Transverse Front Instability in Bistable Systems with Long-Range Interactions

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Using the Ginzburg-Landau equation with a long-range interaction, we study the stability of a planar front with respect to transverse perturbations in bistable systems. It is well known that when a bistable system has competing short-range and long-range interactions, the front connecting two stable states can exhibit transverse instability. We focus on the effects of the nonlocal nature of the interaction, using long-range interactions with exponential decay (weak nonlocality) and power-law decay (strong nonlocality). It is found that in the former case, the planar front can be stabilized by varying a parameter value, while in the latter case, the strong nonlocal nature of the interaction prevents stabilization of the front.

§1. Introduction

In nature, various patterns appear.\textsuperscript{1)} Independently of the details of the specific systems, spatial patterns with common morphological features are observed in different types of systems.\textsuperscript{2)} It is well known that such spatial patterns originate from a competition between interactions: short-range attractive interactions, which promote spatial homogeneity, and long-range repulsive interactions, which promote spatial heterogeneity.

In the case of bistable systems, competition between interactions can cause the instability of a front connecting the two stable states with respect to transverse perturbations. This instability is called the transverse front instability (TFI). It eventually leads to a labyrinth structure. The process of labyrinth formation has been observed in bistable reaction-diffusion systems,\textsuperscript{3)–7)} Langmuir films\textsuperscript{8)} and ferrofluids.\textsuperscript{9)} Recently, Hagberg et al. reported that the Swift-Hohenberg equation (SH),\textsuperscript{10)}

$$\partial_t \psi(r, t) = \varepsilon \psi - \psi^3 - (\nabla^2 + 1)^2 \psi,$$

(1.1)

exhibits the TFI, and that a planar front can be stabilized by increasing $\varepsilon$.\textsuperscript{11)} Here, $\psi(r, t)$ represents a real order parameter in 2D space. Although Eq. (1.1) contains terms representing a repulsive interaction ($-2\nabla^2 \psi$) and an attractive interaction ($-\nabla^2 \nabla^2 \psi$), these interactions are local.

In the present paper, we study the TFI in systems with long-range repulsive and short-range attractive interactions. The simplest equation describing such systems

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Fig. 1. Labyrinth formation processes in a 2D space of linear size $L = 64\pi$ with the grid spacing $\Delta x = \pi/16$. Snapshots for (top) the SH with $\varepsilon = 2.5$ at $t = 70, 300$ and $2500$, (middle) the WNS with $\varepsilon = 1.5$ at $t = 50, 250$ and $3800$, and (bottom) the SNS with $\varepsilon = 7.5 - 2\pi$ at $t = 70, 200$ and $5000$.  

may be  

$$
\partial_t \psi(r, t) = \epsilon \psi - \psi^3 + \nabla^2 \psi - \int K(r - r') \left[ \psi(r', t) - \psi(r, t) \right] dr',
$$

where the diffusion and the integral term represent the attractive and the repulsive interactions, respectively. The long-range nature of the systems in question has roots in electromagnetic forces, fast diffusion, and so on, and the degree of nonlocality depends on the system. The nonlocality of Eq. (1.2) is determined by the integral kernel $K(r)$. For example, reaction-diffusion systems in the fast-inhibitor limit can be described by Eq. (1.2) with $-\nabla^2 K(r) + a^2 K(r) = J\delta(r)$, where $a$ and $J$ are positive constants. In 2D space, we have $K(r) = K_w(r) \equiv \frac{J}{2\pi} K_0(a|\mathbf{r}|)$, where $K_0$ is the modified Bessel function of the second kind of order zero. The interaction has the asymptotic form $K_w(r) \sim e^{-a|\mathbf{r}|}$ ($|\mathbf{r}| \to \infty$), which represents a “weakly” nonlocal property. For another example, in magnetic systems, the long-range repulsive interaction is the dipole-dipole interaction, given by $K(r) = K_s(r) \equiv 1/|\mathbf{r}| - 1/\sqrt{|\mathbf{r}|^2 + 1}$. This interaction has the asymptotic form $K_s(r) \sim 1/|\mathbf{r}|^3$, which is “strongly” nonlocal.

