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Short Note

Attitude, velocity, and depth of a plane refractor from two line profiles

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INTRODUCTION

Many refraction seismologists believe that a minimum of two reversed profiles, preferentially oriented at right angles to each other, are necessary to determine the attitude, velocity, and depth of a plane subsurface refractor. Russell et al. (1982) demonstrated that traveltimes recorded along three unreversed spreads can be analyzed to yield this same information. They state that similar measurements made on only two such profiles cannot define the three-dimensional (3-D) attitude of the dipping horizon. This statement, however, is incorrect. Russell et al. did not fully use the information contained in the intercept times of the traveltime curves. The purpose of the present note is to demonstrate that, in many cases, two refraction profiles are sufficient to define the 3-D attitude, true velocity, and depth of a plane refractor. Generally speaking, the main condition required is that the two lines provide independent traveltime information about the subsurface.

EARTH MODEL AND RECORDING GEOMETRY

The earth model consists of a single layer with P-wave speed V_1 overlying a half-space with P-wave speed V_2. The plane interface separating the two media possesses, in general, a 3-D dipping attitude. Let the xy plane of a right-handed rectangular coordinate system be coincident with the free surface of the layer; depth z is measured positive in the downward direction. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector of an arbitrary point on the subsurface interface, then the equation defining this dipping plane is

$$\mathbf{r} \cdot \mathbf{n} = d. \tag{1}$$

d is the perpendicular distance from the coordinate origin O to the plane, and $\mathbf{n}$ is a unit vector at O normal to the plane. Figure 1 illustrates that $\mathbf{n}$ is conveniently described in terms of polar and azimuthal angles ϕ and θ:

$$\mathbf{n} = (\sin\phi\cos\theta)\mathbf{i} + (\sin\phi\sin\theta)\mathbf{j} + (\cos\phi)\mathbf{k}. \tag{2}$$

ϕ $(0 \le \phi < \pi/2)$ is the dip angle of the interface and θ $(0 \le \theta < 2\pi)$ is the updip direction angle. If the $+x$ and $+y$ axes are taken to point toward geographic north and east, respectively, then the strike angle of the interface is $\psi = \theta + \pi/2$ (modulo 2π).

Figure 2a is a plan view of the surface recording geometry. The position vectors of the source S and receiver R are given by

$$\mathbf{r}_S = x_S\mathbf{i} + y_S\mathbf{j} \tag{3a}$$

and

$$\mathbf{r}_R = x_R\mathbf{i} + y_R\mathbf{j}. \tag{3b}$$

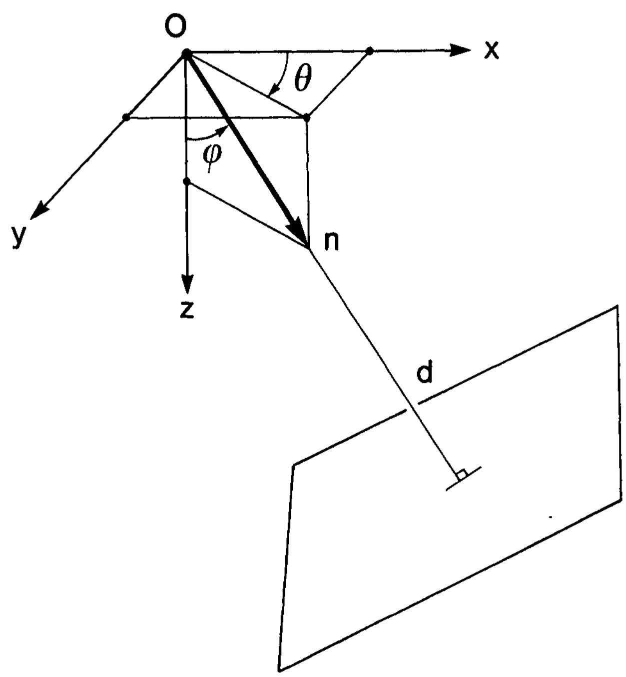

FIG. 1. Earth model and coordinate reference frame.

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The azimuth angle of the receiver relative to the source is α ($0 \le \alpha < 2\pi$). Azimuth is measured positive in the clockwise sense from geographic north. If the horizontal offset between source and receiver is denoted by X ($X \ge 0$), then the receiver coordinates can be expressed in terms of the source coordinates as

$$x_R = x_S + X \cos \alpha \tag{4a}$$

and

$$y_R = y_S + X \sin \alpha. \tag{4b}$$

Profile recording geometry is defined by the condition that the azimuth angle α remains fixed for a set of receivers that record energy from a single source.

