The Generalized Problem of Lamb

I. The Reflection and Transmission of Pressure and Shear Waves Generated by an Explosion in a Two Layered Spherical Earth

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* Received in original form 1965 October 9.
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### Symbol | Definition | Equation reference
--- | --- | ---
\( Q' \) | generalized Stoneley function associated with the transmitted \( S \)-potentials | 5.7
\( r \) | radial coordinate of observer | Fig. 1
\( \psi \) | angle of refraction for a \( P \) ray | —
\( R \) | \((r^2 + b^2 - 2br \cos \theta)^{1/2}\) | Fig. 1
\( t \) | time | —
\( U \) | displacement vector | 1.3
\( u \) | \( Re(\eta) \) | —
\( z \) | complex variable of integration | 5.5
\( \alpha \) | \( c_1/c \) | —
\( \gamma \) | shear wave number | 1.6
\( \delta_3, \delta_4 \) | transmission factors | 9.2
\( \eta \) | variable of integration for reflected potentials | 5.3
\( \theta \) | angular coordinate of observer | Fig. 1
\( \iota \) | angle of incidence for a \( P \) ray | —
\( \lambda, \mu \) | Lamé's constant for the elastic medium | 2.5
\( \nu \) | order of modified Bessel functions, \( n + \frac{1}{2} \) | 2.2
\( \xi \) | variable of integration for transmitted \( S \)-potentials | 5.7
\( \Pi \) | transmitted shear potential | 1.3
\( \Pi_3, \Pi_4 \) | potentials associated with \( \Pi \) in the Laplace transform plane | 4.7
\( \rho \) | density of elastic medium | —
\( \sigma \) | Poisson’s ratio | 2.8
\( \Phi \) | acoustic potential | 1.1
\( \Phi_j \) | potentials associated with \( \Phi \) in the Laplace transform plane | 4.6
\( \phi \) | \( \theta \) or \( 2\pi - \theta \) | 4.4
\( \Psi \) | transmitted pressure potential | 1.3
\( \Psi_3, \Psi_4 \) | potentials associated with \( \Psi \) in the Laplace transform plane | 1.3
\( \omega \) | \( c_1^2/c_2^2 \) | 2.8
\( \Omega \) | \((\sin \theta)^{-1} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right)\) | 2.8

### Summary

A model of the Earth comprising a fluid core and an infinite elastic mantle is considered. An explosive sound source initiated at an arbitrary point in the fluid core generates reflected acoustic waves in the core, Stoneley waves at the interface, and transmitted \( P \) and \( S \) waves in the elastic mantle.

The general solution in the Laplace transformed plane is composed of series of modified spherical Bessel and Hankel functions. Watson-type integral representations of the series are obtained and their asymptotic estimates are deduced by the method of Laplace. It is shown that the stress wave motion in the mechanical system is analogous to the geometrical optical system.
Introduction

In 1904, Sir Horace Lamb (5) considered the problem of disturbances generated by impulsive point forces applied on the surface of a plane semi-infinite elastic medium. The present investigation is a generalization of Lamb's problem. We attempt to study the phenomenon of wave propagation generated by an explosive sound source at an arbitrary point in a spherical fluid core surrounded by an infinite elastic medium.

The phenomenon of propagation resolves itself into three categories:
(a) The problem of internal reflection in the acoustic sphere,
(b) Propagation of pressure and shear waves which are transmitted into the elastic zone, and
(c) Generalized Stoneley waves in the fluid-solid interface.

If we assume that an atomic blast is essentially a pressure-type explosive source, then this problem should cast some light on the nature of the propagation of underground blasts beyond the detonation enclosure into the elastic zone. In this paper we shall consider only parts (a) and (b) of these phenomena.

1. Formulation of the problem

Let us suppose that an acoustic spherical region designated in Fig. 1 by I is surrounded by a homogeneous, isotropic elastic infinite medium, II.

Let \( P(x_0, y_0, z_0) \) = source location,
\( Q(x, y, z) \) = observer location,
\( C \) = acoustic velocity in I,
\( \Phi \) = acoustic potential in I.

![Source-observer location](Fig. 1)
Then the equation representing an explosion in I is given by
\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Phi = -\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\delta(t).
\] (1.1)

As the disturbance strikes the interface \( r = a \), it generates a Stoneley wave, elastic wave propagation in region II and acoustic wave reflection into region I. From Tanyi (6), the phenomenon of propagation in II is characterized by the system of equations
\[
\left( \nabla^2 - \frac{1}{c_1^2} \frac{\partial^2}{\partial t^2} \right) \Psi = 0; \quad \left( \nabla^2 - \frac{1}{c_2^2} \frac{\partial^2}{\partial t^2} \right) \Pi = 0,
\] (1.2)

where \( \Psi \) and \( \Pi \) are the \( P \)- and \( S \)-potentials and \( c_1 \) and \( c_2 \) are the pressure and shear wave velocities respectively.

