Improved Renormalization Group Analysis for Yang-Mills Theory

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We apply an improved renormalization group analysis for pure Yang-Mills theory in four dimensions in the Landau gauge and O\((N)\) nonlinear sigma model in two dimensions. We find a pole in the gluon two-point function at one-loop order, which is generated non-perturbatively. In Yang-Mills theory this may correspond to the expectation value of a gauge invariant operator and emerges as \(M_P/\Lambda_{\text{MS}} \approx 0.81\), where \(\Lambda_{\text{MS}}\) is the \(\overline{\text{MS}}\) scale.

\section*{§1. Introduction}

The improved perturbation theory formulated by Gell-Mann and Low\textsuperscript{1}) is a theoretical framework with which, using the ideas of the renormalization group with the results of perturbation theory to a given order, one can determine something about the next order of perturbation theory. Employing the same type of philosophy, variational approximation methods, by which one can extract certain nonperturbative information from the results of perturbation theory, have been developed in both theoretical and numerical approaches.

The essential idea of the variational approximation scheme, which is often called an improved mean field approximation in the literature, can be summarised simply by the expression “the principle of minimal sensitivity”.\textsuperscript{2}) Suppose that we have a Lagrangian \(\mathcal{L}\) of interest. For concreteness, we use the \(\phi^4\) theory as an example. We first introduce a mean-field Lagrangian \(\mathcal{L}_m = m^2 \phi^2/2\), where \(m^2\) is a parameter to be tuned such that the quantity we would like to evaluate is independent of this parameter. We then rewrite the original Lagrangian as

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m_0^2}{2} \phi^2 - \frac{g}{4!} \phi^4 \\
\Rightarrow \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 + \lambda \left( -\frac{m_0^2}{2} \phi^2 + \frac{m^2}{2} \phi^2 - \frac{g}{4!} \phi^4 \right),
\]

where we have introduced the formal coupling constant \(\lambda\). We then regard the first two terms as the unperturbed action and the terms in the parentheses as a perturbation with respect to \(\lambda\), and calculate the quantity of interest using perturbation theory. This \(\lambda\) will be set to 1 at the end of calculation, and thus, in principle, any physical quantity calculated with this Lagrangian cannot depend on \(m^2\), but, as we will see, the \(m^2\) dependence remains if terms of higher order in \(\lambda\) are dropped. Finally, we tune the parameter \(m^2\) for the calculated quantity to be independent of it.
as should be the case. In other words, one should choose the parameter for the quantity of interest to be on a plateau whose emergence signals that the approximation scheme works well. This is the meaning of the expression “the minimal sensitivity”.

When we apply the variational approximation for a massless theory, it is convenient to start with the massive counterpart $L_{\text{massive}}(m^2, g)$ to the massless theory and employ the ordinary perturbation theory with respect to the coupling $g$. All we have to do is to replace the mass and coupling as $m^2 \rightarrow (1 - \lambda)m^2$ and $g \rightarrow \lambda g$ in the result of the perturbation theory and then keep the desired order in the formal coupling constant $\lambda$ and set $\lambda = 1$. Here, the mass plays the role of the parameter to be tuned.

There have been many works carried out with this method. For the case of anharmonic oscillators, it was shown that the new perturbation series obtained with this recipe is convergent.\(^3\) (See also Ref. 4.) Some progress has been made in the study of matrix models and reduced Yang-Mills models.\(^5\)–\(^15\) In Ref. 16), $D$-dimensional pure Yang-Mills theory is studied in the context of the reduced model and convergence of Monte Carlo data is demonstrated. In Ref. 7) and, subsequently, Ref. 8), the IIB matrix model, which is defined as the reduced maximally supersymmetric Yang-Mills theory and is also conjectured to be a nonperturbative definition of superstring theory, is studied, and the emergence of four-dimensional space-time as a vacuum of string theory is suggested.\(^*\)

Regarding applications to quantum field theory, in Ref. 17) the variational method is combined with the renormalization group method. The mass gap of the Gross-Neveu model in two dimensions is calculated and is found to exhibit good agreement with the exact solution. Furthermore, in a subsequent series of papers,\(^18\) the method is applied to QCD in order to study, among other things, the dynamical origin of the quark mass. In this paper, we call the method developed there “the improved renormalization group analysis”, and we review it in §2.1 with some refinement.

This improved mean field approximation can be regarded as an “improved Taylor expansion”, which is a general scheme to improve the convergence of a Taylor expansion series of interest.\(^8\) We note here that when we attempt to apply the improved mean field approximation to quantum field theories, we usually face the problem of determining at which stage we should make the replacement, i.e., how to make it compatible with the renormalization procedure. This problem arises because both involve the choice of the unperturbed part of the action, and usually they do not seem to be compatible. The proposal made in 17) and 18) is that the replacement is carried out for the bare Lagrangian, as the original mean field approximation suggests, and then the same renormalization scheme as for the original Lagrangian is applied. It was shown that this leads a consistent improved series; that is, the divergences are removed by this renormalization. Here we employ another approach. Inspired by the spirit of the improved Taylor expansion, we first apply usual renormalization scheme and improve the quantity of interest by using a

\(^*\) In Ref. 12), higher-order calculations are given and the result strengthens the original idea. (See also Ref. 14) and references therein.)
renormalization group technique, and then apply the improved Taylor expansion to functions that admit a Taylor expansion. Surprisingly, our method gives the same improved function as the method in Refs. 17) and 18). We therefore believe that our general scheme still can be interpreted as an improved mean field approximation.

