Improved gravimetric terrain corrections

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SUMMARY
The objective of this paper is to improve gravimetric terrain corrections by: (1) investigating the effect of different topographic representations that are suitable for efficient processing of high volumes of data (e.g. the mass prism and the mass line models) on terrain corrections and on geoid computations; (2) accelerating the convergence of fast Fourier transform (FFT)-based terrain-correction formulae; and (3) developing a set of new formulae corresponding to the mass prism topographic model, which can be evaluated efficiently with the 2-D FFT.

Terrain corrections were computed on a grid of 600 by 600 points with spacing 30" by 60" in the Rocky Mountains of British Columbia, Canada. The effect of using the mass line model instead of the mass prism model is 7.4 mGal (maximum) and 0.7 mGal (rms) on the terrain corrections, and 24 cm (maximum) and 16 cm (rms) on the geoid undulations. The optimizations made on the FFT-based terrain-correction formulae effectively speed up the convergence. The newly developed mass prism terrain-correction formula significantly reduced the required computer time and provided identical results with those from the rigorous numerical integration. On an IBM/RISC machine running AIX, the computation of the 15 convolutions (the matrix size was expanded to 1200 by 1200 after 100 per cent zero-padding) via the new formulae only took 15 min (user time), while the numerical summation method required 83.5 days.

Key words: digital topographic representation, geoid undulation, terrain correction.

1 INTRODUCTION
Extensive theoretical and numerical investigations indicate that in order to improve the accuracy of the predicted gravimetric geoid undulations in mountainous areas, more attention should be paid to the short-wavelength topographic effect, in which the terrain correction has a dominant contribution (Moritz 1968, 1983; Schwarz 1984; Li 1993; Sideris 1993).

The conventional approaches of computing the terrain corrections subdivide the area around the measurement point into zones and compute the terrain correction by adding the contributions of the zones (Nagy 1966; Ferland 1984). Because they are very time consuming, these approaches are not convenient for applications where a dense coverage in a large area is required, as for example, in geodetic boundary value problems. This problem has been successfully overcome since the development of the fast Fourier transform (FFT)-based techniques at the University of Calgary (Sideris 1984, 1985; Sideris & Li 1993). Because of the very high efficiency, nowadays, the FFT-based methods are taken as the standard ones in the computation of gravity field convolutions, such as geoid undulations, vertical deflections and terrain corrections (Sideris & Tziavos 1988; Harrison & Dickinson 1989; Schwarz et al. 1990; Sideris 1990). Consequently, it is now possible to compute by FFT grids of terrain corrections covering countries as large as Canada or whole continents on a personal computer in a single run.

The terrain-correction formulae currently used are in the form of a series, approximating the rigorous equation. The physical meaning of the linear approximation is that the topographic mass within each mass prism is concentrated along its vertical symmetric axis; in other words, the topography is approximately represented by the mass line topographic model. When the grid spacing is small enough, e.g. 100 m, the effect of this approximation on geoid prediction may be negligible. In practice, however, most available DTM are sampled with spacing of 1 km or even larger (such as in Canada). Therefore, it is necessary to investigate whether it is acceptable to use the mass line topographic model when the objective accuracy of the geoid prediction is 10 cm or better.

This paper refines the gravimetric terrain-correction techniques in terms of both the computational efficiency and accuracy. First, the convergence of the series will be improved by introducing an optimal parameter in the formulae. Secondly, a set of
new formulae that corresponds to the more rigorous mass prism topographic model will be given. Unified formulae are provided for the evaluation of either the conventional or the newly developed formulae by means of the fast Fourier transform. Numerical examples are given to show the effectiveness of these refinements.

2 THE RIGOROUS TERRAIN-CORRECTION FORMULAE

The terrain correction at a point \((x_i, y_j)\) is (Heiskanen & Moritz 1967)

\[
v(i, j) = -\frac{G}{E} \int_0^{h_{ij}} \int_0^{h_{ij}} \frac{\rho(x, y, z)(h_{ij} - z)}{r^3(x_i - x, y_j - y, h_{ij} - z)} \, dx \, dz,
\]

where \(G\) is Newton's gravitational constant, \(\rho(x, y, z)\) is the topographic density at the running point, \(h_{ij}\) is the topographic height at point \((i, j)\), \(E\) denotes the integration area, and \(r(x, y, z)\) is the distance kernel defined as

\[
r(x, y, z) = (x^2 + y^2 + z^2)^{1/2}.
\]

Using a gridded digital topographic model and taking the density as constant, eq. (1) can be written as

\[
c(i, j) = -\sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \int_{x_n - \Delta x/2}^{x_n + \Delta x/2} \int_{y_m - \Delta y/2}^{y_m + \Delta y/2} \frac{(h_{ij} - z)}{r^3(x_i - x, y_j - y, h_{ij} - z)} \, dx \, dz,
\]

or, equivalently,

\[
c(i, j) = \rho \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \int_{x_n - \Delta x/2}^{x_n + \Delta x/2} \int_{y_m - \Delta y/2}^{y_m + \Delta y/2} \left[ \frac{1}{r(x_i - x, y_j - y, 0)} - \frac{1}{r(x_i - x, y_j - y, h_{ij} - h_{nm})} \right] \, dx \, dz.
\]

With different topographic representations, \((i, j)\) can be expressed in different forms.

2.1 Two different topographic models

In practical applications, the topography is digitized on a regular grid. The height within each cell is represented by a prism with mean height and mean density of the topography as shown in Fig. 1(a), which is called the mass prism topographic model. If the mass of the prism is mathematically concentrated along its vertical symmetric axis, then the topography within the prism is represented by a line as shown in Fig. 1(b), which gives the mass line topographic model.