The aim of this paper is to investigate the effect of the nonlocality on the TFI by using Eq. (1.2) with the two kernels $K_w$ and $K_s$. We show that there are two types of
the TFI, depending on the degree of the nonlocality of the interaction. Hereafter, we call a system described by Eq. (1.2) with \( K_w \) a “weakly nonlocal system” (WNS) and a system with \( K_s \) a “strongly nonlocal system” (SNS). The TFI and the labyrinth formation process studied below are observed in both a WNS and a SNS, as well as in the SH (Fig. 1).

The present paper is organized as follows. In §2, the TFI in the SH is briefly reviewed, and the TFIs in the WNS and the SNS are studied. In the WNS, the TFI has the same characteristics as that in the SH, while in the SNS, the strong nonlocality changes the characteristics. In §3, we discuss why this difference appears. Concluding remarks are given in §4.

§2. Stability of planar front solutions with respect to transverse perturbations

In this section, we consider the WNS and the SNS in 2D space. For \( \epsilon \) larger than a threshold value, Eq. (1.2) has the two stable homogeneous states \( \psi_{\pm} = \pm \sqrt{\epsilon} \). Then, a front solution \( f(x) \) connecting the two stable states can exist and can become unstable with respect to transverse perturbations; that is, the TFI occurs. In particular, we focus on the TFI of a planar front solution, in which 1D fronts \( f(x) \) are aligned along the \( y \) axis in 2D space.

The TFI of a planar front can be described by the equation of motion of the front position, \( X(y, t) \) (Fig. 2). Let the planar front solutions be represented as \( \psi(x, y, t) = f(x - X(y, t)) \), with \( X(y, t) = \text{const} \). When the locus \( X(y, t) \) of the front varies slowly with \( y \), we can write

\[
\psi(x, y, t) = f(x - X(y, t)) + u(r, t). \tag{2.1}
\]

Here \( u(r, t) \) represents the deformation of the front profile, which is caused by the \( y \) dependence of \( X(y, t) \). We assume that the deformation \( u \) is a small quantity and has the form \( u(r, t) = u(x - X(y, t), \partial_y X, \partial_y^2 X, \cdots) \). Substitution of Eq. (2.1) into Eq. (1.2) yields

\[
Lu = -f' \partial_t X + f' \partial_y^2 X + f'' [\partial_y X]^2 - \int K(r - r')g(r', y, t)dr', \tag{2.2}
\]

where \( g(r', y, t) = f(x' - X(y', t)) - f(x' - X(y, t)) \) and \( LA(r) = \epsilon A - 3f^2(x)A + \partial_y^2 A - \int K(r - r')[A(r') - A(r)]dr' \). Here we ignore the terms nonlinear in \( u \) and \( \partial_t u \) because they are much smaller than the terms in Eq. (2.2). It should be noted that the relation \( Lf' = 0 \) is satisfied, and therefore \( (f', Lu) = (Lf', u) = 0 \), where \( (A, B) \equiv \int A(x)B(x)dx \). The solvability condition of Eq. (2.2) yields the evolution
equation of $X(y,t)$,
\[
\partial_t X = \partial_{yy} X - \int \frac{f'(x - X(y,t))K(r - r')g(r',y,t)}{(f',f')} dr'dx.
\] (2.3)

Here we have used the relation $f(x) = -f(-x)$.

The TFI of the planar front has been investigated in the case of the SH.\cite{11}

Before considering the case of Eq. (1.2), we briefly review the results for the SH.

Next we investigate the TFI in the case of Eq. (1.2) by using Eq. (2.3).

2.1. \textit{In the case of the Swift-Hohenberg equation}

In the case of the SH, (1.1), the equation of motion for $X(y,t)$ is given by
\[
\partial_t X = -2(1 - \alpha)\partial_{yy}^2 X - \beta_1 \partial_{yy}^4 X + \beta_2 \partial_y^2 X \partial_{yy}^2 X,
\] (2.4)

where $\alpha$, $\beta_1$ and $\beta_2$ are positive constants determined by the planar front solution $f(x)$ (see Appendix A). Equation (2.4) has the same form as the equation describing the zigzag instability in electrohydrodynamic convection in nematic liquid crystals.\cite{16}