In this paper, we apply the improved renormalization group analysis to the pure Yang-Mills theory in four dimensions in order to explore the nonperturbative nature of the theory, such as confinement. Though lattice theory provides a method for nonperturbative study, there is still a great need for analytic methods that we could trust. Recently, there has been growing interest in the condensate of a mass dimension two operator in QCD and Yang-Mills theory. Such condensation is believed to be a key to understanding the confinement problem.19)–22),∗) The operator considered here is $\Delta = \frac{1}{2}(\text{volume})^{-1}\langle \text{min}_U \int d^4x (A_U^\mu)^2 \rangle$, where $U$ represents a gauge transformation, (volume) denotes the space-time volume, and “min” represents the operation of taking minimum with respect to gauge transformations. This operator is indeed gauge invariant, but non-local in a general gauge. In the Landau gauge, this is the expectation value of the gauge-variant local operator $\int \langle \frac{1}{2} A_\mu A^\mu \rangle$.22) Then, in the Landau gauge, the mass term of the gluons, $\text{Tr} m^2 A_\mu A^\mu$, might be induced nonperturbatively through the four-point interaction.**) This “induced mass” may appear as a pole of the gluon propagator. Thus, we perform the improved renormalization group analysis to evaluate the mass pole of the gluon propagator in the Landau gauge. We do not demand that this “mass” be the physical mass of the gluon, since the mass term is not gauge invariant. However, what we calculate is assumed to be the expectation value of a gauge invariant quantity. Therefore, though this does not have the meaning of a mass, except in the Landau gauge, the quantity itself is thought to have a gauge invariant meaning. In other words, we simply propose a possible way to calculate $\Delta$ in Yang-Mills theory. The condensation of these kinds of operators has been discussed by means of lattice simulations21) as well as from a phenomenological point of view.19),22)

This paper is organized as follows. In the next section we explain the improved renormalization group method developed in Refs. 17) and 18), with our own refinement, employing the idea of the improved Taylor expansion. Then, in §3 we demonstrate an application to the $O(N)$ nonlinear sigma model in two dimensions. In §4 we calculate the mass pole of the gluon propagator in the Landau gauge by using the Curci-Ferrari model26) as a massive counterpart of Yang-Mills theory. We perform the calculation in a general covariant gauge, in which we can obtain the result in the Landau gauge by setting the gauge parameter $\xi$ to zero. Section 5 is devoted to conclusions and discussion.

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∗) For recent developments, see, e.g., Refs. 23), 24) and references therein.

**∗) Note that no clear explanation has been given of the connection between mass generation in gauge theory and the condensate of the operators. There have been several arguments regarding this issue. (See Refs. 20) and 25).) Here we simply assume the mass generation of the Landau gauge.
§2. Improved renormalization group analysis

We first describe an improved renormalization group analysis that we subsequently apply to Yang-Mills theory on the basis of Refs. 17) and 18). We start with a Lagrangian density $\mathcal{L}(m^2, g, \mu)$, where $m^2$ is the mass parameter, $g$ is a dimensionless coupling constant, and $\mu$ is a mass scale which is introduced to keep the coupling $g$ dimensionless. This massive theory should be regarded as the massive counterpart of a massless theory of interest. As we have seen in the Introduction, in order to obtain an improved perturbation series in $\lambda$ for a given quantity, such as propagators, in the massless theory, it is sufficient to replace $m^2$ with $(1 - \lambda)m^2$ and $g$ with $\lambda g$ in the quantity calculated using perturbation theory with the massive action. After setting $\lambda = 1$, one can, at least naively, expect that this new quantity describes that of the massless theory. Therefore, we first review the method for obtaining a pole mass, in which we are interested here, improved by a renormalization group equation in the massive theory. As stated in the Introduction, we realize this substitution by use of the improved Taylor expansion, which is in general expected to improve the convergence of a series and to extract information regarding the original function from its perturbation series of finite degree. We can easily apply this method to a perturbation series in quantum field theory, even after we have improved the perturbation series by use of the renormalization group technique. In this case, we simply improve the quantity of interest, which is written as a series in a coupling constant, by replacing the original parameters with new ones, including a formal expansion parameter denoted by $\lambda$ in the Introduction. Here we note that there can be some quantities that do not admit a Taylor expansion, for example, a function of a fractional power in the coupling constant appearing through a nonperturbative refinement like the renormalization group. We do not consider such quantities and will improve only the parts that can be written as a power series with integer powers in the coupling constant. This is the general criterion for the improvement we propose in this paper. This procedure provides the same formula as that proposed in Refs. 17) and 18), where the improve mean-field approximation is applied to the bare Lagrangian, and then the renormalization and the renormalization group method are applied carefully. We thus conclude that our method is actually an improved approximation of a massless theory of interest.

Here, let us start by explaining how to improve the pole mass expression in which we are interested in this paper. As explained in Appendix B, a pole mass can be written in terms of these parameters as

$$M_p^2 = m^2 \sum_{n=0}^{\infty} \sum_{L=0}^{n} A_{n,L} \left[ \ln \left( \frac{m^2}{\mu^2} \right) \right]^{n-L} g^{2n}, \quad (2.1)$$

where the quantities $A_{n,L}$ are constant, that is, are independent of $m^2$, $\mu$ and $g$. The coefficients with $L < n$ are governed by the renormalization group equation and can be determined recursively with the condition $A_{0,0} = 1$, while the coefficients $A_{n,n}(\equiv A_n)$ need to be fixed by perturbative calculations and are called “non-logarithmic corrections”.
If we set the scale $\mu$ equal to $m$, which is accomplished at the specific point

$$\mathcal{M}^2 = m^2(\mu = \mathcal{M}),$$

then all the logarithmic terms vanish, and we obtain

$$M_P^2 = m^2(\mathcal{M}) \left( 1 + A_1 g^2(\mathcal{M}) + A_2 g^4(\mathcal{M}) + \cdots \right),$$

where we denote the running mass and coupling constant at the scale $\mu$ as $m^2(\mu)$ and $g^2(\mu)$, respectively. Although both $g^2(\mathcal{M})$ and $m^2(\mathcal{M})$ should be determined by the full renormalization group equation (RGE), we can approximate them by replacing them with $g^2_{1\text{-loop}}$ and $m^2_{1\text{-loop}}$, which are solutions to the RGE at one-loop order, namely

$$m^2_{1\text{-loop}} = \frac{m^2(\mu)}{1 + b_0 g^2(\mu) \ln \left( \frac{m^2_{1\text{-loop}}}{\mu^2} \right)}, \quad g^2_{1\text{-loop}} = \frac{g^2(\mu)}{1 + b_0 g^2(\mu) \ln \left( \frac{m^2_{1\text{-loop}}}{\mu^2} \right)},$$

where $b_0$ and $\gamma_0$ are the coefficients of the renormalization group functions $\beta(g^2) = -2b_0 g^4 - O(g^6)$ and $\gamma_m(g^2) = \gamma_0 g^2 + O(g^4)$. Thus we have

$$M_P^2 = m^2_{1\text{-loop}} \left( 1 + g^2_{1\text{-loop}} A_1 + g^4_{1\text{-loop}} A_2 + \cdots \right).$$