2.2 The terrain-correction formula with the mass prism topographic model

With the mass prism topographic model, assuming the mass within a prism is homogeneous and carrying out the double integration in eq. (4) (Haaz 1953), the expression for the terrain correction \(c(i, j)\) is obtained as

\[
c(i, j) = \rho \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \left\{ x \ln \left[ y + r(x, y, z) \right] + y \ln \left[ x + r(x, y, z) \right] - z \arctan \frac{xy}{zr(x, y, z)} \right\} \frac{1}{r(x_i - x_n, y_j - y_m, 0)} - \frac{1}{r(x_i - x_n, y_j - y_m, h_{ij} - h_{nm})}.
\]

2.3 The terrain-correction formula with the mass line topographic model

When the mass within a prism is concentrated along a line, instead of carrying out the double integration in eq. (4), the terrain correction is simply expressed as

\[
c(i, j) = \rho \Delta x \Delta y \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \left[ \frac{1}{r(x_i - x_n, y_j - y_m, 0)} - \frac{1}{r(x_i - x_n, y_j - y_m, h_{ij} - h_{nm})} \right].
\]

It is easy to understand that the mass line model is less realistic than the mass prism model from the physical point of view; therefore, it is worth investigating how big the effect on the terrain corrections will be when the mass line model is used instead of the mass prism model.

\[\text{Figure 1. Two different topographic representations.}\]
2.4 The mathematical relation between the two expressions

By expanding the integrand $1/r$ in eq. (4) into a Taylor series and completing the integration, the mathematical relation between the two terrain-correction expressions can be derived as

$$c_{MP}(i, j) = c_{ML}(i, j) + E_c(i, j),$$

where $c_{MP}(i, j)$ and $c_{ML}(i, j)$ denote terrain corrections corresponding to the mass prism and the mass line topographic model as expressed by eqs (5) and (6), respectively, and $E_c(i, j)$ is

$$E_c(i, j) = \frac{Gp \Delta x \Delta y}{24} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \left[ \frac{2 \Delta x_m^2 - \Delta y_m^2 - z^2}{r^5(\Delta x_m, \Delta y_m, \zeta)} \Delta x^2 + \frac{2 \Delta y_m^2 - \Delta x_m^2 - z^2}{r^5(\Delta x_m, \Delta y_m, \zeta)} \Delta y \right]_{z=h_0-h_{nm}},$$

where $\Delta x_m = x_i - x_n$ and $\Delta y_m = y_j - y_m$.

In fact, $E_c(i, j)$ can be taken as the error introduced by the use of the mass line topographic model instead of the mass prism topographic model. Eq. (8) indicates that the magnitude of the error is dependent on the grid size $\Delta x$ and $\Delta y$, the roughness of the topography, and the distance between the computation point and the running point. The bigger the $\Delta x$ and $\Delta y$ are and the rougher the topography is, the bigger the errors will be.

Because the evaluation of eqs (5) and (6) by numerical summation is very time consuming, an FFT-based technique should be used instead. Different equations can be formulated for different requirements for accuracy and computational efficiency.

3 COMPUTATION OF TERRAIN CORRECTIONS VIA 2-D FFT

3.1 Formulae with the mass line topographic model

With the mass line topographic model, the terms containing $1/r(x_i - x_n, y_j - y_m, 0)$ can be computed directly as will be seen later. The only thing we have to do is to express the terms containing $1/r(x_i - x_n, y_j - y_m, h_{ij} - h_{nm})$ as 2-D convolutions. Expanding $1/r(x_i - x_n, y_j - y_m, h_{ij} - h_{nm})$ in eq. (4) into a Taylor series (Sideris 1990; Li 1993), the $c(i, j)$ can be expressed as

$$c(i, j) = c_0(i, j) + c_1(i, j) + c_2(i, j) + c_3(i, j) + \cdots,$$

where

$$c_0(i, j) = \frac{Gp \Delta x \Delta y}{24} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \left[ \frac{1}{r(x_i - x_n, y_j - y_m, 0)} - \frac{1}{r(x_i - x_n, y_j - y_m, \alpha)} \right],$$

$$c_k(i, j) = (-1)^{k+1} Gp \Delta x \Delta y \frac{(2k - 1)!!}{(2k)!!} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \frac{[(h_{ij} - h_{nm})^2 - \alpha^2]^{k-1}}{r^{2k+1}(x_i - x_n, y_j - y_m, \alpha)},$$

$c(i, j)$ is the same expression as in Sideris (1984) for $\alpha = 0$. The objective of adding the parameter $\alpha$ is to speed up the convergence of the series in eq. (9). $\alpha$ was chosen as the average height in the computation area of the difference between the maximum and the minimum height (Dorman & Lewis 1974; Tziavos et al. 1988). From the mathematical point of view, these values do not provide the fastest convergence speed for the series, because they do not result in the smallest differences between $r(x_i - x_n, y_j - y_m, h_{ij} - h_{nm})$ and $r(x_i - x_n, y_j - y_m, \alpha)$. The optimal value for $\alpha$ can be determined by minimizing the variation function

$$J = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} [(h_{ij} - h_{nm})^2 - \alpha^2]^2.$$

Using the average height $\bar{h}$ instead of $h_{ij}$, we get

$$\alpha = \left[ \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} (\bar{h} - h_{nm})^2 \right]^{1/2} = \sigma_h.$$