In Ref. 11), the linear stability of the planar front was investigated by considering the linearized equation for $X(y,t)$, which is identical to Eq. (2.4) without the nonlinear term. The linear dispersion relation for the planar front ($X = \text{const}$) is given by
\[
\sigma_{\text{SH}}(k) = 2(1 - \alpha)k^2 - \beta_1 k^4.
\] (2.5)

For $\alpha \geq 1$, $\sigma_{\text{SH}}$ is non-positive for arbitrary $k$, and thus the planar front is linearly stable. Figure 3 shows the dependence of $\alpha$ on $\varepsilon$. (This figure corresponds to Fig. 5 in Ref. 11.) As shown in Fig. 3, the critical point is $\varepsilon_c \approx 4.8$, at which we have $\alpha = 1$. Above (below) the critical point $\varepsilon_c$, $\alpha$ is greater (less) than unity; that is, the planar front is linearly stable (unstable).

As $\varepsilon$ is increased to $\varepsilon_c$, the spatiotemporal scales of $X(y,t)$ diverge as follows. The spatial scale $\lambda$ and the temporal scale $\tau$ can be estimated as $\lambda = 2\pi / k_{\text{max}}$ and $\tau = 1 / \sigma_{\text{max}}$. Here, as shown in Fig. 4, $k_{\text{max}} = \sqrt{(1 - \alpha) / \beta_1}$ is the wave number that maximizes the dispersion relation (2.5), and $\sigma_{\text{max}} = (1 - \alpha)^2 / \beta_1^2$ is the maximum value. Near the critical point, because $\alpha$ increases almost linearly with $\varepsilon$, we find $k_{\text{max}} \sim (\varepsilon_c - \varepsilon)^{1/2}$ and $\sigma_{\text{max}} \sim (\varepsilon_c - \varepsilon)^2$. Therefore, the spatiotemporal scales exhibit the following critical behavior:
\[
\lambda = \frac{2\pi}{k_{\text{max}}} \sim (\varepsilon_c - \varepsilon)^{-1/2},
\] (2.6)
\[
\tau = \frac{1}{\sigma_{\text{max}}} \sim (\varepsilon_c - \varepsilon)^{-2}.
\] (2.7)
2.2. In the WNS and the SNS cases

Here we consider the stability of the planar front in Eq. (1.2) by studying Eq. (2.3). The linear dispersion relation for Eq. (2.3) in the case of a planar front is given by

\[ \sigma_{\text{NS}}(k) = -k^2 + L_k, \]

where \( L_k = \bar{K}_k - \bar{K}_0 \), and \( \bar{K}_k \) is the Fourier transform of \( \bar{K}(y-y') = -\int f'(x)K(r-r')f'(x')dx'dx'/\langle f', f' \rangle \).

Let us again define \( k_{\text{max}} \) and \( \sigma_{\text{max}} \) as in Fig. 4. When both \( k_{\text{max}} \) and \( \sigma_{\text{max}} \) vanish, the planar front \((X = \text{const})\) is linearly stable. Contrastingly, when \( k_{\text{max}} \) and \( \sigma_{\text{max}} \) are positive, the planar front is linearly unstable. We numerically calculated \( k_{\text{max}} \) and \( \sigma_{\text{max}} \) for both the WNS and the SNS (Fig. 5). In the WNS, as \( \epsilon \) is increased, \( k_{\text{max}} \) and \( \sigma_{\text{max}} \) decrease and vanish at \( \epsilon = \epsilon_c \approx 15.34 \). Furthermore, we find \( k_{\text{max}} \sim (\epsilon_c - \epsilon)^{1/2} \) and \( \sigma_{\text{max}} \sim (\epsilon_c - \epsilon)^2 \) near the critical point as shown in Fig. 5(a). Therefore, the spatial and the temporal scales exhibit the same behaviors as Eqs. (2.6) and (2.7), describing the SH.