According to Appendix B, this approximation corresponds to the leading-logarithm approximation with perturbative non-log corrections. At this stage, we assert that the approximation is valid if the running coupling constant $g^2_{1\text{-loop}}$ remains small at $\mu = \mathcal{M}$. Thus, we can apply this approximation to asymptotically free theories, if $\mathcal{M}$ is large enough. Thus we should choose the initial value of the running mass $m^2(\mathcal{M})$ in order for these conditions to be satisfied.

2.1. One-loop improved RG analysis

Let us consider a pole mass at one-loop order in ordinary perturbation theory,

$$M_P^2 = m^2(\bar{\mu}) \left[ 1 - \frac{\gamma_0}{2} g^2(\bar{\mu}) \ln \left( \frac{m^2}{\bar{\mu}^2} \right) + A_1 g^2(\bar{\mu}) + O(g^4) \right],$$

where $\gamma_0$ must be such that this expression is consistent with the expression (2.5), and $A_1$, which comes from a finite part in the renormalization prescription, is the non-logarithmic correction. Note that we have used a “mass-independent renormalization scheme”, like the $\overline{MS}$ or $\overline{MS}$ scheme.

The leading logarithm contributions can be included as explained above, and then the pole mass becomes

$$M_P^2 = m^2_{1\text{-loop}}(m) \left( 1 + g^2_{1\text{-loop}}(m^2) A_1 \right) = \frac{m^2(\bar{\mu})}{1 + b_0 g^2 \ln \left( \frac{m^2_{1\text{-loop}}(m^2)}{\bar{\mu}^2} \right)} + m^2_{1\text{-loop}}(m) g^2_{1\text{-loop}}(m) A_1.$$
It is convenient to rewrite this as

\[ M_P^2 = \frac{m^2(\bar{\mu})}{(b_0g^2(\bar{\mu})F)^{\frac{m_2}{2\bar{\mu}}}} \left[ 1 + \frac{A_1}{b_0F} \right], \tag{2.8} \]

where

\[ F = \frac{1}{b_0g^2(\bar{\mu})} + \ln \left( \frac{m^2_{1\text{-loop}}}{\bar{\mu}^2} \right). \tag{2.9} \]

Note that \( F \) satisfies the recursive relation

\[ F = -\ln \left( \frac{m^2_{1\text{-loop}}}{2\bar{\mu}^2} \right), \tag{2.10} \]

with the basic scale \( \Lambda_{\text{MS}}^2 = \bar{\mu}^2 e^{-\frac{1}{b_0\bar{\mu}^2}} \), where \( \bar{\mu}^2 = 4\pi e^{-\gamma} \). Let us define the dimensionless parameter \( x = (b_0g^2)^{-\frac{\gamma_0}{2\bar{\mu}^2}} \frac{m^2}{\Lambda_{\text{MS}}^2} \). Then the expression for the pole mass and the above defining equation for \( F \) become

\[ M_P^2 = \Lambda_{\text{MS}}^2 \frac{x}{F^{\frac{m_2}{2\bar{\mu}}}} \left[ 1 + \frac{A_1}{b_0F} \right], \tag{2.11} \]

and

\[ F = -\frac{\gamma_0}{2b_0} \ln F + \ln x. \tag{2.12} \]

Note that this function has a logarithmic cut starting from \( x = 0 \) in the complex \( x \)-plane. We choose this cut so that \( F \) is analytic for \( x > 0 \).

Now we have a renormalization group expression for the pole mass at one-loop order, and the next task is to apply the variational method around a massless theory in the manner reviewed in the Introduction.

The improved renormalization group analysis for the massless theory is realized by the substitution

\[ m^2 \to (1 - \lambda)m^2, \quad g^2 \to \lambda g^2, \tag{2.13} \]

in the parts that admit power series expansions of integer powers in \( \lambda \). Note that we have improved the pole mass expression by use of the renormalization group, and actually its recursive form is a result of this refinement. As a result, there are some terms in \( M_P^2 \) that do not admit a Taylor expansion. Because the coupling constant appears in a nonperturbative fashion, it is not difficult to see that it is sufficient for the improvement to replace \( x \) with \( (1 - \lambda)x \) and to leave the rest unchanged. Then the mass \( M_P^2 \) comes to depend on the formal coupling \( \lambda \) and is expanded in power series in \( \lambda \) as

\[ M_P^2(\lambda) = \sum_{m=0}^{\infty} a_m \lambda^m. \tag{2.14} \]
We define the $n$-th order mass after setting $\lambda = 1$ as

$$M^2_{(n)} = \sum_{m=0}^{n} a_m$$

$$= \oint \frac{dz}{2\pi i} \left( \frac{1}{z} + \cdots + \frac{1}{z^{n+1}} \right) M^2_P(z)$$

$$= \Lambda^2_{MS} \int_L \frac{du}{2\pi i} \left( -1 + \frac{1}{u} \right) \frac{1}{F^{\gamma_0/2b_0}} \left[ 1 + \frac{A_1}{b_0 F} \right]. \quad (2.15)$$

Note that the analyticity of $F$ leads to the conclusion that $[F((1 - z)x)]^{-\gamma_0/2b_0}$ and $[F((1 - z)x)]^{-\gamma_0/2b_0}$ cannot be singular at the origin of the complex $z$ plane when $x > 0$, and therefore the first term in the integrand must vanish. Let us define $u \equiv n(1 - z)$. Then we have

$$M^2_{(n)} = \Lambda^2_{MS} x \int_L \frac{du}{2\pi i} \left( 1 - \frac{u}{n} \right)^{-(n+1)} \frac{1}{F^{\gamma_0/2b_0}} \left[ 1 + \frac{A_1}{b_0 F} \right]. \quad (2.16)$$

Here, the contour path $L$ is taken around the cut on the negative real axis in the $u$-plane, as shown in Fig. 1.

