combined nomograph and curves in Fig. 3. The nomograph evaluates the parameter

$$15.37\left(\frac{a}{f_0^2 h}\right)$$

which appears in Equations [7] and [9]. This is transferred by a horizontal projection to the co-ordinate system for the curves; values for the ratio of natural frequencies Ω/Ω_0 are read from the lower abscissa scale, and values for the deflection ratio δ/h are read from the upper abscissa scale. The method of employing Fig. 3 is illustrated by the following example:

Application of Nomograph

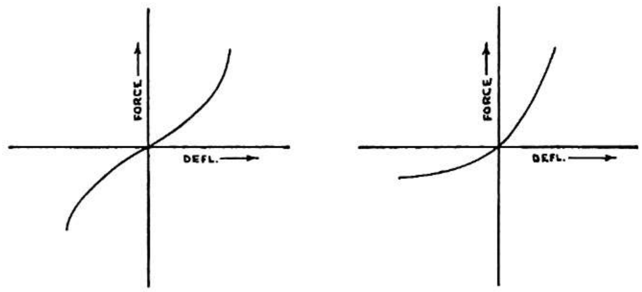
An isolator using rubber in compression $^1/_2$ in. thick has a natural frequency at zero deflection of 10 cps; it is subjected to a sustained acceleration of 11 g. It is desired to determine the transmissibility of the isolator to vibration having a forcing frequency of 100 cps, and to determine the deflection of the isolator under the sustained acceleration. Referring to Fig. 3, a straight line is drawn from 10 on the "natural frequency" scale to 0.5 on the "thickness" scale. A second straight line is drawn from the intersection of the first line with the R scale, and is extended through the value 11 on the "sustained acceleration" scale. The second line intersects the left side of the co-ordinate system and is extended horizontally to the intersection with the solid and dotted curves. This indicates a ratio of natural frequency Ω to natural frequency Ω_0 at zero deflection equal to 3.5, and a ratio of deflection δ to thickness h of 0.81. The deflection of the isolator as a result of the sustained acceleration is $0.81 \times 0.5 = 0.405$ in. The transmissibility is calculated from Equation [8] by substituting $\omega = 100 \times 2\pi$ and $\Omega = 10 \times 3.5 \times 2\pi$. Making these substitutions, the transmissibility T is 0.140. In the absence of the sustained acceleration, the corresponding transmissibility would be obtained by substituting $\omega = 100 \times 2\pi$ and $\Omega_0 = 10 \times 2\pi$. The transmissibility of 0.011 obtained from this latter calculation compares with the transmissibility of 0.140 when the system is subjected to sustained acceleration of 11 g.

The expression for the force-deflection characteristic of an isolator, Equation [1], is an approximation. The characteristic for a rubber member in compression may be made to coincide quite closely with this expression by care in design. In general, however, some deviation from the characteristic defined by the expression may be expected. This deviation does not destroy the usefulness of the foregoing data as a design procedure, because the dynamic properties of rubber are such that other uncertainties of equal magnitude are introduced into the determination of natural frequency on the basis of static properties of the rubber.

Discussion

D. C. Kennard, Jr.[5] The system treated in this paper has considerable practical significance in vibration protection of delicate equipment. When the protected equipment is subjected to a sustained acceleration while vibrating, the appropriate stiffening spring must be used in mounting the equipment to minimize static deflections and yet provide the necessary degree of vibration protection.

In this paper it has been assumed that the vibratory displacements are small enough to permit the dynamic spring action to be considered as linear. Hence only one resonant frequency results at a given sustained acceleration in a single-degree-of-freedom system. However, in many applications the vibratory displacements are large enough to warrant consideration of the nonlinear spring effects.



Symmetrical nonlinear spring Asymmetrical nonlinear spring

Fig. 4

This type of nonlinearity is "asymmetrical" about the neutral position of the mass, i.e., the position of rest when the mass does not vibrate. In contrast, the usual nonlinearity treated in text books is symmetrical, Fig. 4 of this discussion.

Assume that asymmetrical nonlinearity of spring rate may be approximated by two straight lines forming an obtuse angle, as shown in Fig. 5(a). Two simple systems having such a spring-rate characteristic are shown in Fig. 5(b), herewith, where a simple pendulum has a different length on one side of its swing than on the other or where a spring-suspended mass contacts another spring at its neutral position. In such systems, free oscillations take place where the half-sine excursion has a different period and amplitude on one side of the neutral position than on the other, as shown in Fig. 6.

The kinetic energy of the mass passing through neutral position just prior to contact with the stiffened-spring rate is $^1/_2\, mv^2$.

The potential energy stored in both springs at the point of maximum downward excursion is

$$^1/_2(K_1 + K')d_2^2 = \frac{1}{2} K_2 d_2^2$$

Equating the energies

$$mv^2 = K_2 d_2^2 \quad [10]$$

For the upward excursion above the neutral position, the energy equation is

$$mv^2 = K_1 d_1^2 \quad [11]$$

From Equations [10] and [11] the following expression for amplitude ratio in terms of spring-rate ratio is obtained

$$\frac{d_1}{d_2} = \sqrt{\frac{K_2}{K_1}} \quad [12]$$

The periods may be expressed as

$$2T_1 = 2\pi\sqrt{\frac{m}{K_1}} \quad \text{and} \quad 2T_2 = 2\pi\sqrt{\frac{m}{K_2}}$$

Hence

$$\frac{T_1}{T_2} = \sqrt{\frac{K_2}{K_1}} \quad [13]$$

With a forcing frequency having a period $T_f = T_1 + T_2$, a net amount of energy can be fed into the system each cycle, thus producing resonance at a frequency which corresponds to the natural frequency treated in the paper.