In the SNS, however, both \( k_{\text{max}} \) and \( \sigma_{\text{max}} \) decrease exponentially with increasing \( \epsilon \) [Fig. 5(b)]. This fact implies that in the SNS, \( k_{\text{max}} \) and \( \sigma_{\text{max}} \) are positive for finite \( \epsilon \); that is, the planar front is always linearly unstable. In this case, both the spatial and temporal scales increase exponentially with \( \epsilon \).
§3. Discussion

Here we discuss the difference between the TFIs in the WNS and the SNS. It is found that the linear dispersion relation for a planar front has the form displayed in Fig. 4, and that the instability is determined by the stability of the long-wavelength modes. Therefore, we focus on the case $k \sim 0$ in the following discussion.

First, let us consider the case of the SH, (1.1), in which the attractive and repulsive interactions are represented by $-\nabla^2 \nabla^2 \psi$ and $-2\nabla^2 \psi$, respectively. For $k \sim 0$, the contribution of the former to the dispersion relation (2.5) is $-2\alpha k^2$, and that of the latter is $2k^2$. Because both contributions are of order $k^2$, the stability of the planar front is determined by the sign of the sum of the coefficients $2(1 - \alpha)$.

Because $\alpha$ increases with $\varepsilon$ as shown in Fig. 3, there is a critical point $\varepsilon_c$ at which $\alpha = 1$.

Next, we consider Eq. (1.2), in which the attractive and repulsive interactions are represented by $\nabla^2 \psi$ and $-\int K(r - r')[\psi(r') - \psi(r)]dr'$, respectively. The contribution of the former to the dispersion relation (2.8) is $-k^2$, and that of the latter is $L_k = \bar{K}_k - \bar{K}_0$, which vanishes at $k = 0$ and satisfies $dL_k/dk = d\bar{K}_k/dk = 0$, due to the symmetry $\bar{K}(y) = \bar{K}(-y)$.

In the SNS case, $\bar{K}(y)$ has the asymptotic form $|\bar{K}(y)| \sim 1/|y|^3 (|y| \to \infty)$, which leads to $d^2L_k/dk^2|_{k=0} = d^2\bar{K}_k/dk^2|_{k=0} \sim 1/|y|dy = \infty$. This divergence gives $L_k/k^2 \to \infty (k \to 0)$. Therefore, the contribution $L_k$ of the repulsive interaction to the dispersion relation is much greater than that of the attractive interaction, $-k^2$, and is dominant for $k \sim 0$. Thus, a planar front is always unstable. In addition, estimation of the Lyapunov functional of Eq. (1.2) shows that in the SNS, $L_k$ is proportional to $-k^2 \log k$ for $k \sim 0$, which is consistent with the above result. Actually, we numerically confirmed $L_k \sim -k^2 \log k$ for $k \sim 0$, as shown in Fig. 6.

In the WNS case, we find $d^2L_k/dk^2|_{k=0} < \infty$. This result implies that the contribution $L_k$ of the repulsive interaction is of order $k^2$, as in the SH case. This is the reason that the TFI in the WNS can be stabilized and exhibits the critical behavior (2.6) and (2.7) observed in the SH.

As discussed above, the strong nonlocality of the kernel $K_s$ causes the divergence of $d^2L_k/dk^2$ at $k = 0$, which prevents the stabilization of the planar front. When the long-range repulsive interaction decays to zero faster than $|r|^{-3}$, the divergence does not occur as in the WNS case. Therefore, we expect that in systems with a repulsive interaction that decays more rapidly than $|r|^{-3}$, a planar front can be stabilized by varying $\varepsilon$, and the same critical behavior, described by Eqs. (2.6) and (2.7), can be
observed near the stabilization point.

§4. Concluding remarks

We have studied the TFI of a planar front in the WNS and the SNS, focusing on the effect of the nonlocality of the interaction. We found that in the WNS, the TFI exhibits the same characteristics as those in the SH, which has no long-range interaction. However, in the SNS, the TFI possesses different properties from those in the WNS and the SH. We found that this difference results from the strong nonlocality of the interaction.

After the TFI, the region with stripe structures expands and covers the whole area. In this process, the stripe region invades the homogeneous regions uniformly in the WNS and the SNS, and randomly in the SH (see the middle column of Fig. 1). Although we expect that the nonlocality of the interaction aligns the boundary between the stripe and homogeneous regions, the details of this phenomenon should be examined in future. When the stripe region covers the whole area, an ordering process of the labyrinth structure is observed. For this ordering process, the nonlocal phase dynamics of stripe structures have been derived.18,19) Other differences between local and nonlocal dynamics may be found also in this process.