The approximation becomes better as higher-order terms are taken in the perturbative expansion in the formal coupling $\lambda$. Thus it is natural to take the limit $n \to \infty$, and we then obtain

$$M^2 \equiv \lim_{n \to \infty} M^2_{(n)} = \Lambda^2_{MS} x \int_L \frac{du}{2\pi i} e^{u} \frac{1}{F^{\gamma_0/2b_0}} \left[ 1 + \frac{A_1}{b_0 F} \right]. \quad (2.17)$$

From the defining relation, $F$ can be expanded around $x = 0$ (hence $u = 0$) and we have

$$[F(u)]^{-B} = (xu)^{-1} \left[ 1 + (xu)^{1/B} + \frac{B - 2}{2B} (xu)^{2/B} + O((xu)^{3/B}) \right], \quad (2.18)$$

where $B = \gamma_0/(2b_0)$. We find an approximate expression for the pole mass with the help of the formula

$$\frac{1}{F(z)} = \int_L \frac{dt}{2\pi i} e^{t/z}.$$
This expression is
\[
\frac{M^2(x)}{\Lambda^2_{\overline{MS}}} = 1 + \frac{1 + B A_1}{B b_0} + \frac{A_1}{b_0} \frac{x^{-1/B}}{\Gamma \left(1 + \frac{1}{B}\right)} + \left(1 + \frac{B^2 - 1 A_1}{2B^2 b_0}\right) \frac{x^{1/B}}{\Gamma \left(1 - \frac{1}{B}\right)}
\]
\[
+ \left(\frac{B - 2}{2B} + \frac{B^3 - 3B^2 + 4 A_1}{6B^3 b_0}\right) \frac{x^{2/B}}{\Gamma \left(1 - \frac{2}{B}\right)} + \cdots.
\] (2.20)

We would like to find a plateau with respect to the parameter \(x\) \(^a\) because the dynamically induced mass can be accurately approximated by the value on such a plateau.

We can further refine the above result by introducing a scale-changing parameter \(a\), which is defined as a scaling parameter that rescales \(\bar{\mu}\) to \(a\bar{\mu}\). Although the rescaling of a renormalization point does not affect the renormalization invariant quantities, such as the one-loop mass \(m^2_{1\text{-loop}}\), the perturbative mass does depend on the parameter \(a\). However, the pole mass must be independent of \(a\), and thus it is natural to tune \(a\) in such a way that a plateau clearly emerges. Once we introduce the parameter \(a\), the defining equation for \(F\) becomes
\[
F = \ln \left(\frac{xu}{a}\right) - \frac{\gamma_0}{2b_0} \ln F.
\] (2.21)

Finally, we arrive at
\[
\frac{M^2_P}{\Lambda^2_{\overline{MS}}} = x \int_{\mathcal{L}} \frac{du}{2\pi i} e^{u F - \frac{\gamma_0}{2b_0}} \left[1 + \left(\frac{A_1}{b_0} + \frac{\gamma_0}{2b_0} \ln a\right) \frac{1}{F}\right].
\] (2.22)

Thus, if we know \(b_0\), \(\gamma_0\) and \(A_1\), we can calculate \(M^2_P/\Lambda^2_{\overline{MS}}\) as a function of \(x\) and \(a\). We then search for a plateau of this function with respect to \(x\) by varying the parameter \(a\) and take the value on the plateau as the approximated value of \(M^2_P/\Lambda^2_{\overline{MS}}\).

§3. Improved RG analysis for the nonlinear sigma model

Having reviewed the basic strategy, we move on to the \(O(N)\) nonlinear sigma model in two dimensions as a preliminary example. The \(O(N)\) invariant Lagrangian density with an external field is
\[
\mathcal{L} = \frac{1}{2g^2} (\partial_{\mu} \mathbf{S} \cdot \partial^\mu \mathbf{S} - 2 \mathbf{h} \cdot \mathbf{S}),
\] (3.1)
where \(\mathbf{S}\) is the sigma model field of magnitude 1 (i.e. \(\mathbf{S} \cdot \mathbf{S} = 1\)), and \(\mathbf{h}\) is an \(N\)-dimensional external vector field. It is convenient to choose \(\mathbf{h} = (h, 0, \ldots, 0)\) by \(O(N)\) symmetry and decompose \(\mathbf{S}\) as \(\mathbf{S} = (\sigma = \sqrt{1 - \pi^2}, \pi^i)\), where \(\pi^i\) are the components of an \((N - 1)\)-dimensional vector field with the condition \(\pi^2 \leq 1\). Then

\(^a\) Recall that \(x\) corresponds to the mass parameter to be tuned.
the action can be rewritten as

\[ S = \frac{1}{2g^2} \int d^2x \left\{ (\partial_\mu \pi^i)^2 + \frac{\pi \cdot \partial_\mu \pi}{1 - \pi^2} - 2h\sigma \right\} \]

\[ = \frac{1}{2g^2} \int d^2x \left\{ (\partial_\mu \pi^i)^2 + (\pi \cdot \partial_\mu \pi)^2 + \cdots - 2h + h\pi^2 + \frac{h}{4}(\pi^2)^2 + \cdots \right\}, \quad (3.2) \]

where the external magnetic field \( h \) plays the role of the infrared regularization to give a mass to the \( \pi \) fields. We regard this as a massive counterpart of the original \( O(N) \) nonlinear sigma model without the external field.