Therefore, the optimal value for $\alpha$ is the standard deviation of the heights.
Expanding the numerator of eq. (11) into a series, \( c_\ell(i, j) \) can be equivalently expressed as a set of 2-D convolutions to which the fast Fourier transform can be applied. The final expressions are

\[
c_\ell(i, j) = Gp F^{-1} \{ H_0 R_0 \},
\]

\[
c_\ell(i, j) = \frac{Gp}{2} \left[ (h_{ij}^2 - \alpha^2) F^{-1} \{ H_0 R_0 \} - 2h_{ij} F^{-1} \{ H_1 R_1 \} + F^{-1} \{ H_2 R_2 \} \right],
\]

\[
c_\ell(i, j) = -\frac{3Gp}{8} \left[ (h_{ij}^2 - \alpha^2)^2 F^{-1} \{ H_0 R_0 \} - 4h_{ij} (h_{ij}^2 - \alpha^2) F^{-1} \{ H_1 R_1 \} + (6h_{ij}^2 - 2\alpha^2) F^{-1} \{ H_2 R_2 \} - 4h_{ij} F^{-1} \{ H_3 R_3 \} + F^{-1} \{ H_4 R_4 \} \right],
\]

\[
c_\ell(i, j) = \frac{15Gp}{48} \left[ (h_{ij}^2 - \alpha^2)^3 F^{-1} \{ H_0 R_0 \} - 6h_{ij} (h_{ij}^2 - \alpha^2)^2 F^{-1} \{ H_1 R_1 \} + 3(h_{ij}^2 - \alpha^2)(5h_{ij}^2 - \alpha^2) F^{-1} \{ H_2 R_2 \} - (20h_{ij}^2 - 12\alpha^2) h_{ij} F^{-1} \{ H_3 R_3 \} + (15h_{ij}^2 - 3\alpha^2) F^{-1} \{ H_4 R_4 \} - 6h_{ij} F^{-1} \{ H_5 R_5 \} + F^{-1} \{ H_6 R_6 \} \right],
\]

where

\[
H_k = F[h^k], \quad k = 0, 1, 2, 3, 4, 5, 6,
\]

\[
R_k = F \left\{ \Delta x \Delta y \frac{x^2 + y^2 + \alpha^2}{(x^2 + y^2 + \alpha^2)^{k+1}} \right\}, \quad k = 1, 2, 3,
\]

\[
\alpha = \sigma_h.
\]

Considering the fact that eq. (10) represents the vertical attraction of a mass layer with thickness \( \alpha \), \( c_\ell(i, j) \) can be identically expressed as (Li 1993; Li & Sideris 1993)

\[
c_\ell(i, j) = Gp \left[ x \ln(r(x, y, z)) + y \ln(r(x, y, z)) \right]_{x=(N-1)/2}^{x=(N-1)} + \alpha \arctan\frac{xy}{\sigma r(x, y, \alpha)} \right]_{x=(N-1)/2}^{x=(N-1)} + \alpha \arctan\frac{xy}{\sigma r(x, y, \alpha)}
\]

For non-edge points, \( c_\ell(i, j) \) can be approximated as the attraction of a mass cylinder with height \( \alpha \). When the radius of the cylinder tends to infinite, \( c_\ell(i, j) \) can be simply evaluated by (Heiskanen & Moritz 1967)

\[
c_\ell(i, j) = 2\pi Gp \alpha.
\]

The conventional method to derive the 2-D convolutions is, first, to expand \( 1/r \) in eq. (1) into series with respect to \( z \), then to carry out the integration as done in Tziavos et al. (1992). This procedure is equivalent to expanding both \( 1/r(x_i-x_m, y_i-y_m, 0) \) and \( 1/r(x_i-x_m, y_i-y_m, h_{nm}) \) into series. Consequently, the terrain correction \( c(i, j) \) is

\[
c(i, j) = c_1(i, j) + c_2(i, j) + c_3(i, j) + \cdots,
\]

with

\[
c_1(i, j) = Gp \left[ h_{ij}^2 F^{-1} \{ H_0 R_0 \} - 2h_{ij} F^{-1} \{ H_1 R_1 \} + F^{-1} \{ H_2 R_2 \} \right]
\]

\[
c_2(i, j) = -\frac{3Gp}{8} \left[ (h_{ij}^2 - \alpha^2)^2 F^{-1} \{ H_0 R_0 \} - 4h_{ij} (h_{ij}^2 - \alpha^2) F^{-1} \{ H_1 R_1 \} + (6h_{ij}^2 - 2\alpha^2) F^{-1} \{ H_2 R_2 \} - 4h_{ij} F^{-1} \{ H_3 R_3 \} + F^{-1} \{ H_4 R_4 \} \right]
\]

\[
c_3(i, j) = \frac{15Gp}{48} \left[ (h_{ij}^2 - \alpha^2)^3 F^{-1} \{ H_0 R_0 \} - 6h_{ij} (h_{ij}^2 - \alpha^2)^2 F^{-1} \{ H_1 R_1 \} + 3(h_{ij}^2 - \alpha^2)(5h_{ij}^2 - \alpha^2) F^{-1} \{ H_2 R_2 \} - (20h_{ij}^2 - 12\alpha^2) h_{ij} F^{-1} \{ H_3 R_3 \} + (15h_{ij}^2 - 3\alpha^2) F^{-1} \{ H_4 R_4 \} - 6h_{ij} F^{-1} \{ H_5 R_5 \} + F^{-1} \{ H_6 R_6 \} \right]
\]

where \( H_k \) and \( R_k \) are the same as in eqs (18) and (19). The optimal value for the parameter \( \alpha \) in this case, however, should provide the smallest differences between \( r(x_i-x_m, y_i-y_m, h_{ij}-h_{nm}) \) and \( r(x_i-x_m, y_i-y_m, \alpha) \) as well as between \( r(x_i-x_m, y_i-y_m, 0) \) and \( r(x_i-x_m, y_i-y_m, \alpha) \), which can be determined by minimizing the following variation function instead of eq. (12):

\[
J = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \left[ (h_{ij} - h_{nm})^2 - \alpha^2 \right]^2 + (0 - \alpha^2)^2.
\]

Correspondingly, the optimal value for \( \alpha^2 \) is one-half of the standard deviation of the heights, i.e.

