Some Remarks on the Charge Conjugation

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The charge conjugation is studied, and it is represented by a reflection in some space, i.e., the charge space. The discontinuity of this transformation gives rise to some arbitrariness and so several transformation types. Regarding these transformation types as the intrinsic property of the individual elementary particles, we shall investigate the possibility that we could introduce the qualitative difference between elementary particles and the conservation law of the heavy particles, more generally, transition rule between elementary particles would be supported by such a superselection rule that recently Wigner et al. proposed in association with the time reversal.

§ 1. Introduction

The charge conjugation is usually considered as a following transformation.

(I)
\begin{align}
\phi^a_a \rightarrow \phi^-_a,
\phi^-_a \rightarrow \phi^+_{a'}
\end{align}

where \(a\) and \(\bar{a}\) describe some particle and antiparticle, respectively.

(II)
\begin{align}
\phi^a_a \rightarrow \phi^+_a,
\phi^-_a \rightarrow \phi^-_{a'}
\end{align}

where \(a\) describes a tensor suffix.

(III) The coupling constant of the interaction of \((\bar{\phi}^a_\alpha \phi^b_\beta)\) type between the fermion and boson is transformed as

\begin{align}
\mathcal{g}_{S,P\pm, P\mp} \rightarrow \mathcal{g}_{S,P\pm, P\mp},
\mathcal{g}_{V,T,(P\pm)} \rightarrow -\mathcal{g}_{V,T,(P\pm)},
\end{align}

where \(\mathcal{g}_{S,P\pm, P\mp}\) denotes the coupling constant for the scalar, pseudoscalar and pseudovector coupling and \(\mathcal{g}_{V,T,(P\pm)}\) for the vector, tensor and pseudotensor coupling, respectively. It should be noted here that this transformation may be interpreted as a reflection in some space called a charge space as we shall show in a following section and that the relation between the representation \(\phi^+_a, \phi^-_{a'}\) of \(a\) and \(\bar{a}\) particles:

\begin{align}
\phi^-_{a'} = \rho \phi_{a}
\end{align}

*) The dagger describes the hermitian conjugate.

**) This transformation of the coupling constant is unique, when the boson field is neutral and the spinor particles \(a\) and \(b\) are identical, because otherwise the interaction term vanishes.
is not unique and involves an arbitrary factor \( p = \pm 1, \pm i \) where \( K \) is some unitary operator consisted of \( E_\mu \)-matrices.\(^1\) Further, the discontinuity associated with a reflection and the appearance of arbitrary factor are closely connected with each other.

In this paper we shall treat the charge conjugation from a general point of view which involves the conventional one above mentioned as a special case.

§ 2. The charge conjugation

As a simple example of the charge conjugation of the spinor field, we shall treat the electron field, whose interaction with the electromagnetic field is invariant under this transformation. However, the following discussion may be independent from the electric charge and holds for such a field as the neutron which does not interact with the electromagnetic field in a vector type.

The interaction Hamiltonian density between the electron and the electromagnetic field has a following form.

\[ H' = ie / 2 \{ \phi_\alpha E_\mu \phi_\beta - \bar{\phi}_\beta E_\mu \bar{\phi}_\alpha \} A_\mu, \]  

where \( \phi \) and \( \bar{\phi} \) describe the electron and positron fields, respectively, and whose relation is considered as fixed by a specific value \( \rho \) of the arbitrary factor in (1.4). Now, let us consider \( \phi_\alpha \) and \( \bar{\phi}_\beta \) as two components of the representation of the electron-positron field by a specified reference system in a certain two dimensional space which we shall hereafter generally \( P-A \) space and denote this set as

\[ \phi = \begin{pmatrix} \phi_\alpha \\ \bar{\phi}_\beta \end{pmatrix}. \]  

Using \( \phi \) and Pauli’s isotopic spin matrix \( \tau_3 \) whose operand is the above mentioned \( P-A \) space, we can rewrite (2.1) in a following simple form.

\[ H' = i\bar{\epsilon} (2) \{ \phi E_\mu \tau_3 \phi \} A_\mu^{(0)}, \]  

where \( \epsilon^{(0)} \) and \( A_\mu^{(0)} \) are defined by

\[ \epsilon^{(0)} = \epsilon, \]  

\[ A_\mu^{(0)} = A_\mu. \]  