Now assume a forcing frequency having a half period equal to

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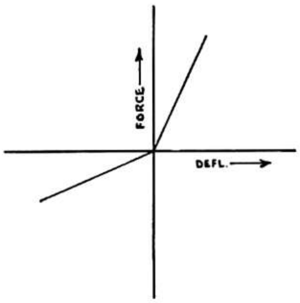
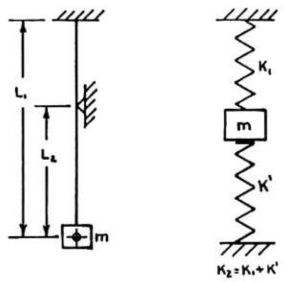


(a)
Asymmetrical nonlinear spring approximation



(b)
Typical systems with spring characteristics shown in (a)

FIG. 5

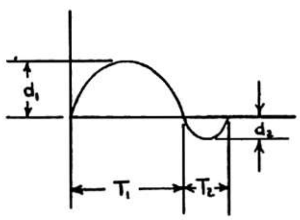


FIG. 6 FREE OSCILLATION OF ASYMMETRICAL NONLINEAR SYSTEM

T_1, i.e., $T_f = 2T_1$. The forcing frequency will be in phase with the first half cycle of velocity only when the following ratio is integral

$$n\left(\frac{T_{f1}}{T_1 + T_2}\right) = N; \quad N = 1, 2, 3 \ldots\ldots\ldots\ldots \quad [14]$$

where n is the number of forcing cycles required to make the ratio integral. In other words, T_{f1} will be in phase with the velocity of T_1 every nth cycle of the forcing frequency.

If the forcing frequency has a half period $T_{f2} = 2T_2$, the forcing frequency will be in phase with the second half of the velocity cycle only when the following ratio is integral

$$n\left(\frac{T_{f2}}{T_1 + T_2}\right) = N \ldots\ldots\ldots\ldots\ldots\ldots \quad [15]$$

The same number n is required in Equations [14] and [15] to make the ratios integral.

This system responds with a pseudobeating effect at two frequencies of maximum excursion. The first frequency has a period T_{f1} which corresponds to the slow part of the cycle T_1. But as the forcing frequency enters the fast portion of the cycle, its phasing with the velocity response is lost so that even when the slow cycle repeats, the forcing frequency is not phased with response although the half-cycle periods are the same. Hence the slow-cycle displacement is decreased by the opposing effect of the unphased force cycle. This opposing effect continues as the oscillations progress until an integral number of response cycles has taken place during an integral number of force cycles. At this point the velocity cycle receives an in-phase force cycle which boosts the displacement response to a maximum, thus completing the pseudobeat cycle. A similar beat is encountered with the higher frequency resonance where T_{f2} corresponds to the fast part of the cycle T_2.

Thus with large amplitudes such a system has three resonant frequencies or frequencies of maximum excursion. These are the resonant frequency having a period $T_f = T_1 + T_2$, a lower beating-type resonance where $T_{f1} = 2T_1$, and a higher beating-type resonance where $T_{f2} = 2T_2$.

This discussion is intended to provide a simple explanation for the presence of beats which otherwise are unexplainable in nonlinear systems. It in no way detracts from the importance, validity, and timeliness of the paper.

C. D. PENGELLEY.[6] The title of the paper is somewhat misleading since it seems to indicate a general study of the natural frequency of nonlinear systems. In actual practice, it provides a solution for a specific system consisting of a mass supported by a pad of rubber in compression.

A valuable contribution has been made for design purposes. The nomograph, presented in Fig. 3 of the paper, provides much useful information, and the procedures are set forth clearly. Of particular interest is the empirical relationship contained in Equation [1] of the paper which has been set forth graphically in Fig. 2. This provides an approximate force-deflection function for typical rubber springs in compression. It would have been interesting if the author had discussed in more detail the limitations of this expression. What are the effects of changes in the ratio of spring height to cross-sectional area? May it be applied with equal confidence to a "sheet" of rubber and a "block?"

The frequency equations are based upon the assumption that under any given static deflection, the stiffness may be assumed constant during vibration. This, of course, limits the application to vibrational amplitudes which are very small compared with the initial static deflection. Needless to say, a more precise nonlinear analysis which would be applicable to large amplitudes probably would be too complex for practical purposes. Nevertheless, a discussion based on empirical data or qualitative reasoning, which would indicate whether the nonlinearities tend to raise or lower the natural frequencies, would be desirable. An analogy of this would be of considerable practical importance.

AUTHOR'S CLOSURE

It is well known that a complete analysis of a nonlinear system involves many complex phenomena. Certain of these can be simplified by assuming small displacements. The present paper implies but does not state explicitly that the analysis refers only to vibration of small displacement. Mr. Kennard's interesting discussion points out some of the considerations that must be taken into account when analyzing the motion of a nonlinear system with large displacements.

Mr. Pengelley points out that the title of the paper suggests a more general treatment of nonlinear systems than that included in the paper. This criticism is justified in a sense. The explicit results set forth in the nomograph and curves are, of course, limited to one particular type of nonlinearity. In a broader sense, however, the paper points out generally the effect of a nonmassive load and suggests an analytical approach that may be applied to any known type of nonlinearity.

It is suggested by Mr. Pengelley that information be included on the application of Equation [1] to rubber shapes in general. This equation is empirical and has been found to be approximate for rubber elements loaded in compression. The stiffnesses of rubber blocks of the same elastomer and same load-carrying area tend to vary inversely as the thickness. There is an additional consideration resulting from the fact that Poisson's ratio for rubber is one half; i.e., the rubber is incompressible and deflects only by bulging. The lateral area of the block is thus a very important factor in determining the stiffness. Equation [1] may be made to fit the resulting force-deflection curves of blocks and pads empirically by assignment of appropriate values to k_0 and h.

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