\[
\alpha = \sigma_h / \sqrt{2}.
\]
3.2 Formulae with the mass prism topographic model

In eq. (5), keeping the terms containing \( z = 0 \) unchanged and expanding the terms containing \( z = h_i - h_m \) into a series, the terrain-correction formulae with the mass prism topographic model can be expressed as

\[
c(i, j) = c_0(i, j) + c_1(i, j) + c_2(i, j) + c_3(i, j) + \cdots ,
\]

where \( c_0(i, j) \) can be evaluated directly according to eqs (21) or (22). \( c_1(i, j) \), \( c_2(i, j) \) and \( c_3(i, j) \) can be efficiently evaluated by means of the fast Fourier transform as

\[
c_1(i, j) = \frac{G_p}{2} \left[ (h_i^2 - \alpha^2)F^{-1}(H_0F_0) - 2h_iF^{-1}(H_1F_0) + F^{-1}(H_2F_0) \right],
\]

\[
c_2(i, j) = -\frac{G_p}{8} \left[ ((h_i^2 - \alpha^2)^2F^{-1}(H_0F_0) - 4h_i(h_i^2 - \alpha^2)F^{-1}(H_1F_0) + (6h_i^2 - 2\alpha^2)F^{-1}(H_2F_0) - 4h_iF^{-1}(H_3F_0) + F^{-1}(H_4F_0) \right],
\]

\[
c_3(i, j) = \frac{G_p}{48} \left[ ((h_i^2 - \alpha^2)^3F^{-1}(H_0F_0) - 6h_i(h_i^2 - \alpha^2)^2F^{-1}(H_1F_0) + 3(h_i^2 - \alpha^2)(5h_i^2 - 7\alpha^2)F^{-1}(H_2F_0) - (20h_i^2 - 12\alpha^2)h_iF^{-1}(H_3F_0) 
+ (15h_i^2 - 3\alpha^2)F^{-1}(H_4F_0) - 6h_iF^{-1}(H_5F_0) + F^{-1}(H_6F_0) \right],
\]

where \( H_k \) is defined by eq. (14) and

\[
F_1 = F_1(x, y, \alpha) + F_1(y, x, \alpha) - F_2(x, y, \alpha),
\]

\[
F_2 = F_2(x, y, \alpha) + F_2(y, x, \alpha) - F_3(x, y, \alpha),
\]

\[
F_3 = F_3(x, y, \alpha) + F_3(y, x, \alpha) - F_4(x, y, \alpha),
\]

\[
\alpha = \sigma, \quad \sigma = \frac{x}{r}.
\]

The expressions of \( c_1, c_2 \) and \( c_3 \) in the spatial domain are given in Appendix A.

Similarly, if the first term of \( c_0(i, j) \) in eq. (29) is also expanded into series, the following formulae can be derived:

\[
c(i, j) = c_0(i, j) + c_1(i, j) + c_2(i, j) + \cdots ,
\]

\[
c_1(i, j) = \frac{G_p}{2} \left[ h_i^2F^{-1}(H_0F_0) - 2h_iF^{-1}(H_1F_0) + F^{-1}(H_2F_0) \right],
\]

\[
c_2(i, j) = -\frac{G_p}{8} \left[ (h_i^2 - \alpha^2)^2F^{-1}(H_0F_0) - 4h_i(h_i^2 - \alpha^2)F^{-1}(H_1F_0) + (6h_i^2 - 2\alpha^2)F^{-1}(H_2F_0) - 4h_iF^{-1}(H_3F_0) + F^{-1}(H_4F_0) \right],
\]

\[
c_3(i, j) = \frac{G_p}{48} \left[ (h_i^2 - \alpha^2)^3F^{-1}(H_0F_0) - 6h_i(h_i^2 - \alpha^2)^2F^{-1}(H_1F_0) + 3(h_i^2 - \alpha^2)(5h_i^2 - 7\alpha^2)F^{-1}(H_2F_0) - (20h_i^2 - 12\alpha^2)h_iF^{-1}(H_3F_0) 
+ (15h_i^2 - 3\alpha^2)F^{-1}(H_4F_0) - 6h_iF^{-1}(H_5F_0) + F^{-1}(H_6F_0) \right].
\]

3.3 The unified terrain-correction formulae via 2-D FFT

The four sets of terrain-correction formulae, namely, eq. (9) with eqs (14)–(17), eqs (23)–(26), eq. (29) with eqs (21), (30)–(32) and (43)–(46), can be uniformly expressed as