HEAD-WAVE RAYPATH

Consider the critically refracted raypath from surface source S to surface receiver R (Figure 3). This raypath is confined to a single plane referred to herein as the raypath plane. In general, the raypath plane is *not* a vertical plane (parallel to the z axis). Only in the specific situation where the profile line is oriented directly updip or downdip is the raypath plane vertical. The traveltime of a head wave propagating along the critical raypath from source to receiver can be worked out from simple geometric considerations. It is

$$T = \frac{L}{V_2} + (d_S + d_R)\frac{\cos i_c}{V_1}, \qquad L \ge L_c. \tag{5}$$

L is the source-receiver range measured parallel to the refracting interface and d_S and d_R are perpendicular distances from S and R to this interface. The critical refraction angle i_c is given by $\sin i_c = V_1/V_2$. Of course, the head wave exists only for ranges exceeding the critical range L_c,

$$L_c = (d_S + d_R) \tan i_c. \tag{6}$$

TRAVELTIME EQUATION

The head-wave traveltime formula is more useful to the geophysicist if it is expressed in terms of the horizontal offset X. A simple analytical derivation of the desired relation is given in this section. This method has the advantage of not requiring visualization of the 3-D geometry. Starting with the general equation for the dipping plane, it is easy to demonstrate that the perpendicular distances from S and R to this plane are given by

$$d_S = d - \mathbf{r}_S \cdot \mathbf{n}$$

and

$$d_R = d - \mathbf{r}_R \cdot \mathbf{n}.$$

Hence,

$$d_R = d_S + (\mathbf{r}_S - \mathbf{r}_R) \cdot \mathbf{n}.$$

Substituting equations (2), (3), and (4) and simplifying yields

$$d_R = d_S - X \sin \phi \cos (\alpha - \theta).$$

Defining δ ($-\pi/2 < \delta < \pi/2$) as $\sin \delta = \sin \phi \cos (\alpha - \theta)$,

$$d_R = d_S - X \sin \delta. \tag{7}$$

The angle δ is *not* the apparent dip of the refracting interface along the profile line azimuth α; apparent dip γ is related to true dip ϕ via

$$\tan \gamma = \tan \phi \cos (\alpha - \theta). \tag{8}$$

Obviously, $\delta \ne \gamma$. The difference between δ and γ arises from the fact that δ is measured in the raypath plane, whereas apparent dip γ is measured in a vertical plane. For small values of true dip ϕ, $\delta \approx \gamma$.

The range L is equal to the distance PQ in Figure 3. Hence,

$$\begin{aligned} L^2 = PQ^2 &= |(\mathbf{r}_S + d_S\mathbf{n}) - (\mathbf{r}_R + d_R\mathbf{n})|^2 \\ &= |(\mathbf{r}_S - \mathbf{r}_R) + (d_S - d_R)\mathbf{n}|^2 \\ &= |\mathbf{r}_S - \mathbf{r}_R|^2 + 2(d_S - d_R)(\mathbf{r}_S - \mathbf{r}_R) \cdot \mathbf{n} + (d_S - d_R)^2 \\ &= |\mathbf{r}_S - \mathbf{r}_R|^2 - (d_S - d_R)^2 \\ &= X^2 - X^2 \sin^2 \delta. \end{aligned}$$

Thus,

$$L = X \cos \delta. \tag{9}$$

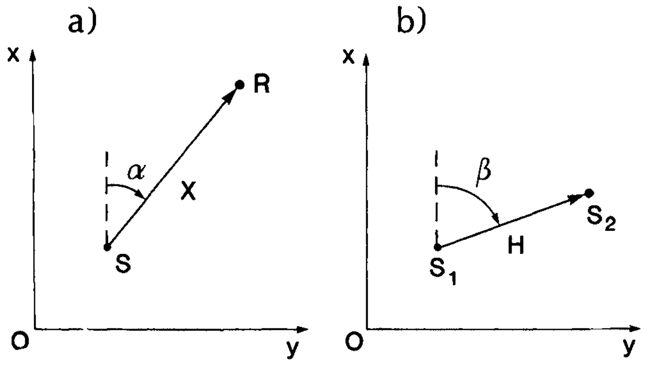

FIG. 2. (a) Plan view of surface recording geometry. S and R denote a point source and a point receiver, respectively. (b) Spatial relationship of the two sources S_1 and S_2.

