} + q^{-2n}\frac{A}{r}\right) d_{m,q}^{(n,A)} \]
for the coefficient \( d_{m,q}^{(n,A)} \). The coefficients of the \( d_{m+1,q}^{(n,A)} \) terms vanish identically for \( m = -1 \). From this observation, we find the equation \( d_{-1,q}^{(n,A)} = 0 \) and the restriction \( m \geq 0 \). Accordingly, we obtain the recursion formula
\[ d_{m,q}^{(n,A)} = d_{0,q}^{(n,A)}(-q)^m \prod_{l=1}^{m} \frac{1 + q^{-2(n-l+1)}\frac{A}{r}}{1 - q^{-2l}} \quad \text{(for } m = 0, 1, 2, \ldots \text{)} \] (4.13)
for the coefficients \( d_{m,q}^{(n,A)} \).

From the above, we find that the eigenfunctions of the two inequivalent sectors \( (A = -r \text{ and } r \leq A < rq^2) \) can be written by
\[ \psi_n^{(A)}(x) = (-is)^n q^{n/2}d_{0,q}^{(n,A)} f(x)g(x) e^{-insx}H_n^{(A)}(x;e^{s^2}), \] (4.14)
where \( H_n^{(A)}(x;e^{s^2}) \) is a generalization of the \( q \)-Hermite function defined by
\[ H_n^{(A)}(x;e^{s^2}) = \left(\frac{i}{s}\right)^n \sum_{m=0}^{\infty} (-1)^m \left(\prod_{l=1}^{m} \frac{1 + q^{-2(n-l+1)}\frac{A}{r}}{1 - q^{-2l}}\right) \times \exp \left((n - 2m)\left[isx - \frac{1}{2} s^2 \right]\right). \] (4.15)

For \( A = -r \), our \( q \)-Hermite function \( H_n^{(A)}(x;e^{s^2}) \) is related to the well-known \( q \)-Hermite function \( H_n(q^2 \sin sx|q) \) for the Fock sector.

Applying the operator \( A(x)^\dagger \) to the eigenfunctions, we find, from Eq. (3.9), that the coefficients \( d_{0,q}^{(n,A)} \) are expressed by
\[ d_{0,q}^{(n,A)} = \left(\frac{1}{ie^{ia_0 - s^2}}\right)^n \prod_{m=1}^{\infty} \left[\frac{1}{\sqrt{1 + q^{-2m}\frac{A}{r}}}\right] d_{0,q}^{(0,A)}, \quad n > 0 \]
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and

$$d_{0,q}^{(-n,A)} = \left(\frac{1}{i} e^{ia_0-s^2}\right)^{-n} \prod_{m=1}^{n} \left[\sqrt{1 + q^{2(m-1)}A/r}\right] d_{0,q}^{(0,A)}, \quad n > 0.$$  

The normalization constant $d_{0,q}^{(0,A)}$ is given by

$$d_{0,q}^{(0,A)} = \left[\int_I dx |f(x)g(x)|^2\right]^{-\frac{1}{2}}, \quad (4.16)$$

where $I$ is the domain of integration, which depends on the global structure of the $q$-deformed oscillator.\(^{18}\)

4.2. The $x$-representation of D-oscillator

For D-oscillator, there exists only one irreducible representation space, $F_0 (A = -r)$, since the deformation parameter is given by $q = e^{-s^2} \leq 1$. From the result obtained in §3, we find the equation

$$A(x)^\dagger A(x)\psi_n^{(A)}(x) = (r + q^{-2n}A)\psi_n^{(A)}(x). \quad (n = 0, 1, 2, \cdots) \quad (4.17)$$