From the expression of (2.3), we can interpret \( H' \) as a quantity associated with the third

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\(^1\) In this paper, we use the notation introduced by S. Watanabe.\(^2\) Namely, four basic matrices \( E_1, E_2, E_3 \) and \( E_0 \) are defined by

\[ 1/2 \cdot (E_\mu E_\nu + E_\nu E_\mu) = \epsilon_{\mu\nu}, \]

and the matrix \( K \) is such one that

\[ K^{-1} E_\mu K = -E_\mu^T; \]

\[ K^T = -K. \]

Further, \( \bar{\phi} \) is defined by

\[ \bar{\phi} = -\phi^t E_0 \]

in a hermitian system, where \( E_1, E_2, E_3 \) and \( i E_0 \) are hermitian.
axis in a certain space, which we shall call hereafter the charge space. Considering (1.1), 
(1.2) and (1.3), it finds out that the charge conjugation corresponds to the following 
transformation

\[
\begin{align*}
(1a) & \quad \{\bar{\phi} E_{\mu} \tau_3 \phi\} \longrightarrow \{\bar{\phi} E_{\mu} \tau_3 \phi\} \\
(2.5) & \quad A_\mu^{(3)} \longrightarrow A_\mu^{(3)} \\
(3) & \quad e^{(3)} \longrightarrow -e^{(3)}.
\end{align*}
\]

Further, from the expression of (2.3), (2.5) is equivalent to the following transformation.

\[
\begin{align*}
(1b) & \quad \{\bar{\phi} E_{\mu} \tau_3 \phi\} \longrightarrow \{\bar{\phi} E_{\mu} \tau_3 \phi\} \\
(2.5)' & \quad A_\mu^{(3)} \longrightarrow -A_\mu^{(3)} \\
(3) & \quad e^{(3)} \longrightarrow e^{(3)}.
\end{align*}
\]

When we adopt the latter expression (2.5)' and regard \(A_\mu^{(3)}\) and \(\bar{\phi} E_{\mu} \tau_3 \phi\) as the 
third components of the vectors in the charge space, (2.5)' shows that the charge 
conjugation is a reflection of the third axis in the charge space.

Since it seems natural from a physical point of view to consider the charge conjugation 
as (2.5)' rather than (1.1) - (1.3), we define it by

\[
\begin{align*}
(1) & \quad I_\phi \{\bar{\phi} E_{\mu} \tau_3 \phi\} = -\{\bar{\phi} E_{\mu} \tau_3 \phi\} \\
(2.6) & \quad I_\phi A_\mu^{(3)} = -A_\mu^{(3)},
\end{align*}
\]

where \(I_\phi\) is the charge conjugation operator in the charge space. However, it is important 
to note that the unitary operator \(C\) in the \(P-A\) space which corresponds to \(I_\phi\) in the 
charge space, i.e.,

\[
I_\phi \{\bar{\phi} E_{\mu} \tau_3 \phi\} = \{\bar{\phi} C^{-1} E_{\mu} \tau_3 C\phi\}
\]

is not uniquely determined. From (2.6) and (2.7), \(C\) must satisfy the following relations :

\[
C^{-1} \tau_3 C = -\tau_3.
\]

Therefore the general form of \(C\) is given by

\[
C = e^{i\theta} \begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix},
\]

where \(\theta\) and \(\varphi\) are the real arbitrary factors, respectively.

It should be noted here that the charge conjugation \(C\) which is equivalent to the 
certain unitary transformation \(I_\phi\) of the system is restricted only to the one whose deter-

*) In order that the charge conjugation is equivalent to some unitary transformation, this interpretation 
of the charge conjugation becomes unique, because the coupling constant remains invariant under the unitary 
transformation.
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\[ \text{Det. } C = -1, \]

that is,
\[ C = \begin{pmatrix} 0 & \rho & \\
\rho^* & 0 & \\
0 & 0 & 1 \end{pmatrix}, \quad \rho \rho^* = 1. \quad (2.9) \]

This fact is easily explained as follows. Let us assume that there exists the following unitary transformation:
\[ \begin{pmatrix} \phi \\ \phi^* \end{pmatrix} = R \begin{pmatrix} \phi \\ \phi^* \end{pmatrix}, \quad (2.10) \]

where
\[ C = \begin{pmatrix} 0 & \rho & \\
\rho^* & 0 & \\
0 & 0 & 1 \end{pmatrix}. \]