\[
c(i, j) = \beta c_0(i, j) + c_1(i, j) + c_2(i, j) + c_3(i, j) + \cdots ,
\]
with \( c_0(i, j) \) is expressed as in eq. (24),
\[
c(i, j) = \frac{G_p}{2} \left[ (h_{ij}^2 - \beta \alpha^2) \mathbf{F}^{-1}(H_0K_1) - 2h_{ij}(H_1K_1) + \mathbf{F}^{-1}(H_2K_1) \right],
\]
(48)
\[
c_2(i, j) = -\frac{G_p}{8} \left[ (h_{ij}^2 - \alpha^2)^2 - (1 - \beta)\alpha^2 \right] \mathbf{F}^{-1}(H_0K_2) - 4h_{ij}(h_{ij}^2 - \alpha^2) \mathbf{F}^{-1}(H_1K_2)
+ (6h_{ij}^2 - 2\alpha^2) \mathbf{F}^{-1}(H_2K_2) - 4h_{ij} \mathbf{F}^{-1}(H_3K_2) + \mathbf{F}^{-1}(H_4K_2),
\]
(49)
\[
c_3(i, j) = \frac{G_p}{48} \left[ (h_{ij}^2 - \alpha^2)^3 - (1 - \beta)\alpha^2 \right] \mathbf{F}^{-1}(H_0K_3) - 6h_{ij}(h_{ij}^2 - \alpha^2)^2 \mathbf{F}^{-1}(H_1K_3) - 3(h_{ij}^2 - \alpha^2)(5h_{ij}^2 - \alpha^2) \mathbf{F}^{-1}(H_2K_3)
- (20h_{ij}^2 - 12\alpha^2)h_{ij} \mathbf{F}^{-1}(H_3K_3) + (15h_{ij}^2 - 3\alpha^2) \mathbf{F}^{-1}(H_4K_3) - 6h_{ij} \mathbf{F}^{-1}(H_5K_3) + \mathbf{F}^{-1}(H_6K_3),
\]
(50)
where the parameter \( \beta \) is
\[
\beta = \begin{cases} 1, & \text{if } c_0(i, j) \text{ is computed directly}, \\ 0, & \text{otherwise}, \end{cases}
\]
(51)
the optimal value for \( \alpha \) is
\[
\alpha = \begin{cases} \sigma_h, & \text{when } \beta = 1 \\ \sigma_h/\sqrt{2}, & \text{when } \beta = 0, \end{cases}
\]
(52)
\( K_i \) \((i = 1, 2, 3)\) is the Fourier transform of the kernel function defined by
\[
K_i = \begin{cases} R_i, \text{ (eq. 19) for the mass line topographic model}, \\ F_i, \text{ (eqs 33–35) for the mass prism topographic model}, \end{cases}
\]
(53)
\( H_i \) \((i = 0, 1, 2, 3, 4, 5, 6)\) is the Fourier transform of the heights with power \( i \), as expressed in eq. (18).

It is worth pointing out that all the above formulae are based on a flat-earth assumption. Our experience has shown that this approximation does not introduce significant errors. This is due to the fact that, since the kernel function of the terrain-correction formula drops with the cube of the distance, only the topography close to the computation point contributes significantly to the terrain correction. Nevertheless, if one wants to work with the spherical formulae, the application of FFT techniques is still possible as proposed by Haagmans et al. (1993).

### 4 NUMERICAL EXAMPLES

Terrain corrections were computed on a 600 by 600 height grid in a mountainous area in British Columbia, bounded by latitude 49°N to 54°N and longitude 236°E to 246°E. The grid spacing is 30 arcsec (0.93 km) in a N–S direction and 60 arcsec (1.15 km) in a W–E direction. Fig. 2 shows the topography and the statistical information of the heights.

![Figure 2](https://academic.oup.com/gji/article-abstract/119/3/740/626726)

**Figure 2.** Topography in the Rocky Mountains of British Columbia. (Maximum height: 3573 m. Minimum height: 0 m. Mean height: 1358 m.)
4.1 The effect of the different topographic models

To investigate the effect of the different representations of the topographic models, terrain corrections were computed with eqs (5) and (6) by the numerical summation method. Eq. (5) represents the mass prism topographic model, while eq. (6) corresponds to the mass line model. The computations were done on an IBM RISC6000 computer with the AIX operating system. Based on the test computations, Table 1 gives the required computer time with different numbers of computation points and different integration cap sizes.

**Table 1.** Computer time required for the numerical summation method.

<table>
<thead>
<tr>
<th>topographic model</th>
<th>number of c</th>
<th>cap size</th>
<th>CPU time</th>
<th>real time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass line model</td>
<td>10 x 10</td>
<td>400 x 400</td>
<td>1m 30s</td>
<td>5m 00s</td>
</tr>
<tr>
<td></td>
<td>10 x 10</td>
<td>600 x 600</td>
<td>3m 22s</td>
<td>11m 00s</td>
</tr>
<tr>
<td></td>
<td>600 x 600</td>
<td>600 x 600</td>
<td>8d 10h</td>
<td>28d 03h</td>
</tr>
<tr>
<td>Mass prism model</td>
<td>10 x 10</td>
<td>400 x 400</td>
<td>14m 46s</td>
<td>2h 03m</td>
</tr>
<tr>
<td></td>
<td>10 x 10</td>
<td>600 x 600</td>
<td>33m 13s</td>
<td>4h 36m</td>
</tr>
<tr>
<td></td>
<td>600 x 600</td>
<td>600 x 600</td>
<td>83d 10h</td>
<td>69d 21h</td>
</tr>
</tbody>
</table>

Table 1 indicates that the mass prism model requires much more computer time than the mass line model does. If the terrain corrections need to be computed in a large area (for example, with the early template method, the cap size is usually about 800 km by 800 km), however, neither of the models can be practically used with the numerical summation method.

To show the effect of the different topographic representations, terrain corrections were computed at all the 600 by 600 points with a limited cap size of 100 km by 100 km. The differences between the terrain corrections computed with the two different topographic models basically represent the errors due to the use of the mass line topographic model. Table 2 summarizes the statistics of both the terrain corrections and their contribution to the geoid undulation. Fig. 3 shows the terrain-correction differences and Fig. 4 shows the effect of these differences on the geoid undulation.

**Table 2.** Effects of different models on terrain corrections and on geoid undulations.

<table>
<thead>
<tr>
<th>model</th>
<th>terrain correction (mGal)</th>
<th>effect on geoid prediction (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>max</td>
<td>min</td>
</tr>
<tr>
<td>MP</td>
<td>85.832</td>
<td>0.001</td>
</tr>
<tr>
<td>ML</td>
<td>81.351</td>
<td>0.001</td>
</tr>
<tr>
<td>MP-ML</td>
<td>4.481</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Figure 3. Terrain correction differences (mGal). (Mass prism topographic model versus mass line topographic model.)
Table 2 indicates that the rms terrain-correction error introduced by the mass line topographic model is 0.7 mGal and the maximum value is 4.5 mGal. Comparing Fig. 3 with Fig. 2, it is obvious that the differences are correlated with the topography. Thus, the conclusion is the same as that from eq. (8): the rougher the topography is, the bigger the differences will be.