In analogy to the case of M-oscillator, we find that the eigenfunctions of $A(x)^\dagger A(x)$ are given by

$$\psi_n^{(A)}(x) = (-is)^n d_{0,q}^{(n,A)} f(x)g(x)H_n^{(A)}(x; e^{-s^2}), \quad (4.18)$$

where $H_n^{(A)}(x; e^{-s^2})$ is the $q$-Hermite function\(^9,11,19\) which takes the form

$$H_n^{(A)}(x; e^{-s^2}) = s^{-n}H_n(sin sx; e^{-2s^2})$$

$$= \left(\frac{i}{s}\right)^n \sum_{m=0}^{\infty} (-1)^m \left(\prod_{l=1}^{m} \frac{1 + q^{2(n-l+1)}x}{1 - q^{2l}}\right) e^{i(2m-n)sx} \quad (4.19)$$

and

$$d_{0,q}^{(n,A)} = \left(\frac{1}{i} e^{ia_0}\right)^n \prod_{m=1}^{n} \left[\sqrt{1 + q^{2m}r}\right] d_{0,q}^{(0,A)},$$

and the normalization constant $d_{0,q}^{(0,A)}$ is given by

$$d_{0,q}^{(0,A)} = \left[\int_I dx |f(x)g(x)|^2\right]^{-\frac{1}{2}}. \quad (4.20)$$

§5. Discussion

To construct the eigenfunctions corresponding to the irreducible representations of $q$-deformed oscillators of the Macfarlane and Dubna types, we followed the Kulish formulation given in terms of the boson oscillator algebra $A_q$. The approach employing the algebra $A_q$ reveals the existence of non-trivial structure of the $q$-deformed oscillator. The introduction of the boson oscillator algebra $A_q$ leads to redundant...
degrees of freedom in the representation space. We interpret such degrees of freedom as a kind of gauge freedom, and by fixing this gauge freedom, we specify a definite physical state in the representation space. There exist two inequivalent irreducible representation spaces, \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \), for \( q > 1 \), and one irreducible representation space, \( \tilde{\mathcal{F}}_0 \), for \( q < 1 \). The spaces \( \mathcal{F}_0 \) and \( \tilde{\mathcal{F}}_0 \) are characterized by vanishing central charge and a discrete spectrum. By contrast, the eigenvectors on \( \mathcal{F}_1 \) possess non-vanishing central charge and a continuous spectrum.

We examined the classical limit (c-deformed limit) of q-deformed oscillators in a previous paper.\(^{19}\) The upper bound energy \( E = (q + q^{-1})/2(q - q^{-1}) \) in \( \mathcal{F}_0 \), as well as the lower bound energy in \( \mathcal{F}_1 \), is reduced to the distinct energy between the closed and open orbits of the M-oscillator. We can interpret the states on \( \mathcal{F}_0 \) and \( \tilde{\mathcal{F}}_0 \) as corresponding to closed classical orbits and the states on \( \mathcal{F}_1 \) as corresponding to open classical orbits in the phase space.

The construction carried out in \( \S 3 \) shows that the q-deformed oscillator for \( q > 1 \) possesses two irreducible (abstract) representations, which consist of the Fock and non-Fock representations \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \), and that for \( q > 1 \) there exists only one irreducible (abstract) representation, \( \tilde{\mathcal{F}}_0 \). Employing the abstract constructions in \( \S 3 \), the eigenfunctions on \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) (M-oscillator) and the eigenfunctions on \( \tilde{\mathcal{F}}_0 \) (D-oscillator) were constructed in \( \S 4 \). We emphasize that this is the first derivation of the eigenfunctions on the non-Fock type \( \mathcal{F}_1 \). The eigenstates on \( \mathcal{F}_1 \) correspond to open orbits in the classical deformed oscillator. We identify the eigenfunctions on \( \mathcal{F}_1 \) with the solutions of the Bloch type studied in Ref. 19). As an illustration, let us consider a simple specification of Eq. (4.8) by choosing \( b_0 = 1, b_k = 0 (k \neq 0) \) and

\[
\eta(x) = \sum_{m = -\infty}^{\infty} e^{(x - ims)^2/2}.
\]

We then obtain the following compact form for the part-function \( f(x) \):

\[
f(x) = \sum_{m = -\infty}^{\infty} e^{-m^2s^2} e^{-imsx}.
\] (5.1)

This example shows that the wavefunctions on \( \mathcal{F}_1 \) appearing in Eq. (4.14) are in this case oscillating solutions with period \( 2\pi/s \). We identify these oscillating solutions with wavefunctions of the Bloch type\(^{19}\) with a continuous spectrum and period \( 2\pi/s \).