Then the relation between \( \rho \) and \( \rho_1 \) is determined from the fact that the two components \( \phi_a \) and \( \phi_a^* \) are connected by (1.4) with a specific value \( \rho_a \). Using (1.4) and \( R \phi_a R^{-1} = \rho \phi_a^* \) which follows from (2.10), we have
\[ \begin{pmatrix} \phi_a \\ \phi_a^* \end{pmatrix} = \rho_a \begin{pmatrix} \phi_a \\ \phi_a^* \end{pmatrix} \]

From (2.10) and (2.11), we have
\[ \rho_1 = \rho^*. \quad (2.12) \]

Thus, the above statement is verified.

For the boson field, the situation is perfectly analogous to the case of spinor field. For brevity, we treat the scalar or pseudoscalar boson, whose current four vector is given by
\[ j_\mu = ie(u^t a_\mu u - u \partial_\mu u^t). \quad (2.13) \]

Here again, we can rewrite (2.13) as
\[ j_\mu = ie(u^t \partial_\mu \tau_3 u), \quad (2.14) \]

introducing \( u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \), where \( u_1 \) and \( u_2 \) are defined by
\[ u_1 = u \]
and
\[ u_2 = \eta u_1^\dagger, \quad (2.15) \]

where \( \eta \) is an arbitrary factor such as \( \rho \) in the case of spinors.

The equations corresponding to (2.5) - (2.12) for the spinor field hold for the boson field, only if \( \phi, \phi^*, E_\mu, K \) and \( \rho \) are replaced by \( u, u^t, \partial_\mu, 1 \) and \( \eta \), respectively.

\section{§ 3. The super-selection rule}

In this section we shall show that the phase factor in the charge conjugation plays

\[ *) \] For the neutral boson, \( \eta \) is restricted to be real, that is, \( \pm 1 \). However, it is not the case for the neutral boson which is represented by the complex variable.\[ \]
an important role in connection with the time reversal of Wigner's type. The time reversal of Wigner's type is defined by the following transformation;

\[ \Psi \rightarrow K_0 R_w \Psi \]  

where \( \Psi \) is a state vector, \( K_0 \) is an operator which converts the operand into its complex conjugate, and further \( R_w \) is a unitary operator which has a following properties:

\[ R_w \psi_a(t) R_w^{-1} = c \ E_0 K \bar{\psi}_a(-t) \]  
\[ R_w \psi_\bar{a}(t) R_w^{-1} = \rho_0^2 \ c^* \ E_0 K \bar{\psi}_\bar{a}(-t) \]  
\[ R_w \eta_1(t) R_w^{-1} = c' \ \eta_1(-t) \]  
\[ R_w \eta_2(t) R_w^{-1} = \eta_2^* \ c'^* \ \eta_2^T(-t) \]  

where (3.3) and (3.5) are obtained by using (3.2), (3.4) and \( \rho_0 = \rho \ K \bar{\psi}_a \), \( \eta_2 = \eta \eta_1^T \), while \( c \) and \( c' \) are arbitrary factors which satisfy the relations \( cc^* = 1 \) and \( c'c'^* = 1 \).

According to (2.9), (2.10) and (3.2)-(3.5), the product of the time reversal of Wigner's type and the charge conjugation \( K_0 R_w R_o = K_0 R_p \), the time reversal of Pauli's type has following properties, operating to the states \( \psi_a \ \Psi_o, \ \phi_\bar{a} \ \Psi_o, \) etc., where \( \Psi_o \) denotes a vacuum state.

\[ K_0 R_p \psi_a(t) \Psi_o = e^{i \alpha \lambda /2} \bar{\psi}_a(-t) \Psi_o^* \]  
\[ K_0 R_p \psi_\bar{a}(t) \Psi_o = e^{-i \alpha \lambda /2} \bar{\psi}_\bar{a}(-t) \Psi_o^* \]  
\[ K_0 R_p \bar{\psi}_a(t) \Psi_0^* = -e^{-i \alpha \lambda /2} \psi_a(-t) \Psi_0 \]  
\[ K_0 R_p \bar{\psi}_\bar{a}(t) \Psi_0^* = -e^{i \alpha \lambda /2} \psi_\bar{a}(-t) \Psi_0 \]  