If \( U \) is the displacement vector and \( i_1 \) a unit vector in the radial direction, then
\[
U = -\text{grad} \Psi + \text{curl} \text{curl} i_1 (r \Pi).
\] (1.3)

Across the I–II interface, we must ensure the continuity of the stress vector and the normal component of the displacement field. These conditions are reducible to
\[
\sigma_{rr} \big|_I = \sigma_{rr} \big|_\Pi; \quad \sigma_{r\theta} \big|_I = 0; \quad U_r \big|_I = U_r \big|_\Pi,
\] (1.4)
at \( r = a \).

Let us introduce the bilateral Laplace transform with respect to time. Then in I,
\[
(-\nabla^2 + q^2) \Phi = \delta(x-x_0),
\] (1.5)

and in II,
\[
(-\nabla^2 + k^2) \Psi = 0; \quad (-\nabla^2 + \gamma^2) \Pi = 0,
\] (1.6)

where
\[
q^2 = p^2/c^2, \quad k^2 = p^2/c_1^2, \quad \gamma^2 = p^2/c_2^2,
\]
\[
F(x, p) = \int_{-\infty}^{\infty} e^{-pt} F(x, t) dt,
\]
\[
F(x, t) = (2\pi i)^{-1} \int_B e^{pt} F(x, p) dp,
\]
x = (x, y, z),

and \( B \) is the Bromwich contour from \( \alpha - i\infty \) to \( \alpha + i\infty \) in the complex \( p \) plane.

The continuity conditions (1.4) are transformed accordingly.

2. Formal solutions

G. Tanyi has shown (6) that for \( r > b \),
\[
\Phi = \Phi_p + \frac{(br)^{-k}}{4\pi} \sum_{n=0}^{\infty} R_n(2n+1)I_n(qb)I_n(qr)P_n(\cos \theta)
\] (2.1)
where \( \Phi_p \) is the particular integral of (1.5) and is given by
\[
\Phi_p = \frac{(br)^{-1}}{4\pi} \sum_{n=0}^{\infty} (2n+1) I_{n}(q) K_{n}(qr) P_n(\cos \theta)
\]
\[
= (4\pi R)^{-1} \exp (-qR),
\]
\[ R = (r^2 + b^2 - 2br \cos \theta)^{1/2}, \quad \text{and} \quad \nu = n + \frac{1}{2}.
\] (2.2)

In space-time, the particular integral,
\[
\Phi_p(r, \theta, t) = (4\pi R)^{-1} \delta(t-R/c)
\] (2.3)
is a retarded potential with spherical wavefronts, \( R = ct \).

In region II, the homogeneous solutions of equation (1.6) may be given in the forms,
\[
\Psi_p = \frac{(br)^{-1}}{4\pi} \sum_{n=0}^{\infty} A_n (2n+1) I_{n}(q) K_{n}(kr) P_n(\cos \theta),
\]
\[
\tilde{\Psi}_p = \frac{(br)^{-1}}{4\pi} \sum_{n=0}^{\infty} B_n (2n+1) I_{n}(q) K_{n}(yr) P_n(\cos \theta).
\] (2.4)

It will be observed that the solutions (2.4) tend to zero as \( r \) tends to infinity. We will now express the continuity conditions in terms of the solutions (2.1) and (2.4):

At \( r = a \),
\[
\tilde{\sigma}_{rr} |_{I} = -\lambda I q^2 \tilde{\Phi} = \tilde{\sigma}_{rr} |_{II} = -\lambda k^2 \Psi - 2\mu \left[ \left( k^2 - \frac{\Omega}{r^2} - \frac{2}{r} \frac{\partial}{\partial r} \right) \Psi - \frac{\Omega}{r^2} + \frac{1}{r} \frac{\partial}{\partial r} (\Omega \tilde{\Psi}) \right],
\]
\[
\tilde{\sigma}_{r\theta} |_{I} = 0 = \tilde{\sigma}_{r\theta} |_{II} = -2\mu \frac{\partial}{\partial \theta} \left[ \frac{\partial}{\partial r} \left( \frac{\Psi}{r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\Omega + 1}{r^2} - \frac{\gamma^2}{2} \right) \tilde{\Psi} \right],
\]
\[
\tilde{U}_r |_{I} = -\frac{\partial \tilde{\Phi}}{\partial r} = \tilde{U}_r |_{II} = -\left[ \frac{\partial \tilde{\Psi}}{\partial r} + \frac{\Omega \tilde{\Psi}}{r} \right].
\] (2.5)

Using the expressions (2.1) and (2.4) the system (2.5) becomes
\[ 2A_n [(\omega a^2 + x + 1) K_{n}(ka) - 2ka K_{n}(ka)] 
+ xB_n [3K_{n}(qa) - 2\gamma a K_{n}(qa)] - R_n [2q^2 a^2 \omega f I_{n}(qa)] = 2q^2 a^2 \omega f K_{n}(qa), \] (2.6)
\[ A_n [2ka K_{n}(ka) - 3K_{n}(ka)] + B_n [2\gamma a K_{n}(qa) - (\gamma^2 a^2 + 2x - 1) K_{n}(qa)] = 0, \] (2.7)
\[ A_n [K_{n}(ka) - 2ka K_{n}(ka)] + 2x B_n K_{n}(qa) 
- R_n [I_{n}(qa) - 2qa I_{n}(qa)] = K_{n}(qa) - 2qa K_{n}(qa). \] (2.8)