It is not unnatural to conjecture that the eigenfunctions of M-oscillator and D-oscillator have a common parametric representation.\(^{18}\) In fact, the eigenfunctions given in Eqs. (4.14) and (4.18) can be written

\[
\psi_n^{(A)}(x) = (-is)^n e^{n(s + t)^2/2} d_{0,q}^{(n,A)} f(x) g(x)e^{-in(s + t)x} H_n^{(A)}(x; e^{s^2 + t^2 + 3st}),
\] (5.2)

using the generalization of the unified \( q \)-Hermite function \( H_n^{(A)}(x; e^{s^2 + t^2 + 3st}) \) given by

\[
H_n^{(A)}(x; e^{s^2 + t^2 + 3st}) = \left(\frac{i}{s}\right)^n \sum_{m=0}^{\infty} (-1)^m \left( \prod_{l=1}^{m} \frac{1 + e^{-2st(n-l+1)} A}{1 - e^{-2lst}} \right)
\]
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\[ \times \exp \left\{ (n - 2m) \left[ isx - \frac{1}{2} (s + t)^2 \right] \right\}, \]  

(5.3)

where $H_n^{(A)}(x; e^{s^2 + t^2 + 3st})$ satisfies the tri-normal relation

\[ H_{n+1}^{(A)}(x; e^{s^2 + t^2 + 3st}) = \frac{i}{s} (e^{-\frac{1}{2} (s+t)^2 + isx} - e^{\frac{1}{2} (s+t)^2 - isx}) H_n^{(A)}(x; e^{s^2 + t^2 + 3st}) \]

\[ - \frac{1}{s^2} \left( 1 + e^{-2ns^2 A} r \right) H_{n-1}^{(A)}(x; e^{s^2 + t^2 + 3st}) \]  

(5.4)

and the recursion formula

\[ is \exp \left[ isx - \frac{1}{2} (s + t)^2 \right] H_n^{(A)}(x; e^{s^2 + t^2 + 3st}) \]

\[ = \left\{ e^{-ns^2} \exp \left[ 2isx - (s + t)^2 \right] H_n^{(A)}(x - is; e^{s^2 + t^2 + 3st}) - H_n^{(A)}(x; e^{s^2 + t^2 + 3st}) \right\} . \]  

(5.5)

Obtaining this generalization of the unified $q$-Hermite function (5.3) is a consequence of the formulation of the new sector $\mathcal{F}_1$.

We have attempted to describe the $q$-deformed oscillators in a unified formalism employing difference equations. In this formalism, it is possible to elucidate similarities and differences between the M-oscillator ($q > 1$) and D-oscillator ($q < 1$). Specifically, it is found that these two types of $q$-deformed oscillators possess similar wavefunctions in the Fock representation with a discrete energy spectrum. In the case of the M-oscillator, however, there also exist wavefunctions of the non-Fock representation with a continuous spectrum. The wavefunctions of the D-oscillator appearing in Eq. (4.18) are parity eigenstates, but the wavefunctions of the M-oscillator appearing in Eq. (4.14) are not invariant under the parity transformation. The global structures of both oscillator systems are determined by selecting the part-function $f(x)^{19}$ The specification simplifying Eq. (4.8) realized by choosing $\eta(x) = 1$, $b_k = 1$ and $b_l = 0 (l \neq k)$ for a certain fixed integer $k$ leads to a localized state centered at the point $2k\pi/(s - t)$. With the special choice of $f(x)$ given in Eq. (5.1), the wavefunctions of both oscillators become oscillating functions with period $2\pi/s$. Because the wavefunctions do not vanish as $x \to \infty$, they are interpreted as non-localized states. In particular, in the case of the non-Fock type on $\mathcal{F}_1$, a non-localized wavefunction with period $2\pi/s$ corresponds to a wavefunction of the Bloch type $^{19}$ that is related to open orbit in the $c$-deformed limit. With the generalized $q$-Hermite function $H_n^{(A)}(x; e^{s^2 + t^2 + 3st})$ in Eq. (5.3), we are able to describe all of the localized and non-localized solutions for the D-oscillator and the M-oscillator in a unified manner.

In this paper, we have studied the fundamental dynamics of one-dimensional $q$-deformed oscillators related to quantum group symmetries as one exactly solvable model. Interesting applications of $q$-deformed oscillators are found in studies of the phonon spectrum of $^4$He $^{21}$ $q$-thermostatistics $^{22}$ and in attempts to resolve divergence problems in quantum field theory. $^{23,24}$
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