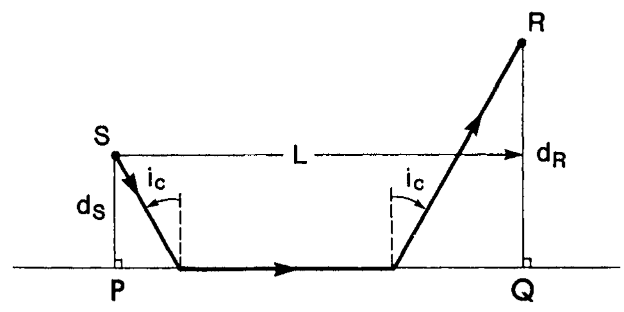

FIG. 3. Head-wave raypath. The plane of this diagram is perpendicular to the refracting interface and thus is not necessarily vertical.

Equations (7) and (9) are the desired expressions. It is evident that these formulas could have been derived by purely geometrical reasoning based on the raypath diagram of Figure 3. However, the relationship of the angle δ to the profile line azimuth and the orientation angles of the refracting plane would not have been easy to ascertain. Substituting expressions (7) and (9) into equation (5) and reducing yields an expression for the head-wave traveltime:

$$T = X\frac{\sin(i_c - \delta)}{V_1} + 2d_S\frac{\cos i_c}{V_1}, \qquad X \geq X_c. \tag{10}$$

An expression for the critical offset distance X_c is obtained by similar manipulations. Evaluating equations (7) and (9) at the critical distance and then substituting into equation (6) gives

$$X_c = 2d_S\frac{\sin i_c}{\cos(i_c - \delta)}. \tag{11}$$

The traveltime formula (10) indicates that refracted arrivals propagate along the receiver spread with an apparent velocity V_a given by

$$V_a = \frac{V_1}{\sin(i_c - \delta)} = \frac{\sin i_c}{\sin(i_c - \delta)}V_2. \tag{12}$$

The variation of apparent velocity with profile azimuth is depicted in Figure 4 for $V_1/V_2 = 3/5$.

Equations (10) through (12) indicate that the 3-D refraction formulas are straightforward generalizations of those appropriate for the 2-D problem: the angle δ replaces the true dip φ in the relevant expressions. It is emphasized again that δ is not the apparent dip of the subsurface interface along the profile line. However for a gently dipping refractor, the practical difference between the two is small.

TRAVELTIME INVERSION

Traveltimes recorded along a set of refraction profiles can be inverted to recover the 3-D attitude and true velocity of the refractor. The measured slope and intercept time of the ith such traveltime curve, m_i and τ_i, respectively, are related to the assumed earth model parameters as follows:

$$m_i = \frac{\sin(i_c - \delta_i)}{V_1}, \tag{13a}$$

$$\tau_i = 2d_{S_i}\frac{\cos i_c}{V_1}, \tag{13b}$$

with

$$\sin\delta_i = \sin\phi\cos(\alpha_i - \theta), \tag{13c}$$

$$d_{S_i} = d - \sin\phi(x_{S_i}\cos\theta + y_{S_i}\sin\theta). \tag{13d}$$

(x_{S_i}, y_{S_i}) is the position of source i, and α_i is the azimuth angle of profile line i. The perpendicular depth to the interface below source i is designated d_{S_i}.

How many refraction lines are required in order to successfully invert for the earth-model parameters? Assuming that the overburden velocity V_1 is known (perhaps from borehole data or traveltime analysis of the direct arrivals), then the earth model is defined by the four parameters V_2, φ, θ, and d. Intuition suggests that two refraction profiles will yield the four data values (two slopes and two intercepts) required to solve the problem unambiguously. Indeed, if the line index i is set equal to 1 and then 2, the system (13) above becomes a set of eight equations in eight unknowns. Since the equations are nonlinear in the unknowns, a definitive statement on the existence and uniqueness of a solution cannot be made. However, it can be demonstrated by algebraic techniques that, in many cases, two refraction profiles are sufficient to solve the problem.