The effect of the terrain-correction differences on geoid prediction, as indicated by Table 2, is characterized by a bias of 16 cm and a maximum value of 24 cm. The maximum effect on the geoid, as shown in Fig. 4, is at the centre of the mountains. It can then be concluded that the terrain-correction difference between the use of the two topographic models can be significant; also, its effect on geoid undulations is not negligible, especially when the objective accuracy of geoid prediction is around 10 cm or better and the topography is very rough. Therefore, the newly developed terrain-correction formulae should be used instead of the conventionally used ones.

### 4.2 The convergence of the series with the mass line model

In Section 3, the terrain-correction formula was linearized with different methods and different parameter $\alpha$. To show their effect on the convergence of the series, Table 3 gives the differences between the terrain corrections computed by the numerical summation method (eq. 6), and by the first term of the linearized formulae (eq. 48) with the mass line topographic model. The first column shows the values used for the parameters in eq. (48). The computation is done in the whole area, with an integration cap size of 100 km by 100 km. Taking the values computed by the summation method as the standard, the differences represent the errors introduced by the linearized formula due to the truncation of the series. Therefore, the smaller the differences are, the faster the convergence of the series is.

**Table 3.** Effects of the linearization method on terrain correction and geoid undulation (with the mass line topographic model).

<table>
<thead>
<tr>
<th>parameter</th>
<th>terrain correction diff. (mGal)</th>
<th>effect on geoid prediction (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>max</td>
<td>min</td>
</tr>
<tr>
<td>$\beta = 0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\frac{\sigma}{\sqrt{2}}$</td>
<td>$0.518$</td>
<td>$-1.483$</td>
</tr>
<tr>
<td>$1$</td>
<td>$1.696$</td>
<td>$-7.435$</td>
</tr>
<tr>
<td>$\bar{h}$</td>
<td>$13.966$</td>
<td>$-0.001$</td>
</tr>
</tbody>
</table>
Table 4. Effects of series terms on terrain corrections and geoid undulations (with the mass line topographic model).

<table>
<thead>
<tr>
<th>terms used</th>
<th>terrain correction diff. (mGal)</th>
<th>effect on geoid prediction (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>max</td>
<td>min</td>
</tr>
<tr>
<td>$c_1$</td>
<td>0.518</td>
<td>-11.483</td>
</tr>
<tr>
<td>$c_1+c_2$</td>
<td>3.924</td>
<td>-0.002</td>
</tr>
<tr>
<td>$c_1+c_2+c_3$</td>
<td>0.052</td>
<td>-1.746</td>
</tr>
</tbody>
</table>

Table 3 indicates that the parameters $\beta = 0$ and $\alpha = \sigma_h / \sqrt{2}$ give the best results, as compared with the numerical summation method. The use of these values for parameters $\alpha$ and $\beta$ means that both terms in the right-hand side of eq. (4) are expanded into series, with parameter $\alpha$ as $1/\sqrt{2}$ times the standard deviation of the heights. One the other hand, the method of only expanding the second term in the right-hand side of eq. (4) while computing the first term rigorously, i.e. with $\beta = 1$ and $\alpha = \sigma_h$, does not provide the best results. This is due to the fact that by expanding both terms into series and then truncating them, the effects of the truncation are cancelled out due to the subtraction of the two terms. Table 3 also shows that $h$ should not be used to improve the convergence of the series. Similarly, $h_{\max} - h_{\min}$ should not be used for this purpose either.

Table 4 lists the statistical values of the differences between the terrain corrections computed by the numerical summation method and those by the linearized formula with different terms. The parameters used are $\beta = 0$ and $\alpha = \sigma_h / \sqrt{2}$.

It can be seen from Table 4 that the convergence of the series is much faster when the optimal parameters are used. If the computation is done including the third-order term, the linearized formulae provide identical results to those of the rigorous summation method. Fig. 5 shows the terrain corrections from the third term, which, as compared with Fig. 2, shows very strong correlation with the topography.

It must be mentioned that the effects on the geoid undulations listed in Tables 3 and 4 are only due to the linearization of the formula and the truncation of the series. In other words, these values do not include the errors due to the use of the mass line topographic model. The total geoid errors due to both the linearization and the mass line model used are much bigger than those given in the above two stables.

4.3 The convergence of the series with the mass prism model

Table 5 gives the statistical information related to the use of different terms in the linearized formulae with the mass prism topographic model. The cap size used in the computations is also 100 km by 100 km.

Comparing Table 5 with Table 4, it can be seen that the convergence of the series with the mass prism topographic model is slower than with the mass line model. However, due to the fact that the mass prism model represents the topography much more realistically than the mass line model does, the geoid error introduced by truncation of the series is only 2 cm if the

Figure 5. The third term of the terrain correction formulae with the mass line topographic model (mGal).
Table 5. Effects of different terms used on terrain corrections and geoid undulations (with mass prism model).

<table>
<thead>
<tr>
<th>terms used</th>
<th>terrain correction diff. (mGal)</th>
<th>effect on geoid prediction (m)</th>
<th>prediction (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>max</td>
<td>min</td>
<td>mean</td>
</tr>
<tr>
<td>C1</td>
<td>1.102</td>
<td>-13.614</td>
<td>0.166</td>
</tr>
<tr>
<td>C1+C2</td>
<td>6.754</td>
<td>-0.002</td>
<td>0.054</td>
</tr>
<tr>
<td>C1+C2+C3</td>
<td>0.229</td>
<td>-4.580</td>
<td>0.008</td>
</tr>
</tbody>
</table>

4.4 The effect of the integration cap size

In the preceding three subsections, the integration, or summation cap size, was limited to 100 km by 100 km. To show the effect of the integration cap size, terrain corrections were computed by the linearized formulae by means of the fast Fourier transform without limitation on the cap size, i.e. for each computation point, the contributions of all the 600 by 600 heights are included. The computer time (user time) required was 15 min for the computation of all three terms (15 convolutions each with size 1200 by 1200) of the series with either the mass prism model or the mass line model. This, as compared with Table 1, is about 800 and 8000 times less than the computer time needed by the summation method with the mass line model and the mass prism model, respectively.