We note in the above system that \( \lambda \) and \( \mu \) are the Lamé constants for the elastic medium,
\[ c^2 = (\lambda/\rho), \quad c_t^2 = [(\lambda + 2\mu)/\rho], \quad c_g^2 = (\mu/\rho), \]
\[ \Omega = (\sin \theta)^{-1} \frac{\partial}{\partial \theta} \left( \sin \frac{\partial}{\partial \theta} \right), \]
\[ \omega = 1 + \frac{\lambda}{2\mu} = (1-a)/(1+2a) = c_t^2/c_g^2, \]
\[ x = n(n+1), \quad f = \rho f / a^2 \rho. \]
and $c_1^2 = \alpha^2 c^2$, where $\alpha$ is a numerical constant, and a prime denotes differentiation with respect to the argument. We observe that for all values of Poisson's ratio, $\sigma$ such that $0 \leq \sigma \leq \frac{1}{2}$, $\omega \geq 1$. The equations (2.6) to (2.8) can now be solved simultaneously for the coefficients $A_n$, $B_n$ and $R_n$.

Define:

$$
\begin{align*}
    d_1 &= q^2 a^2 \omega f - (\omega k^2 a^2 + x + 1), \\
    d_2 &= 1 - q^2 a^2 \omega f \\
    d_3 &= \omega k^2 a^2 + x + 1 - 2ka \frac{K_v'(ka)}{K_v(ka)}, \\
    d_4 &= 3 - 2ya \frac{K_v'(ya)}{K_v(ya)}, \\
    d_5 &= 4xq^2 a^2 \omega f - 3x + 2xya \frac{K_v'(ya)}{K_v(ya)}. \\
    \beta &= \left[ \gamma^2 a^2 + 2x - 1 - 2ya \frac{K_v'(ya)}{K_v(ya)} \right]^{-1}, \\
    Q(\nu) &= \beta \left[ 2d_1 + 4d_2 k a \frac{K_v'(ka)}{K_v(ya)} + 4d_3 qa \frac{I_v'(qa)}{I_v(qa)} \right] + 2xd_4 qa \frac{I_v'(qa)}{I_v(qa)} + d_5, \\
    Q_2(\nu) &= \beta \left[ 2d_1 + 4d_2 k a \frac{K_v'(qa)}{K_v(qa)} + 4d_3 qa \frac{K_v'(qa)}{K_v(qa)} \right] + 2xd_4 qa \frac{K_v'(qa)}{K_v(qa)} + d_5, \\
    R &= Q_2/Q; \quad T_1 = \beta T_2; \quad T_2 = 2qa\omega f/Q.
\end{align*}
$$

Whence, we find that

$$
\begin{align*}
\Phi &= (4\pi R)^{-1} \exp(-qR) \left( \sum_{n=0}^{\infty} (2n+1) \frac{I_v(qr)}{I_v(qa)} K_v(qa) P_n(\cos \theta), \\
\Psi &= \frac{(br)^{-1}}{4\pi} \sum_{n=0}^{\infty} T_2(2n+1) \frac{I_v(qb)}{I_v(qa)} K_v(qa) P_n(\cos \theta), \\
\Pi &= \frac{(br)^{-1}}{4\pi} \sum_{n=0}^{\infty} T_2(2n+1) \frac{I_v(qb)}{I_v(qa)} K_v(qa) P_n(\cos \theta).
\end{align*}
$$

The second expression in the acoustic potential is the reflected wave function and the potentials $\Psi$ and $\Pi$ represent the wave functions transmitted into the elastic medium. The function $Q(\nu)$ may be termed the 'Generalized Stoneley Function'. Its zeros will give rise to Stoneley Waves along the fluid–solid interface.

3. Discussion of formal solutions

A. The case of the liquid infinite mantle and fluid space

Let us suppose that region II is composed of the same fluid material as region I. Mathematically, this is accomplished in the limiting case as the shear parameter, $\mu$, becomes zero. This further implies that both $\gamma$ and $\omega$ become infinite. At the same time, since the properties of both media are the same, we find that the parameter $f$ given by $f = \frac{c^2_{\rho I}}{c_1^2 \rho II}$ becomes unity and the wave numbers $k$ and $q$ become the same.
Subject to these requirements, we should expect the reflected acoustic and the transmitted shear potentials to vanish, and furthermore, the transmitted pressure potential should be the same as the incident pulse. This is easily shown to be indeed so.