In the above expression, \( e^{i \alpha \lambda /2} \) is given by

\[ e^{i \alpha \lambda /2} = \rho_0^* \rho \ c. \]  

For \( \eta_1 \) and \( \eta_2 \), the similar equations are fulfilled;

\[ K_0 R_p \eta_1(t) \Psi_o = e^{i \alpha \lambda /2} \eta_1(-t) \Psi_0^* \]  
\[ K_0 R_p \eta_2(t) \Psi_o = e^{-i \alpha \lambda /2} \eta_2(-t) \Psi_0^* \]  
\[ K_0 R_p \eta_1(t) \Psi_o^* = e^{-i \alpha \lambda /2} \eta_1(-t) \Psi_0 \]  
\[ K_0 R_p \eta_2(t) \Psi_o^* = e^{i \alpha \lambda /2} \eta_2(-t) \Psi_0 \]  

where

\[ e^{i \alpha \lambda /2} = \eta_2^* \ \eta_1^* \ c'. \]  

Operating the time reversal of Pauli's type twice in succession, we have

\[ (K_0 R_p)^2 \psi_a \Psi_o = -e^{-i \alpha \lambda} \psi_a \Psi_o \]  
\[ (K_0 R_p)^2 \psi_\bar{a} \Psi_o = -e^{i \alpha \lambda} \psi_\bar{a} \Psi_o \]
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\[
(K_0 R_p)^2 u_1 \Psi_0 = e^{-i \alpha} u_1 \Psi_0 \quad (3 \cdot 13)
\]
\[
(K_0 R_p)^2 u_2 \Psi_0 = e^{i \alpha} u_2 \Psi_0 \quad .
\]

It is characteristic to the time reversal of Pauli's type that the phase factor \(e^{\pm i \alpha}\) remains after this transformation operates twice in succession, while in the time reversal of Wigner's type the phase factor is cancelled out in this case.

Now we assume that the several types under the time reversal corresponding to arbitrary \(\lambda\)-values are associated with the intrinsic property of the individual elementary particles and give the qualitative difference together with the mass, spin, charge, etc. Especially let us call them the particle of the first and second kind, whose \(\lambda\)-values are 0 and \(\pm 1\), respectively. As shown in (3.12) and (3.13), the \(\lambda\)-values of the particle and antiparticle are connected with each other by

\[
\lambda_p = -\lambda_{\bar{p}}. \quad (3 \cdot 14)
\]

Now let us describe the states in which the sum of the existing particles of the first kind fermion and second kind boson is even and odd by \(\Psi_E\) and \(\Psi_O\), respectively, and consider the superposed state \(\Psi_E + \Psi_O\). If we operate the time reversal of Pauli's type twice in succession to this state, the result becomes

\[
\Psi_E + \Psi_O \rightarrow \text{const.} \ (\Psi_E - \Psi_O), \quad (3 \cdot 15)
\]

according to the relation (3.12) and (3.13). While the state which is transformed by the time reversal twice in succession must be indistinguishable from the original state, it is not possible to make any statement as to the relative parity between the two states \(\Psi_E\) and \(\Psi_O\), and the measurability of the hermitian operator which has finite matrix element between the both states would lead to a contradiction. Namely, we can say that the superselection rule operates between the both states.\(^4\) Accordingly, the phenomena where the fermions of the first kind are created or annihilated by an odd number or even number are possible only if the bosons of the second kind are created or annihilated by an odd or even number, respectively, at the same time. However, the same statement holds for the fermions of the second kind from the consideration of the spin and statistics, if all spinor particles which are considered to exist in nature are classified into either the first or the second kind.

On the other hand, the heavy particles have been discovered one after another of late years and the stability of these particles has been attached importance to. However, the current theory seems to lack the principle to determine the interaction Hamiltonian or to be phenomenological in this sense. Although these systematic transition rule between elementary particles may be solved positively in a future theory, it seems interesting to note that, if we assume that the above nucleonic charge \(\lambda\) of the individual elementary particles is assigned to be identical with \(\lambda\) introduced by S. Oneda,\(^5\) then the superselection rule plays a role for the stability of the heavy particles from the fact mentioned above.

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