Table 6 summarizes the terrain-correction differences due to the use of different topographic models and different integration cap sizes; (w) means that the summation was done over the whole computation area (600 km by 600 km), while (s) means that a smaller cap size (100 km by 100 km) was used. Fig. 6 shows the differences between terrain corrections when the integration

Figure 6. The effect of integration cap size on terrain corrections (mGal). (Integration cap size: 600 km x 600 km versus 100 km x 100 km.)
Table 7. Statistics of the digitized topographic heights in the same area with different grid spacing (in metres).

<table>
<thead>
<tr>
<th>grid spacing</th>
<th>max</th>
<th>min</th>
<th>mean</th>
<th>RMS</th>
<th>σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1 km x 0.1 km</td>
<td>3136</td>
<td>1164</td>
<td>1721</td>
<td>1755</td>
<td>344</td>
</tr>
<tr>
<td>1.1 km x 1.1 km</td>
<td>2740</td>
<td>1191</td>
<td>1721</td>
<td>1753</td>
<td>332</td>
</tr>
</tbody>
</table>

cap size is 600 km by 600 km and 100 km by 100 km. These differences are almost the same with the two different topographic models.

Table 6 shows that the limitation of the integration cap size to 100 km by 100 km results in a geoid undulation bias with an rms of 10 cm for either the mass prism or the mass line topographic model. Comparing Fig. 6 with Fig. 2, it can be seen that the distribution of the terrain-correction differences is correlated with the topography. Therefore, it is expected that these differences will be smaller in flatter areas.

Table 6 also indicates that the effect on geoid undulation is 24 cm (maximum) and 16 cm (rms) when using the mass line topographic model instead of the mass prism model. These values are of the same order of magnitude as those for the limited integration cap size of 100 km by 100 km. This is expected because this kind of effect is mainly due to the contribution of the topographic heights near the computation point.

### 4.5 The effect of grid spacing

It is clear that very dense topographic heights are needed to compute the terrain correction for accurate geoid prediction. In practice, however, most available DTMs, especially in large areas, are sampled with a grid spacing of 1 km. In order to investigate the effect of the topographic sampling densities, a 0.1 km by 0.1 km grid of heights in the Kananaskis Valley in the Rocky Mountains was used. The total extension of the grid is 44 km by 33 km. A set of 40 by 30 gridded heights with grid spacing 1.1 km by 1.1 km was formed from the original data set. Table 7 gives the statistics of the digitized topographic heights with different grid spacings. Table 8 shows the terrain corrections computed from the two sets of heights, and the terrain-correction differences as well as the effect on geoid undulations. The comparisons are made on the points of the 1.1 km by 1.1 km grids.

Table 8 indicates that the effect of grid spacing on the terrain corrections is considerable. When the grid spacing was 1.1 km instead of 0.1 km, the magnitude of the terrain corrections decreased by about 50 per cent; for example, the rms value decreased from 6.66 mGal to 3.48 mGal. Owing to the limited computational area size, the effects of the two types of terrain correction on geoid undulations are not significantly different in an absolute sense. However, in terms of either the rms value or the maximum (minimum) value, the terrain effect on the geoid decreased by about 50 per cent when the heights of the 1.1 km grid were used.

### 5 CONCLUSIONS AND RECOMMENDATIONS

Summarizing the analysis in this paper, we can conclude that to obtain a gravimetric geoid with an accuracy of 10 cm or better, the mass prism topographic model has to be used instead of the conventionally used mass line topographic model for the computation of terrain corrections. The terrain-correction errors due to the use of the mass line topographic model are proportional to the grid size and the topographic variations. In the Rocky Mountains, the use of the mass line topographic model with a grid spacing of 1 km resulted in additional geoid errors, with rms value of 16 cm and a maximum value of 24 cm.

The optimal parameter introduced in the formulae effectively accelerated the convergence of the terrain-correction series. When only the first term of the series is used with the optimal parameter, the terrain-correction errors and their effect on the geoid undulations were reduced by about 70 per cent. The rms geoid undulation errors introduced by the terrain-correction errors decreased from 6.3 to 2.2 cm when the optimal parameter was used.

The computation of the terrain corrections should be done up to the second term of the terrain correction series for either the mass line or the mass prisms topographic model. The contribution of the second term to the terrain correction is about

Table 8. Effects of height grid spacing on terrain corrections and on geoid undulations.

<table>
<thead>
<tr>
<th>grid spacing (km)</th>
<th>terrain correction (mGal)</th>
<th>effect on geoid (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>max</td>
<td>min</td>
</tr>
<tr>
<td>A: 0.1 x 0.1</td>
<td>42.64</td>
<td>0.21</td>
</tr>
<tr>
<td>B: 1.1 x 1.1</td>
<td>17.36</td>
<td>0.15</td>
</tr>
<tr>
<td>A - B</td>
<td>25.28</td>
<td>0.00</td>
</tr>
</tbody>
</table>
Gravimetric terrain corrections

5 mGal and on the geoid undulation is about 2 cm according to the computations done in the Rocky Mountains of British Columbia. The effect of the third term is negligible.

The integration cap size should be large enough for the computation of terrain corrections. In the same computation area as above, there is a 10 cm (rms) geoid undulation error when the integration cap size is limited to 100 km by 100 km instead of 600 km by 600 km.