Let $z$ be real and positive. As $z \to \infty$,

$$\frac{zK_{\nu}'(z)}{K_{\nu}(z)} \sim z; \quad \frac{zI_{\nu}'(z)}{I_{\nu}(z)} \sim z; \quad I_{\nu}(z) \sim (2\pi z)^{-\frac{1}{4}} e^z; \quad K_{\nu}(z) \sim (\pi/2z)^{\frac{1}{4}} e^{-z}. \quad (3.1)$$

Using these asymptotic representations, it follows easily that in the limit as both $\gamma_\alpha$ and $\omega$ become infinite and subject to the stated requirements on $f$, $k$, and $q$,

$$\Phi = (4\pi R)^{-1} \exp (-qR), \quad \Pi = 0,$$

$$\Psi = \frac{(br)^{-1} \infty}{4\pi} \sum_{n=0}^{\infty} (2n+1) I_{\nu}(kb) K_{\nu}(kr) P_n(\cos \theta),$$

$$= \Phi. \quad (3.2)$$

B. Asymptotic estimates for large order, $\nu$

(a) For large $\nu$, $R(\nu) > 0$,

$$I_{\nu}(z) \sim \frac{(z/2)^{\nu}}{\nu \Gamma(\nu)}; \quad zI_{\nu}'(z) \sim \frac{(z/2)^{\nu}}{\Gamma(\nu)}; \quad K_{\nu}(z) \sim \frac{1}{2} (z/2)^{-\nu} \Gamma(\nu); \quad zK_{\nu}'(z) \sim -\frac{1}{2} (z/2)^{-\nu} \nu \Gamma(\nu). \quad (3.3)$$

Using the above representations, we find that for $\nu$ large and real,

$$Q \sim -\nu^2 (8k^2 \omega + 1). \quad (3.4)$$

(b) Let $\nu = n + \rho$, where $n$ is an integer and $0 < \rho < 1$. Then for $z$ real and positive and $z \ll 1$,

$$\frac{zK_{\nu}'(z)}{K_{\nu}(z)} \sim -\nu; \quad \frac{zI_{\nu}'(z)}{I_{\nu}(z)} \sim \nu.$$

In a similar manner we find from these results that:

$$Q = -(2\nu + 3)^{-1} [\nu^2 (16k^2 \omega + 2) + \nu^2 (16k^2 \omega - 1) + 4\nu K_0 + K_1], \quad (3.5)$$

where

$$K_0 = k^2 a^2 [(\alpha^2 \omega f - \omega) + (\omega + \alpha^2 \omega f)(2\omega k^2 a^2 - 1)] - 1.$$ 

and

$$K_1 = 2(2\omega k^2 a^2 - 1)[\alpha^2 k^2 a^2 \omega f - (\omega k^2 a^2 + 1)].$$

Further examination of the function $Q$ shows that for $\nu > 0$, $\nu \alpha > \epsilon$, ($\epsilon > 0$) and for $\alpha \geq 1$, $Q(\nu)$ is always negative. Fig. 2 is a plot of $-Q(\nu)$ against wave number for the case $\alpha^2 = 2\omega = 4$. We observe that $-Q(\nu)$ increases monotonically with increasing $\nu$.

C. Convergence of series, and integral representations of Watson

Define

$$F_{fe} = \frac{(br)^{-1}}{4\pi} \sum_{n=0}^{\infty} 2\nu f_j(\nu) P_n(\cos \theta), \quad j = 1, 2, 3, \quad (3.6)$$
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where

\[ f_1 = R(v) I_p(qb) \frac{I_p(qr)}{I_p(qa)} K_p(qa), \]

\[ f_2 = T_1(v) \frac{I_p(qb)}{I_p(qa)} K_p(kr), \]

\[ f_3 = T_2(v) \frac{I_p(qb)}{I_p(qa)} K_p(\gamma r). \]

Then, \( F_1 \) represents the reflected function.

In another terminology the explosion or 'Cause' generates three effects—the reflected effect \( \Phi_e \), and the transmitted effects \( \Phi \) and \( \Pi \). Thus, in the light of the above notation,

\[ F_{1e} = \Phi_e; \quad F_{2e} = \Phi; \quad F_{3e} = \Pi. \]

In discussing the convergence of the series (3.6), for large order \( v \), we note that since

\[ |P_n(\cos \theta)| \leq 1 \text{ for } 0 < \theta < \pi, \]

it is sufficient to consider only the behaviour of the functions \( f_j(v) \).

Using the representations (3.3), we find that for large \(|v|\), \( \text{Re}(v) > 0 \),

\[ |2v f_1| \sim \exp \left[ -\text{Re}(v) \log (a^2/br) \right], \]

\[ |2v f_2| \sim (2|v|)^{-1} qa \omega f \exp \left[ -\text{Re}(v) \log (r/b) \right], \]

\[ |2v f_3| \sim \frac{1}{2} qa \omega f |v|^{-2} \exp \left[ -\text{Re}(v) \log (r/b) \right]. \]

\[ (3.7) \]

and

\[ \text{FIG. 2. Stoneley wave function vs transmitted P-wave number.} \]
Thus, $\Phi_e$ converges very slowly provided $b < r < a$. On the other hand, $\Psi$ and $\Pi$ converge comparatively rapidly for all $r > b$.

Because of the poor convergence of these series, particularly $\Phi_e$, a numerical solution would be inadequate. However, we may use the fact that there are no real zeros of $Q(\nu)$ to transform the series to integrals and then seek asymptotic estimates of such integrals.