The 1PI two-point function \( \Pi^{ij} \) of two \( \pi \) fields at one-loop order is

\[ \Pi^{ij}(q^2) = -\delta^{ij} \left[ q^2 + \frac{N - 3}{4} h \right] h^{-\epsilon/2} (4\pi)^{\epsilon/2 - 1} \Gamma(\epsilon/2), \quad (3.3) \]

where \( d = 2 - \epsilon \). It follows that the physical pole mass is

\[ M^2 = h \left[ 1 - \frac{g^2}{4\pi} h^{-\epsilon/2} \frac{N + 1}{4} (4\pi)^{\epsilon/2} \Gamma(\epsilon/2) \right]. \quad (3.4) \]

It is easy to see that there is no non-logarithmic perturbative correction at one-loop order.

We also obtain the renormalization functions

\[ \beta(g^2) = -\frac{N - 2}{2\pi} g^4 + O(g^6), \quad \gamma_m = \frac{N + 1}{8\pi} g^2 + O(g^4), \quad (3.5) \]

and thus we have

\[ b_0 = \frac{N - 2}{4\pi}, \quad \gamma_0 = \frac{N + 1}{8\pi}. \quad (3.6) \]

Substituting these into the pole mass formula (2.22), we find the plateau (in the case \( N = 3 \)) depicted in Fig. 2, in which we have plotted \( M^2/\Lambda_{\overline{MS}}^2 \) with respect to \( h \) with various values of \( a \). We recognise that a plateau emerges when we set \( a = 1 \), and the value on the plateau is \( M^2/\Lambda_{\overline{MS}}^2 = 1 \). This result is indeed identical to the exact value in the large-\( N \) limit, \( M^2/\Lambda^2 = 1 \). Note that also in the Gross-Neveu model, the one-loop result is exact in the large-\( N \) case.\(^{17}\) However, it is rather far from the exact value for \( N = 3 \), \( M/\Lambda_{\overline{MS}} = 8/e \simeq 2.94 \).\(^{\star} \) It can be seen that the deviation from the basic scale \( \Lambda \) depends on the value of the non-logarithmic perturbative correction. It has been previously shown that the two-loop results for the Gross-Neveu model exhibit good agreement with the exact results even for smaller \( N \).\(^{17}\)

Thus, we believe that if we proceed beyond one-loop order, we will obtain a more accurate result in this model as well.

\(^{\star}\) The expression for the exact mass gap evaluated in Ref. 29) is

\[ M = \frac{(8/e)^{\frac{1}{N-2}}}{\Gamma(1 + \frac{1}{N-2}) \Lambda_{\overline{MS}}}. \quad (3.7) \]
Let us comment on the spontaneous symmetry breaking here. Usually a non-zero expectation value of a mean field after the appropriate subtraction indicates that symmetry is broken spontaneously. However in the non-linear sigma model in two dimensions, there never occurs a spontaneous symmetry breaking because of its dimensionality. Thus a non-zero value of a mean field does not imply SSB. Rather, it merely indicates that the starting point of the perturbation theory does not respect the symmetry. In the non-linear sigma model, the generated mass term is indeed $O(N)$ invariant, and thus SSB does not occur. If the generated mass term breaks a symmetry, then one can conclude that SSB takes place and use the generated mass term as an order parameter, as in the Gross-Neveu model, where a fermion mass term breaks the chiral symmetry.

§4. Improved RG analysis for Yang-Mills theory

Let us start with a comment on what we are going to calculate using the improved RG analysis in pure Yang-Mills theory. The improved RG analysis presented here is basically a method to calculate a mass term in a theory. In Yang-Mills theory, the mass term for the gluon is known to be strictly forbidden by gauge symmetry. Indeed, it is shown in Ref. 25) that there is no mass dimension two BRST invariant local operator which can be a source of mass generation. In a specific gauge, however, there can exist a local mass dimension two operator which may be a source of mass generation, though this “mass” does not have any gauge-invariant meaning and is not a physical mass. Below we consider one of these operators, $\text{Tr} A_\mu A^\mu$, in the Landau gauge. This is gauge dependent and becomes nonlocal in a general gauge. However, as long as we start with the gauge-fixed action in the Landau gauge, in which there is no classical gauge invariance, we can think about the condensation of this operator, and through the four-point interaction of gluons, a mass term for the gluon may be generated. In the gauge-fixed action, BRST invariance is an
important symmetry, and therefore it has to be maintained throughout calculation. This operator, with a space-time integral, is indeed BRST invariant in the Landau gauge. In the following section, we consider the Curci-Ferrari model as our massive gauge theory counterpart. Although it seems at first sight that the mass term will undermine the unitarity of the original Yang-Mills theory, our mass term is fictitious. In the prescription of the improved RG method, we introduce a counterterm which cancels the effect of the mass term. Therefore, we expect that pathologies of the massive gauge theory will eventually disappear.

As we will see, the mass term has a nonperturbative nature; that is, it is proportional to the scale given by the dimensional transmutation. A proportional coefficient would be of order 1. This term might be corrected by perturbative quantum effects, like the instanton action. However, due to the lack of a clear understanding regarding the mass generation and condensation of this operator, it is difficult to calculate corrections to determine the exact amount of the generated mass. For this reason, we restrict our investigation to the leading order, and the formulation of a precise argument for these coefficients is postponed to a future study.