The method proposed by Russell et al. (1982) requires three lines to obtain the refractor velocity, attitude, and depth. They use the slopes of the three traveltime curves, m_1, m_2, and m_3, and one intercept time τ_1 as measured data in the inversion. For the case where the index i runs from 1 to 3, system (13) represents 12 equations in 10 unknowns; hence, it would seem that redundant information exists. Although redundant data are always valuable in any practical inversion scheme, the theoretical minimum set of conditions under which the inversion is possible is of interest here.

In the following analysis, only two refraction profiles are employed. First, define the angles μ_1 and μ_2 by

$$\mu_1 = \sin^{-1}(m_1V_1) \tag{14a}$$

and

$$\mu_2 = \sin^{-1}(m_2V_1). \tag{14b}$$

Next, the position of source 2 is expressed in terms of the coordinates of source 1:

$$x_{S_2} = x_{S_1} + H\cos\beta \tag{15a}$$

and

$$y_{S_2} = y_{S_1} + H\sin\beta. \tag{15b}$$

β ($0 \leq \beta < 2\pi$) is the azimuth of source 2 relative to source 1 (measured positive in the clockwise sense from geographic north), and H is the horizontal distance between the two source locations (see Figure 2b). Using these expressions, the angles δ_1 and δ_2 and the depths d, d_{S_1}, and d_{S_2} can be quickly eliminated from the system (13). A reduced system

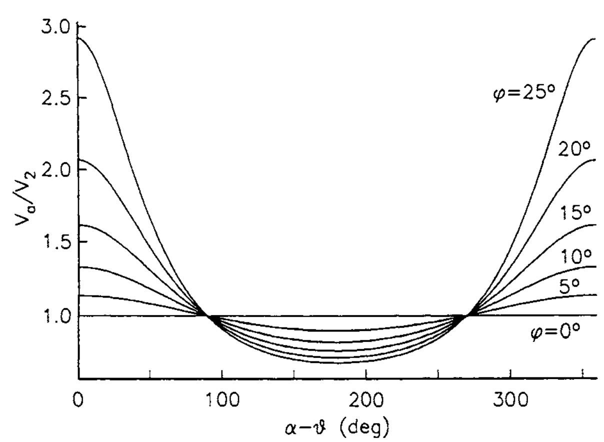

FIG. 4. Normalized apparent refractor velocity V_a/V_2. Each curve refers to a specific value of the interface dip angle φ. The P-wave velocity ratio $V_1/V_2 = 3/5$.

consisting of three nonlinear equations in the three unknown angles i_c, ϕ, and θ is obtained:

$$\sin(i_c - \mu_1) = \sin\phi\cos(\alpha_1 - \theta), \qquad (16a)$$

$$\sin(i_c - \mu_2) = \sin\phi\cos(\alpha_2 - \theta), \qquad (16b)$$

$$\frac{V_1(\tau_1 - \tau_2)}{2} = H\cos i_c \sin\phi\cos(\beta - \theta). \qquad (16c)$$

Three particular cases of this system are examined before a more general solution technique is described in the next section.

(1) *Coincident sources*. If the two sources occupy the same position ($H = 0$), then equation (16c) reduces to $0 = 0$ (coincident source locations imply identical intercept times). The two remaining equations contain three unknowns and cannot have a unique solution. Hence, two refraction profiles emanating from the same source location supply insufficient independent information for a successful inversion. The split recording spread is a particular example of coincident-source refraction profiles.

(2) *Parallel profiles*. If the azimuths of both refraction lines are the same ($\alpha_1 = \alpha_2$), then equations (16a) and (16b) are not independent (identical azimuths imply identical slopes of the traveltime curves and thus $\mu_1 = \mu_2$). Hence, the parallel profile recording configuration is not adequate to solve the problem either.

(3) *Antiparallel profiles*. Suppose that line 2 is recorded in a direction opposite to that of line 1 ($\alpha_2 = \alpha_1 \pm \pi$). Then equations (16a) and (16b) can be solved immediately for the critical refraction angle:

$$i_c = \frac{\mu_1 + \mu_2}{2}. \qquad (17)$$

The true refractor velocity is obtained via $V_2 = V_1/\sin i_c$. This procedure for obtaining refractor velocity is identical to that used with the classical 2-D reversed spread. It is clear that it is generally valid for antiparallel profiles (collinear or otherwise) recorded over a 3-D dipping interface.