From the Cauchy Residue Theorem,

$$F_{se} = \frac{(br)^{-\frac{1}{2}}}{4\pi i} \int_C z f_s(z) P_{z-\frac{1}{2}} [\cos (\pi - \theta)](\cos \pi z)^{-\frac{1}{2}} dz,$$  \hfill (3.8)

where $C$ is the contour surrounding the zeros of $\cos \pi z$ depicted in Fig. 3.

Now, for large $|z|$, $\text{Re}(z) > 0$, $\epsilon < \theta < \pi - \epsilon$, $\epsilon > 0$,

$$P_e(-\cos \theta) \sim (\frac{1}{4}\pi z \sin \theta)^{-\frac{1}{2}} \cos [z(\pi - \theta) + \pi/4].$$  \hfill (Laplace)  \hfill (3.9)

Thus, by equations (3.7) and (3.9), the integrands of (3.8) go to zero as $\text{Re}(z)$ tends to infinity. Furthermore, for large order, there are no poles of the integrands in regions $R_1$ and $R_2$, and the integrands go to zero on $C'$ and $C''$. In view of this, each
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integral over \( C \) is equivalent to its principal value (P.V.) over the imaginary axis:

\[
F_{se} \sim \text{P.V.} \frac{(br)^{-i}}{4\pi} \int_{\Gamma} zf_j(z) P_{s-1}[\cos (\pi - \theta)](\cos \pi z)^{-1} dz,
\]

(3.10)

where \( \Gamma \) is the entire imaginary axis. Our work from now on will be the geometric optical evaluation of the integrals (3.10) and their subsequent representation in space–time.

4. Representations for converging and diverging waves and wave circuit development

In the geometrical optical (large wave number and order) domain of the integrals (3.10), any ray emitted from the source gets partially reflected back into the fluid zone each time it strikes the elastic boundary. In the course of this reflection the ray travels around the sphere. This phenomenon is taken into account (3.4) by introducing the Heaviside representation for converging and diverging waves and developing the integrands into geometric series. The index of such a series would measure the number of internal reflections.

Let \( B_s(t) = \frac{i\pi}{2} e^{i\omega n t} H_s^{(1)}(-it), t \) real and positive. Then

\[
i\pi I_s(t) = B_s(t) - e^{i\omega n} K_s(t); \quad B_s(t) = B_{-s}(t).
\]

(4.1)

Using (4.1), provided \( |A| = \left| e^{i\omega n} \frac{K_s(qa)}{B_s(qa)} \right| < 1,

\[
\frac{K_s(qa)}{I_s(qa)} = \left. \frac{i\pi}{B_s(qa)} \sum_{m=1}^{M} \left[ \frac{K_s(qa)}{B_s(qa)} \right]^{m-1} e^{i\omega (m-1)\pi} + R_M. \right. \)

(4.2)

where \( R_M \) and \( R_M' \) denote the remainder after \( M \) terms of the respective series. Using the asymptotic representation (3.1) for \( qa \) sufficiently large,

\[
|A| \sim \exp \left[ -\pi \text{Im}(z) - 2qa \right].
\]

We shall be concerned with a term by term asymptotic representation of the series (4.2).

To describe the number of times a ray has travelled around the sphere, let consider the identity (3),

\[
P_{s-1}[\cos (\pi - \theta)](\cos \pi z)^{-1} = \tan \pi z \frac{P_{s-1}[\cos (\pi - \theta)]}{\cos \pi z} \frac{\cos(\pi - \theta)}{\sin \pi z}
\]

(4.3)

For \( N \gg 0, \)

\[
(\sin \pi z)^{-1} = \frac{e^{i(N+1)\pi z}}{\sin (N+1)\pi z} \sum_{i=0}^{N} e^{-i\pi z - 2i\pi z}.
\]

Whence,

\[
\frac{\cos(\pi - \theta)}{\sin \pi z} = \frac{e^{i(N+1)\pi z}}{2 \sin (N+1)\pi z} \sum_{i=0}^{N} [e^{-i(2\pi \theta) + \pi z} + e^{i(2\pi \theta) - \pi z}],
\]

and

\[
P_{s-1}[\cos (\pi - \theta)](\cos \pi z)^{-1} = \frac{e^{i(N+1)\pi z}}{2 \sin (N+1)\pi z} \sum_{l} P(l, z),
\]

(4.4)
where
\[ P(l, z) = \tan z \pi \frac{P_{2l-1} \cos(\pi - \theta)}{\cos z(\pi - \theta)} e^{-i(2l+1)\phi} \]
and $\phi$ is $\theta$ or $2\pi - \theta$ depending on whether we are considering clockwise or counterclockwise propagation. The summation index, $l$, is a measure of the number of wave circuits. Changing $I_z$ into the functions $B_z$ and $K_z$ (by virtue of equation 4.2) in all terms in the reflected acoustic potential except the coefficient $R(z)$ and then applying the result (4.4), we obtain

\[ \Phi_e \sim -i/4 \pi^2 \sum_{m=1}^{M} \left[ \sum_{l=0}^{N_{m,1}} \Phi_1 - \sum_{l=0}^{N_{m,2}} \Phi_2 - \sum_{l=0}^{N_{m,3}} \Phi_3 + \sum_{l=0}^{N_{m,4}} \Phi_4 \right]. \] (4.5)