The Lagrangian of the Curci-Ferrari model that is our counterpart of $U(N)$ Yang-Mills theory is

$$
\mathcal{L} = -\frac{1}{4}(\partial_\mu A^a_\mu - \partial_\nu A^a_\nu)^2 - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 - \frac{1}{2}gf^{abc}A^a_\mu A^b_\nu(\partial_\mu A_{\nu c} - \partial_\nu A_{\mu c}) - \frac{1}{4}g^2f^{abc}f^{ade}A_{\mu b}A_{\nu c}A^a_d A^\nu_e \\
+ \partial_\mu c^a \partial_\nu c^a + g f^{abc}(\partial_\mu c^a) A^{\mu \nu} c^c - \frac{g}{2}(1 - \beta)f^{abc}(\partial_\mu A^a_\mu)c^b c^c \\
+ \frac{g^2}{8}\xi(1 - \beta^2)f^{abc}f^{ade}c^b c^d c^e \\
+ \frac{1}{2}m^2 A^a_\mu A^{a\mu} - m^2_{gh} \bar{c}^a c^a,
$$

where $\xi$ is the gauge parameter and $\beta$ is an extra parameter characteristic of the model. Here, the ghost mass is related to the gluon mass as $m^2_{gh} = \xi m^2$. The condition that the model possesses the BRST symmetry is given by

$$
\xi \beta m^2 = 0.
$$

We carry out the calculation by setting $\beta = 1$. The BRST invariance will be manifest in the last stage of the calculation, where we could regard the original mass as going to zero as a result of the subtraction or when we go to the Landau gauge, $\xi = 0$.

To find the perturbative expression for the pole mass of the theory, it is sufficient to calculate the one-particle irreducible (1PI) two-point function. We write the 1PI two-point function as follows:

$$
i \Pi^{\mu\nu}_{ab} \equiv i \delta_{ab} \left[ \pi_1 m^2 \eta^{\mu\nu} + \pi_2 q^2 \Delta^{\mu\nu} \right], 
$$

where $\Delta^{\mu\nu} = \eta^{\mu\nu} - q^\mu q^\nu / q^2$, and $\pi_1$ and $\pi_2$ should be calculated perturbatively.
Then, the full propagator $i \Sigma^{\mu \nu}$ is written in the form

$$i \Sigma^{\mu \nu} = \frac{-i/(1 - \pi_1)}{q^2 - 1 + \pi_1/2} \left( \eta^{\mu \nu} - \frac{q^\mu q^\nu}{1 + \pi_1/2} \right) + \frac{-i/(1 - \pi_1)}{q^2 - (1 + \pi_1/2)} \frac{q^\mu q^\nu}{1 + \pi_1/2}.$$  \hspace{1cm} (4.4)

Thus, the pole mass can be read off as

$$M_P^2 = m^2 \frac{1 + \pi_1}{1 - \pi_2} = m^2(1 + \pi_1 + \pi_2) + O(g^4).$$  \hspace{1cm} (4.5)

The 1PI two-point function is calculated up to one-loop order in a standard way as

$$i \Pi^{ab}_{\mu \nu} = \frac{ig^2 N \delta^{ab}}{(4\pi)^2} \int_0^1 dx \int_0^1 dy \int_0^1 dz \left[ \left( \frac{2}{\epsilon} \right) (m^2 \eta^{\mu \nu} N_{m0} + q^2 \Delta^{\mu \nu} N_{A0}) - \ln \Delta (m^2 \eta^{\mu \nu} N_{m0} + q^2 \Delta^{\mu \nu} N_{A0}) + 2(m^2 \eta^{\mu \nu} N_{m1} + q^2 \Delta^{\mu \nu} N_{A1}) \right],$$

where $\epsilon^{-1} = \epsilon^{-1} - \gamma + \ln(4\pi)$,

$$\Delta = [1 - (1 - \xi) \{ zx + (1 - z)y \}] m^2 - z(1 - z)q^2,$$  \hspace{1cm} (4.6)

and the explicit expressions for the coefficient functions $N_{m0}(x, y, z; \xi)$, $N_{m1}(x, y, z, m^2, q^2; \xi)$, $N_{A0}(x, y, z; \xi)$, and $N_{A1}(x, y, z, m^2, q^2; \xi)$ are listed in Appendix A.

The constant $A_1$ is given by the finite part of $\pi_1 + \pi_2$, which is found from (4.6) to be

$$A_1 = \frac{g^2 N}{(4\pi)^2} \left[ \frac{313}{36} - \frac{11 \sqrt{3} \pi}{8} - \frac{7}{12} \xi + \frac{(4\xi - 1)^{3/2}}{12} \arctan \left( \frac{1}{\sqrt{4\xi - 1}} \right) + \frac{6\xi - 1}{24} \ln \xi \right].$$

In the Landau gauge, we have $\xi \to 0$, and $A_1$ becomes

$$A_1 = \frac{g^2 N}{(4\pi)^2} \left[ \frac{313}{36} - \frac{11 \sqrt{3} \pi}{8} \right].$$

The renormalization functions are

$$\beta(g^2) = -\frac{11}{3} N \frac{g^4}{(4\pi)^2} + O(g^6), \quad \gamma_m = \frac{35 - 3\xi}{12} N \frac{g^2}{(4\pi)^2} + O(g^4),$$

and thus

$$b_0 = \frac{11}{6} \frac{N}{(4\pi)^2}, \quad \gamma_0 = \frac{35 - 3\xi}{12} \frac{N}{(4\pi)^2}.$$  \hspace{1cm} (4.11)

Using the above results, we apply the improved renormalization group method explained above to Yang-Mills theory and are able to calculate the approximated pole mass of the gluon. Note that at one-loop order, the result is independent of $N$. We plot $M_F^2/\Lambda_{MS}^2$ with respect to $x$, which corresponds to the provisional mass $m^2$, for various values of $a$ in Fig. 3. Figure 3 shows that a plateau emerges near $M_F^2/\Lambda_{MS}^2 \simeq 0.66$. 


Improved Renormalization Group Analysis for Yang-Mills Theory

Fig. 3. The plateau in the Landau gauge. The vertical axis represents $M_P^2/\Lambda_{\overline{MS}}^2$, evaluated using (2.22) with the data (4.9) and (4.11). The horizontal axis represents $x$ proportional to $m^2$. We have plotted the graphs for various values of $a$ from $a = 0.6565$ to $a = 0.6635$. The plateau appears near $M_P^2/\Lambda_{\overline{MS}}^2 \simeq 0.66$.