With the critical angle i_c determined, the system (16) can be solved for the remaining unknowns. The azimuth of line 1 is designated by α. Straightforward, but tedious, algebraic manipulation then yields

$$\tan\theta = -\left[\frac{(\tau_1 - \tau_2)\cos\alpha + H(m_1 - m_2)\cos\beta}{(\tau_1 - \tau_2)\sin\alpha + H(m_1 - m_2)\sin\beta}\right], \qquad (18)$$

and

$$\sin\phi = F\sqrt{(\tau_1 - \tau_2)^2 + 2(\tau_1 - \tau_2)H(m_1 - m_2)\cos(\alpha - \beta) + H^2(m_1 - m_2)^2}. \qquad (19)$$

The multiplicative factor F is given by

$$F = \left|\frac{V_1}{2H\cos i_c \sin(\alpha - \beta)}\right|. \qquad (20)$$

A minor ambiguity associated with selecting the proper branch of the inverse tangent function in equation (18) is easily resolved by ensuring that the angle θ so obtained satisfies the original system (16). After θ and ϕ are determined, one of the equations (13b), (13d) can be solved to yield the distance d from the reference point O to the interface.

The solution stated in equations (18) and (19) contains an important indeterminacy when the azimuth angle β of source 2 equals α or $\alpha \pm \pi$. The identity of reciprocal times recorded at two inline shot positions then implies that each equation reduces to 0/0. The recording geometry in this situation consists of two collinear, antiparallel profiles and hence is inadequate to define 3-D refractor attitude. The reversed profile is a typical example of this configuration. As long as there exists some horizontal offset between the two lines recorded in opposite directions, the inverse problem is well posed, at least in a theoretical sense. This data acquisition geometry is a fairly common arrangement for many land and marine seismic surveys.

ARBITRARY LINE AZIMUTHS

A combination of algebraic and algorithmic techniques yields a solution of the system (16) for the case of arbitrary profile line azimuths α_1 and α_2. The special situations treated in the previous section are excluded. Eliminating θ and ϕ results in a quadratic equation in $\tan i_c$:

$$A\tan^2 i_c + B\tan i_c + C = 0. \qquad (21)$$

The constants A, B, and C depend upon measured quantities and recording geometry as follows:

$$A = \frac{V_1(\tau_1 - \tau_2)}{2H}\sin(\alpha_2 - \alpha_1), \qquad (22a)$$

$$B = -\cos\mu_1\sin(\alpha_2 - \beta) - \cos\mu_2\sin(\alpha_1 - \beta), \qquad (22b)$$

and

$$C = \sin\mu_1\sin(\alpha_2 - \beta) - \sin\mu_2\sin(\alpha_1 - \beta) + A. \qquad (22c)$$

In the case of antiparallel profiles, the constant A vanishes and $\tan i_c$ is obtained by solving a linear equation. It is easy to verify that the expression for the critical angle i_c given by equation (17) is reproduced. In general, solution of equation (21) is via the quadratic formula:

$$\tan i_c = \frac{-B + \operatorname{sgn}(B)\sqrt{B^2 - 4AC}}{2A}, \tag{23}$$

where the sign of the radical has been chosen to yield an indeterminacy for $A = 0$. Application of L'Hopital's rule then indicates that equation (17) is recovered as $A \rightarrow 0$. With i_c determined, simple back sequential substitution into the expressions gives the remaining unknowns. Backsubstitution is done numerically rather than symbolically, since the formulas rapidly become unwieldy. Hence,

$$\tan\theta = -\left\{\frac{H\cos i_c \sin(i_c-\mu_1)\cos\beta-[V_1(\tau_1-\tau_2)/2]\cos\alpha_1}{H\cos i_c \sin(i_c-\mu_1)\sin\beta-[V_1(\tau_1-\tau_2)/2]\sin\alpha_1}\right\}. \tag{24}$$

An analogous (and completely equivalent) formula involving μ_2 and α_2 could be used. Again, care must be exercised to ensure that the angle θ obtained by inverting relation (24) satisfies the original system of equations. Any one of the expressions (16a)–(16c) can then be solved for the true dip angle ϕ. Finally, depth d is determined as previously described.

CONCLUSION

There are numerous practical recording geometries where the analysis of critically refracted arrival times recorded on only two unreversed spreads can yield the 3-D attitude, true velocity, and depth of a plane subsurface horizon. Extension of the analytical techniques discussed here to include multilayered earth models and/or nonprofile recording geometries will be quite useful for defining the minimum requirements for a successful inversion of 3-D refraction traveltime data.

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