In the above equation,

\[ \Phi_j = \text{P.V.} \int_{\Gamma} z C_m(z) \mu_j(z) P(l, z) dz, \quad j = 1, 2, 3, 4. \]

Thus

\[ \mu_1 = B_z(qb) B_z(qr) e^{i(m-1)\pi z} \frac{e^{i(N_{m,1}+1)nz}}{2i \sin (N_{m,1}+1)\pi z}, \]
\[ \mu_2 = B_z(qr) K_z(qb) e^{imnz} \frac{e^{i(N_{m,2}+1)nz}}{2i \sin (N_{m,2}+1)\pi z}, \]
\[ \mu_3 = B_z(qb) K_z(qr) e^{imnz} \frac{e^{i(N_{m,3}+1)nz}}{2i \sin (N_{m,3}+1)\pi z}, \]
\[ \mu_4 = K_z(qb) K_z(qr) e^{i(m+1)\pi z} \frac{e^{i(N_{m,4}+1)nz}}{2i \sin (N_{m,4}+1)\pi z}, \] (4.6)

and

\[ C_m(z) = R(z) \left[ \frac{K_z(qa)}{B_z(qa)} \right]^m. \]

For large wave number and order, the modified Bessel function $I_z$ appearing in the coefficient $R(z)$ can be replaced by the auxiliary function $B_z/\lambda z$.

Given a fixed number of reflections, $m$, a ray in travelling from the source to a prescribed observation station undergoes a certain maximum number of complete circuits around the sphere depending on the potential $\Phi_1$, $\Phi_2$, $\Phi_3$ or $\Phi_4$. The precise number becomes apparent once a geometrical optical representation of the potentials is obtained. We shall take up this matter fully in a later section. $N_{m,1}$, $N_{m,2}$, ..., $N_{m,4}$ are such maxima of wave circuits. Upon following each ray as part of it is transmitted into the elastic region, we find in a similar manner that

\[ \Psi_e \sim \frac{(br)^{-i}}{4\pi} \sum_{m=1}^{M} \left[ \sum_{l=0}^{N_{m,3}} \Psi_3 - \sum_{l=0}^{N_{m,4}} \Psi_4 \right], \]
\[ \Pi_e \sim \frac{(br)^{-i}}{4\pi} \sum_{m=1}^{M} \left[ \sum_{l=0}^{N_{m,3}} \Pi_3 - \sum_{l=0}^{N_{m,4}} \Pi_4 \right]. \] (4.7)

where

\[ \Psi_\alpha = \text{P.V.} \int_{\Gamma} z D_m(z) \rho_\alpha(z) P(l, z) dz, \]
\[ \Pi_\alpha = \text{P.V.} \int_{\Gamma} z E_m \lambda_\alpha(z) P(l, z) dz, \quad (\alpha = 3, 4). \]
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\[ \rho_3 = B_2(qb) K_z(kr) e^{i(m-1) \pi} \frac{e^{i(N_{m,3} + 1) \pi}}{2i \sin (N_{m,3} + 1) \pi}, \]
\[ \rho_4 = K_2(qb) K_z(kr) e^{i(N_{m,4} + 1) \pi} \frac{e^{i(N_{m,4} + 1) \pi}}{2i \sin (N_{m,4} + 1) \pi}, \]
\[ \lambda_3 = B_2(qb) K_z(\gamma r) e^{i(N_{m,3} + 1) \pi} \frac{e^{i(N_{m,3} + 1) \pi}}{2i \sin (N_{m,3} + 1) \pi}, \]
\[ \lambda_4 = K_2(qb) K_z(\gamma r) e^{i(N_{m,4} + 1) \pi} \frac{e^{i(N_{m,4} + 1) \pi}}{2i \sin (N_{m,4} + 1) \pi}, \]
\[ D_m(z) = T_1(z) \left[ B_2(qa) K_z(ka) \right]^{-1} \left[ \frac{K_2(qa)}{B_2(qa)} \right]^{m-1}, \]

and
\[ E_m(z) = T_2(z) \left[ B_2(qa) K_z(\gamma a) \right]^{-1} \left[ \frac{K_2(qa)}{B_2(qa)} \right]^{m-1}. \]

We again note that in arriving at the above results, we have employed the relationship (4.2) in all appropriate expressions in the transmitted P and S potentials excepting the coefficients \( T_1(z) \) and \( T_2(z) \). For large wave number and order, these coefficients can be approximated by the auxiliary function \( \frac{1}{i \pi} B_2 \).

5. Geometrical optical representation of potentials

Because of the complicated nature of the integrals (4.6), their exact evaluation for all wave numbers is a very difficult problem. However, physically meaningful results can be obtained by resorting to a geometrical optical approximation. Specifically, we shall use the Debye asymptotic representations (A1) to reduce the integrands to exponential orders.