§5. Conclusion

In this paper, we have presented one method for evaluating the operator $\Delta = \frac{1}{2}(\text{volume})^{-1}(\min_U \int d^4x (A^U_\mu)^2)$. In the Landau gauge, the calculation boils down to evaluating $\langle A_\mu A^\mu \rangle$, which may be connected to the mass pole of the gluon propagator in this gauge. We have thus applied the improved renormalization group method to pure Yang-Mills theory at one-loop order and obtained the result that a nonperturbatively generated pole mass of the gluon, $M_P$, emerges as $M_P^2/\Lambda_{\overline{MS}}^2 \simeq 0.66$, where $\Lambda_{\overline{MS}}$ is the $\overline{MS}$ scale. Some work has been done to calculate the mass of the gluon, for example, in a lattice calculation.\textsuperscript{21} Our result is much smaller than those found in the previous work. However, we cannot compare the previous result with our result directly, because that result is not the pole mass itself but, rather, the vacuum expectation value of the $A_\mu A^\mu$ operator calculated through the operator product expansion. It would be interesting to perform further calculations, two loops or higher, and observe if higher-order calculations might give larger gluon masses. Calculations of the renormalization functions of the Curci-Ferrari model have been carried out to three-loop order.\textsuperscript{27} Those calculations enable us to improve the renormalization group analysis, but we think that perturbative corrections play an important role in producing a nonperturbative mass. Again, we would like to emphasize that though we have done the calculation in the Landau gauge, the result should have a physical meaning, because the original $\Delta$ is a gauge-invariant quantity.

Here, let us comment on a possible relationship between mass and confinement.
If the pole mass of the gluon were infinitely large, we could interpret it as a signal of confinement, since such a large mass would indicate that the gluon does not propagate as a physical mode. This is an interesting viewpoint,*) and we expected to find a large gluon mass. Although one-loop order we have not obtained such a large mass, the gluon mass might become larger and larger as the order of calculation increases.

Yet another perspective concerning confinement may exist. In the operator formalism of Yang-Mills theory, there is a well-known criterion for confinement, the so-called Kugo-Ojima condition,30) which is a sufficient condition for color confinement. The condition mentioned here concerns a BRST transformation property of anti-ghost fields, and the criterion asserts that when this condition is satisfied, one can prove that no color singlet state belongs to the BRST cohomology, and hence it cannot be observed as a physical state. This condition has been interpreted as involving the infrared properties of gluon and ghost propagators in the Landau gauge,31) and these properties have been tested with lattice simulations,32) analytically33) and by employing the Schwinger-Dyson equations.34) Because the infrared behavior of the propagator is involved with the existence of the nonperturbatively generated mass, it would be interesting to study it with our method.**) This viewpoint also provides an interesting interpretation of the remark above, that is, that the confinement via the Kugo-Ojima mechanism requires a finite mass for gluon propagators, and thus it may be consistent with our result.

Another issue concerns the ghost mass generation. In the literature, generation of the ghost mass has been studied intensively, and it might also be possible to evaluate it using the prescription presented in this paper. It would be interesting to calculate the pole mass of the ghost propagator.

We have also applied this method to the $O(N)$ non-linear sigma model in two dimensions. The exact mass gap of this model is known,29) and we can compare it with our result in order to check the effectiveness of this method. However, at one-loop order, the calculation is consistent with the exact result only in the large-$N$ limit, where we have $M_P = \Lambda_{\overline{MS}}$. A similar fact was observed in the Gross-Neveu model17) at one-loop order. In Ref. 17), a two-loop calculation is performed and a result in good agreement with the exact result, even for small values of $N$, is obtained. Considering this fact, it would be interesting to calculate the next order in the non-linear sigma model to make sure of the validity of this method.

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*) The authors owe discussion on this viewpoint to Prof. Kawai.
**) The authors thank an anonymous referee for bringing this interesting idea to their attention. For a review of the infrared properties of QCD propagators, see Ref. 35).
a calculation in a preliminary draft. T. M. is especially grateful to Y. Shibusa for discussions. The authors thank Prof. J. A. Gracey for his comments and for answering their questions. This work is supported by Special Postdoctoral Researchers Program at RIKEN.

**Appendix A**

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**Coefficient Functions**

Below we give explicit expressions for the coefficient functions \(N_{m0}, N_{A0}, N_{m1}\) and \(N_{A1}\):

\[
N_{m0} = \frac{3}{2} z(1-z)(16(1-\xi)x + 16(1-\xi)(1-z)y - 5 + 9\xi), \tag{A.1}
\]

\[
N_{A0} = z(1-z)(80z(1-z) - 3(1+\xi)), \tag{A.2}
\]

\[
N_{m1} = \left\{ -19(1-\xi)^3 z(1-z)((1-z)y + zx)^3 \\
\quad - \frac{1}{4}(63\xi - 181)(1-\xi)^2 z(1-z)((1-z)y + zx)^2 \\
\quad - \frac{3}{4}(1-\xi)(2\xi^2 - 35\xi + 45)z(1-z)((1-z)y + zx) \\
\quad - \frac{2}{3}(1-\xi)(2\xi - 5)z(1-z) \} (m^2)^2 \\
\quad + \{(1-\xi)^2(-46z(1-z) + 3)z(1-z)((1-z)y + zx)^2 \\
\quad + \frac{1}{8}(1-\xi) \times (-174\xi z(1-z) + 446z(1-z) - 3(1-z)y + zx) \\
\quad + \frac{1}{8}z(1-z)(-24\xi^2 z(1-z) - 76z(1-z) + 156\xi z(1-z) - 9\xi - 3) \} m^2 \xi z \]
\]