In rough mountainous areas, the grid spacing of the digitized topographic model has a critical effect on the terrain correction. In the Rocky Mountains of British Columbia, the use of topographic heights with a grid spacing of 1.1 km results in 50 per cent errors in the terrain correction as compared with that with grid spacing 0.1 km. Therefore, the grid spacing should be small enough, especially in the inner zone around the computation point in rough mountainous areas.

When the terrain corrections are computed on gridded points by spectral techniques, the interpolation of randomly distributed gravity points from the gridded points has to be investigated. This is an important problem to be solved in mountainous areas, where very large differences exist between the point and the mean topographic heights. In addition, when the computation area is very large, the effect of a flat-earth approximation should also be investigated.

When terrain corrections of the highest precision are required for some geophysical purposes, further investigations on the use of alternative topographic representations are also necessary, such as the use of a rectangular parallelepiped or vertical prism with inclined top or interpolated surfaces from the sampled heights. The main obstacle is how to reduce the huge computer time required by these models. This is even more critical when the terrain corrections are to be computed in a very large area, such as the whole of Canada.

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REFERENCES


Moritz, H., 1968. On the Use of the Terrain Correction in Solving Molodensky’s Problem, OSU Report 108, Department of Geodetic Science and Surveying, The Ohio State University, OH.

Moritz, H., 1983. Local Geoid Determination in Mountainous Areas OSU Report 353, Department of Geodetic Science and Surveying, The Ohio State University, OH.


APPENDIX: TERRAIN FORMULAE WITH THE MASS PRISM TOPOGRAPHIC MODEL IN THE SPATIAL DOMAIN

The terrain-correction formulae corresponding to eqs (30)–(32) in the spatial domain are

\[ c_1(i, j) = \frac{Gp^{N+1}}{2} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} [(h_{ij} - h_{nm})^2 - \alpha^2][f_{11}(x_i - x, y_j - y, \alpha) + f_{11}(y_j - y, x_i - x, \alpha) - f_{12}(x_i - x, y_j - y, \alpha)], \]  \hspace{1cm} (A1)

\[ c_2(i, j) = -\frac{3Gp^{N+1}}{8} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} [(h_{ij} - h_{nm})^2 - \alpha^2][f_{21}(x_i - x, y_j - y, \alpha) + f_{21}(y_j - y, x_i - x, \alpha) - f_{22}(x_i - x, y_j - y, \alpha)], \]  \hspace{1cm} (A2)

\[ c_3(i, j) = \frac{15Gp^{N+1}}{48} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} [(h_{ij} - h_{nm})^2 - \alpha^2][f_{31}(x_i - x, y_j - y, \alpha) + f_{31}(y_j - y, x_i - x, \alpha) - f_{32}(x_i - x, y_j - y, \alpha)]. \]  \hspace{1cm} (A3)

and

\[ f_{11}(x_i - x, y_j - y, \alpha) = \frac{-x}{[y + r(x, y, \alpha)][r(x, y, \alpha)]} \left[ \frac{x_i - (x_0 + \Delta x/2)}{y_i - (y_0 + \Delta y/2)} \right], \]  \hspace{1cm} (A4)

\[ f_{12}(x_i - x, y_j - y, \alpha) = \frac{xy(r^2 + \alpha^2)}{(x_i^2 + \alpha^2 - r^2)\alpha} - \frac{1}{\alpha} \arctan \left[ \frac{xy[x_i - (x_0 + \Delta x/2)]}{\alpha [y_i - (y_0 + \Delta y/2)]} \right], \]  \hspace{1cm} (A5)

\[ f_{21}(x_i - x, y_j - y, \alpha) = \frac{-x(y + 2r)}{3(y + r)^3} \left[ \frac{x_i - (x_0 + \Delta x/2)}{y_i - (y_0 + \Delta y/2)} \right], \]  \hspace{1cm} (A6)

\[ f_{22}(x_i - x, y_j - y, \alpha) = \frac{xy}{3(x_i^2 + \alpha^2 - r^2)r} \left[ \frac{2(r^2 + \alpha^2)}{(x_i^2 + \alpha^2 - r^2)^2} \frac{r^2 + \alpha^2}{r^2} - 4 \right] - \frac{1}{3\alpha^2} \arctan \frac{xy[x_i - (x_0 + \Delta x/2)]}{\alpha [y_i - (y_0 + \Delta y/2)]}, \]  \hspace{1cm} (A7)

\[ f_{31}(x_i - x, y_j - y, \alpha) = \frac{-x}{15(y + r)^3} \left[ 1 + \frac{y}{y + r} \left( \frac{4}{r} + \frac{2}{y + r} \right) \right] \left[ \frac{x_i - (x_0 + \Delta x/2)}{y_i - (y_0 + \Delta y/2)} \right], \]  \hspace{1cm} (A8)

\[ f_{32}(x_i - x, y_j - y, \alpha) = \frac{xy}{15(x_i^2 + \alpha^2 - r^2)r} \left[ \frac{3r^2 + 6\alpha^2}{a^2} - \frac{r^2 + \alpha^2}{r^2} \frac{2(r^2 + \alpha^2)}{r^2} + \frac{2(3r^2 + \alpha^2)}{x_i^2 + \alpha^2 + r^2} \right] \]  \hspace{1cm} (A9)

\[ \times \left[ 11 - \frac{2\alpha^2}{r^2} + \frac{r^2}{\alpha^2} - \frac{4(r^2 + \alpha^2)^2}{x_i^2 + \alpha^2 + r^2} \right] - \frac{1}{5\alpha^2} \arctan \frac{xy[x_i - (x_0 + \Delta x/2)]}{\alpha [y_i - (y_0 + \Delta y/2)]}, \]  \hspace{1cm} (A9)

and

\[ r = \sqrt{x^2 + y^2 + \alpha^2}. \]  \hspace{1cm} (A10)