A. The reflected function

Following (3), let us set
\[ z = -q \eta, \]
and express all wave numbers in (4.6) in terms of the acoustic wave number, \( q \), through the relations
\[ ka = q \alpha, \quad \gamma a = qa', \]
where \( a' = a(2\omega) / \alpha \) and \( \alpha = a / \omega \).

Then, by (5.1) an asymptotic representation for large argument, \( qr \), also implies one for large order, \( q \eta \), and the integrals (4.6) take the forms
\[ \Phi_j \sim P.V. \left[ -q^2 \int \eta C_m(-q \eta) \mu_j(-q \eta) P(l, -q \eta) d \eta \right]. \quad (j = 1, 2, 3, 4). \]

We may now use the Debye asymptotic representations of the modified Bessel and Hankel functions for large wave number and order (A1) together with the representations (A2) to reduce the potentials (5.3) to their exponential forms. In this regard, we note that for \( q \) large and positive, \( I_{q \eta}(qr) \) is asymptotically equivalent to \( (i \pi)^{-1} B_{q \eta}(qr) \). Since the function \( B_{q \eta}(l) \) is even in \( \eta \), this further implies that
\[ C_m(-q \eta) \sim C_m(q \eta). \]
Let \( \eta = iu \) \((u > 0)\). Then as \( q \to \infty, |u| < \min (b, \tilde{a}), \)

\[
\Phi \sim -q^t P.V. \int_{-a}^{a} A_\eta (\eta) \exp q F_\eta (\eta) d\eta,
\]

where \[
A_1 = -A, \\
A_2 = A_3 = iA, \\
A_4 = R,
\]

\[
A = -i\eta (-i)^m \left[ \frac{Q_0}{Q_{st}} \right] \pi^4 (2|\eta| \sin |\pi - \theta|)^{-4} (\eta^2 + b^2)^{-1} (\eta^2 + r^2)^{-1},
\]

\[
Q_0 = a^2 \omega f(2a^2 + a'^2)(\eta^2 + \tilde{a}^2)^{1/2} - (\eta^2 + \omega \tilde{a}^2)(2a^2 + a'^2)(\eta^2 + \tilde{a}^2)^{1/2}
\]

\[
+ 2\eta^2(\eta^2 + \tilde{a}^2)^{1/2} [(\eta^2 + a'^2)^{1/2} - a^2 \omega f],
\]

\[
Q_{st} = -2\eta^2(\eta^2 + \tilde{a}^2)^{1/2} [(\eta^2 + a'^2)^{1/2} (\eta^2 + \tilde{a}^2)^{1/2} + a^2 \omega f]
\]

\[
+ [(\eta^2 + \omega \tilde{a}^2)(2a^2 + a'^2)(\eta^2 + a'^2)^{1/2} + a^2 \omega f(2a^2 + a'^2)(\eta^2 + a'^2)^{1/2}],
\]

\[
F_1 = -2m f(a) + f(b) + f(r) - i\eta [(m - 1) \pi - (2\pi + \phi)],
\]

\[
F_2 = -2m f(a) - f(b) + f(r) - i\eta [m\pi - (2\pi + \phi)],
\]

\[
F_3 = -2m f(a) + f(b) + f(r) - i\eta [m\pi - (2\pi + \phi)],
\]

\[
F_4 = -2m f(a) - f(b) - d(r) - i\eta [(m + 1) \pi - (2\pi + \phi)],
\]

\[
f(r) = (\eta^2 + r^2)^{1/2} - \eta \arcsin(\eta/r),
\]

and \( \tilde{a} = \min (b, \tilde{a}) \).

\( Q_\eta (\eta) \) now becomes the generalized ‘Stoneley Function’. In addition to the stipulations on equation (5.4), we must add the proviso 

\[
\epsilon < \theta < 2\pi; \quad \theta \neq \pi, \epsilon > 0.
\]

B. Transmitted functions

(a) The transmitted pressure function

In a similar manner, by replacing \( z \) by \(-kz\) and the wave numbers \( q^2 \) and \( \gamma^2 \) in the expression for the transmitted pressure function by \( a^2 k^2 \) and \( 2\omega k^2 \) respectively, we find that

\[
\Psi_\alpha \sim P.V. \left[ -k^2 \int_{\Gamma} zD_m(-kz) \rho_a(-kz) P(l, -kz) dz \right].
\]

For \( |z| < \min (b, a_3), \text{Im} (z) > 0, \epsilon < \theta < 2\pi - \epsilon, \theta \neq \pi \) and \( \epsilon > 0 \), the asymptotic representations (A1) and (A2) yield

\[
\Psi_\alpha \sim k^t P.V. \int_{-i\beta}^{i\beta} B(z) \exp kG_\alpha (z) dz,
\]

\[
\Psi_\gamma \sim -ik^t P.V. \int_{-i\beta}^{i\beta} B(z) \exp kG_\alpha (z) dz,
\]

where

\[
B(z) = \frac{2za_1^2 \omega f(2a^2 + a'^2)(-i)^m}{Q^2_\alpha (z)} \left[ \frac{(x^2 + a_1^2)(x^2 + a'^2)}{(x^2 + a_1^2)(x^2 + b^2)} \right] (2\pi |z| \sin |\pi - \theta|)^{-4},
\]