\[
N_{A1} = \left\{ \frac{1}{2}(-116z(1-z) + 7\xi + 10)(1-z)((1-z)y + zx)^2 \\
\quad + (1-\xi)(-22\xi z(1-z) + 122z(1-z) + \xi^2 - 7\xi - 11) \times z(1-z)((1-z)y + zx) \\
\quad + \frac{1}{4}z(1-z)(12\xi^2 z(1-z) - 258z(1-z) \\
\quad + 86\xi z(1-z) - 3\xi^2 + 15\xi + 24) \} (m^2)^2 \\
\quad - \frac{1}{8}(1-\xi)^2 z(1-z)(4z(1-z)(272z(1-z) - 33) \\
\quad - 28\xi z(1-z) - \xi^2 - 4\xi - 3)((1-z)y + zx) \right\} / \Delta^2, \tag{A.3}
\]
\[ + \frac{1}{8} z(1 - z)(4z(1 - z)(281z(1 - z) - 34) + 4\xi z(1 - z)(33z(1 - z) + 10) \\
+ 8\xi^2 z(1 - z) + 3\xi^2 + 6\xi + 3) \right\} m^2 q^2 \\
+ \frac{1}{8} z^2(1 - z)^2(8\xi z(1 - z) + \xi^2 + 4\xi \\
+ 3 - 8z(1 - z)(76z(1 - z) - 11))(q^2)^2] / \Delta^2. \] 

(A.4)

Appendix B

Renormalization Group Analysis for the Pole Mass

In this appendix, we explain the fact that a pole mass can be written in terms of a logarithm of \( m^2/\mu^2 \), following Ref. 28. We also comment on a leading-logarithm approximation.

Here we concentrate on a theory that has two dimensional parameters, \( m^2 \) and \( \mu \). The physical mass that appears at a pole of the propagator is

\[
M_P^2 = m^2 \sum_{n=0}^{\infty} f_n \left( \frac{m^2}{\mu^2} \right) g^{2n},
\]

where the quantities \( f_n \) are undetermined functions of \( m^2/\mu^2 \) and are calculated by perturbation theory, and \( g \) is the coupling constant of the theory. Note that here we have applied a mass-independent renormalization (MIR) scheme, like minimal subtraction \( MS \) or \( \overline{MS} \). Only when an MIR scheme is applied does the pole mass have a form like (B.1).

By definition, the pole mass \( M_P^2 \) is independent of \( \mu \); that is, it is renormalization group invariant. Then, \( M_P^2 \) satisfies the renormalization group equation (RGE)

\[
\left[ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g^2} - \gamma_{m^2}(g)m^2 \frac{\partial}{\partial m^2} \right] M_P^2 = 0,
\]

where

\[
\beta(g) = \mu \frac{\partial g^2}{\partial \mu}, \quad \gamma_{m^2} = -\mu \frac{\partial m^2}{\partial \mu}.
\]

(B.3)

Substituting (B.1) into this RGE, we obtain

\[
\frac{\partial}{\partial \ln \mu} f_n = \left[ \sum_{n' < n} \left( \gamma_{m^2} m^2 \frac{\partial}{\partial m^2} - \beta \frac{\partial}{\partial g^2} - \gamma_{m^2} \right) f_{n'} g^{2n'} \right]_{\text{coeff of } g^{2n}}.
\]

(B.4)

Integrating with respect to \( \ln \mu \), we obtain

\[
f_n = \left[ \sum_{n' < n} \left( \gamma_{m^2} m^2 \frac{\partial}{\partial m^2} - \beta \frac{\partial}{\partial g^2} - \gamma_{m^2} \right) \right] \int_0^{\ln \mu} d \ln \mu' f_{n'} \left( \frac{m^2}{\mu^2} \right) g^{2n'} \right]_{\text{coeff of } g^{2n}} + \text{const},
\]

(B.5)
where \( \text{const} \) is independent of \( \mu \) and, thus, \( m^2 \). We have the "boundary condition" \( f_0 = 1 \) of this integral equation, and hence

\[
 f_1 = \text{const} - \frac{1}{2} \gamma_0 \ln \left( \frac{m^2}{\mu^2} \right),
\]

where we have defined

\[
 \beta(g) = -2b_0g^4 - 2b_1g^6 + \cdots, \quad \gamma_{m^2} = \gamma_0g^2 + \gamma_1g^4 + \cdots. \tag{B.7}
\]

In this manner, we can determine all the functions \( f_n \) recursively by using of information provided by the RGE, up to constant terms. Thus we obtain

\[
 M_P^2 = m^2 \sum_{n=0}^{\infty} \sum_{L=0}^{n} A_{n,L} \left[ \ln \left( \frac{m^2}{\mu^2} \right) \right]^{n-L} g^{2n}, \tag{B.8}
\]

where the coefficients \( A_{n,L} \) with \( L < n \) are determined through the above procedure, while \( A_{n,n}(\equiv A_n) \) are unknown constants at this stage. These constants are to be calculated within the perturbation theory, and we call them "non-logarithmic corrections".

An interesting fact is that the \( L = 0 \) contribution in the series (B.8), called a leading logarithm approximation, satisfies the renormalization group equation at one-loop order, that is,

\[
 \left[ \mu \frac{\partial}{\partial \mu} - 2b_0g^2 \frac{\partial}{\partial g^2} + \gamma_0g^2 \frac{\partial}{\partial m^2} \right] (M_P^{LL})^2 = 0. \tag{B.9}
\]

Therefore we can solve it in a standard way, and we obtain

\[
 (M_P^{LL})^2 = m^2 \sum_{n=0}^{\infty} A_{n,0} \left[ \ln \left( \frac{m^2}{\mu^2} \right) \right]^n g^{2n}. \tag{B.10}
\]

This indicates that once we know \( b_0 \) and \( \gamma_0 \) at one-loop order, we can calculate the leading-logarithm contribution to the pole mass at all orders in \( g^2 \).

References

1) M. Gell-Mann and F. E. Low, Phys. Rev. 95 (1954), 1300.
4) This method is often called “the delta expansion”.
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