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Appendix A

Derivation of Eq. (2.4)

In this appendix, we derive Eq. (2.4) used in §2.1. This derivation is based on the phase reduction method.20) Let the magnitude of the spatial scale of $X(y, t)$ be $\delta^{-1/2}$, with the smallness parameter $\delta > 0$. Substituting Eq. (2.1) into Eq. (1.1), and expanding $\partial_t X$ and $u$ as $\partial_t X = \partial_t X^{(1)} + \partial_t X^{(2)} + \cdots$ and $u = u^{(1)} + u^{(2)} + \cdots$ where $\partial_t X^{(n)}, u^{(n)} = O(\delta^n)$, we obtain the balance equations

\begin{align}
\mathcal{L}u^{(1)} &= -f'\partial_t X^{(1)} - \mathcal{P}_1 f, \\
\mathcal{L}u^{(2)} &= -f'\partial_t X^{(2)} - \partial_\zeta u^{(1)} \partial_t X^{(1)} + 3f\{u^{(1)}\}^2 \\
&\quad + \sum_{n=1}^{\infty} \partial_{\partial_\zeta^n X^{(1)}} \partial_t X^{(1)} - \mathcal{P}_1 u^{(1)} - \mathcal{P}_2 f, \\
\end{align}

and so on. Here, we have $\zeta = x - X$, $\mathcal{P}_1 = -2(\partial_x^2 + 1)\partial_\zeta^2$, $\mathcal{P}_2 = -\partial_\zeta^4$ and $\mathcal{L} = \varepsilon - 3f(\zeta)^2 - (\partial_x^2 + 1)^2$. Noting the relation $\mathcal{L}f' = 0$, we find $(f', \mathcal{L}u) = (\mathcal{L}f', u) = 0.$
Therefore, the solvability condition of Eq. (A.1) gives
\[ \partial_t X^{(1)} = -2(1 - \alpha)\partial_y^2 X, \] (A.3)
where \( \alpha = (f''', f'')/(f', f') > 0 \). Here we have used the relation \( f(-x) = -f(x) \).
Equation (A.3) for \( \alpha > 1 \) is the diffusion equation, and it exhibits a relaxation process to a planar front solution \( (X = \text{const}) \). For \( \alpha < 1 \), however, Eq. (A.3) cannot describe the dynamics of \( X \), due to the negative diffusion. For this reason, we must consider a higher-order equation.

Equations (A.1) and (A.3) lead to
\[ u^{(1)} = a_1(\zeta)\partial_y^2 X + a_2(\zeta)(\partial_y X)^2, \] (A.4)
where \( a_1 \) and \( a_2 \) obey
\[ \mathcal{L}(\zeta)a_1(\zeta) = -2\{f'''(\zeta) + \alpha f'(\zeta)\}, \] (A.5)
\[ \mathcal{L}(\zeta)a_2(\zeta) = 2f'''(\zeta), \] (A.6)
with the condition \( (f', a_1) = (f', a_2) = 0 \). Noting that \( \mathcal{L} \) and the r.h.s. of Eq. (A.5) are even and the r.h.s. of Eq. (A.6) is odd with respect to \( \zeta \), we find that \( a_1 \) and \( a_2 \) are even and odd functions of \( \zeta \), respectively.

Finally, the solvability condition of Eq. (A.2) yields
\[ \partial_t X^{(2)} = -\beta_1\partial_y^4 X + \beta_2(\partial_y X)^2\partial_y^2 X, \] (A.7)
where \( \beta_1 = 1 - 2(f', a_1'')/(f', f') \), \( \beta_2 = 6\alpha + 6\{f, f a_1 a_2\} + 2(1 - \alpha)(f', a_2')/(f', f') \). Here, we have used the fact that \( a_1 \) and \( f' \) are even functions of \( \zeta \), and \( a_2 \), \( f \) and \( f'' \) are odd. We numerically confirmed that \( \beta_1 \) and \( \beta_2 \) are positive. In this way, we obtain \( \partial_t X(y, t) = \partial_t X^{(1)} + \partial_t X^{(2)} \), that is, Eq. (2.4).

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