\[
\Psi_\alpha \sim P.V. \int_{-i\beta}^{i\beta} B(z) \exp kG_\alpha (z) dz,
\]

\[
\Psi_\gamma \sim -ik^t P.V. \int_{-i\beta}^{i\beta} B(z) \exp kG_\alpha (z) dz,
\]

\[
\Psi_\alpha \sim k^t P.V. \int_{-i\beta}^{i\beta} B(z) \exp kG_\alpha (z) dz,
\]

\[
\Psi_\gamma \sim -ik^t P.V. \int_{-i\beta}^{i\beta} B(z) \exp kG_\alpha (z) dz,
\]

\[
\Psi_\alpha \sim P.V. \int_{-i\beta}^{i\beta} B(z) \exp kG_\alpha (z) dz,
\]

\[
\Psi_\gamma \sim -ik^t P.V. \int_{-i\beta}^{i\beta} B(z) \exp kG_\alpha (z) dz,
\]

\[
\Psi_\alpha \sim k^t P.V. \int_{-i\beta}^{i\beta} B(z) \exp kG_\alpha (z) dz,
\]

\[
\Psi_\gamma \sim -ik^t P.V. \int_{-i\beta}^{i\beta} B(z) \exp kG_\alpha (z) dz,
\]
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\[ Q^0_n(z) = -2z^2(z^2 + a^2) \left\{(z^2 + a_1^2)(z^2 + a_0^2) + a_1^2 \omega f \right\} + (2z^2 + a_0^2)(z^2 + a_1^2)(z^2 + a_2^2) + a_1^2 \omega f(z^2 + a_2^2) \],

\[ G_3(z) = -2(m-1)f(a_1) + f(a) - f(r) - i\xi[(m-1)\pi - (2m + \phi)], \]

\[ G_4(z) = -2(m-1)f(a_1) - f(a) - f(b_1) - f(r) - i\xi[m\pi - (2m + \phi)], \]

\[ a_0 = a(2\omega)^{1}, \quad a_1 = a\bar{a}, \quad \text{and} \quad \beta = \min(b, a_1). \]

Again, we have used the fact that as \( k \to \infty \), the function \( D_n(kz) \) is even in \( z \).

\( b \) The transmitted shear function

For this case we replace \( z \) by \(-\gamma \xi\) and proceed as previously to obtain

\[ \Pi_3 \sim -i\gamma^{-1} P.V. \int_{-i\delta}^{i\delta} C(\xi) \exp \gamma H_3(\xi) d\xi, \tag{5.7} \]

\[ \Pi_4 \sim -i\gamma^{-1} P.V. \int_{-i\delta}^{i\delta} C(\xi) \exp \gamma H_4(\xi) d\xi, \]

provided

\[ \xi < \min(b_1, a_2, a/2), \quad \text{Im} \, (\xi) > 0, \quad \epsilon \leq \theta \leq 2\pi - \epsilon, \]

\[ \theta \neq \pi, \quad \epsilon > 0, \quad \text{and} \quad \delta = \min(b_1, a_2, a/2). \]

In the equation (5.7),

\[ C(\xi) = 4(-i\xi)(-i)^m a^2 \omega f(\xi^2 + a_2)^2 \left\{(\xi^2 + a_2^2)(\xi^2 + a_2) \right\} \left\{(\xi^2 + a_2^2)(\xi^2 + a_2^2) \right\} (2\pi |\xi| \sin |\pi - \theta|)^{-1}, \]

\[ Q_{n}^*(\xi) = -2\xi^2(\xi^2 + a_2^2)^2 \left\{(\xi^2 + a_2^2)(\xi^2 + a_2) \right\} a_1 \omega f \]

\[ + (2\xi^2 + a_2)[(\xi^2 + a_2^2)(\xi^2 + a_2) + a_1^2 \omega f(\xi^2 + a_2^2)] \]

\[ H_3 = -2(m-1)f(a_1) - f(a_2) - f(r) - i\xi[(m-1)\pi - (2m + \phi)], \]

\[ H_4 = -2(m-1) - f(a_1) - f(a_2) - f(b_1) - f(r) - i\xi[m\pi - (2m + \phi)], \]

\[ a_2 = a(2\omega)^{-1} \quad \text{and} \quad a_3 = a_2. \]

The replacement of \( z \) by \(-q\eta\) in the reflection function and by \(-kz\) and \(-\gamma \xi\) in the transmitted pressure and shear functions respectively, implies that

\[ \eta = z/a = \xi(2\omega)^{1}/a. \]

Thus

\[ Q_n(\eta) \to Q_0^*(z) \to Q_{n}^*(\xi). \]

Furthermore, there are no zeros of \( Q_n(\eta) \) on the pure imaginary axis in the domain of validity of the Debye asymptotic representations, \(|\eta| < \min(b, d)|. \) This is the domain of pressure and shear waves.

6. Saddlepoints

The integrals (5.4), (5.6) and (5.7) yield directly to evaluation by the classical method of Laplace (2). For this matter it will be necessary to discuss in some detail the existence of saddlepoints for the phase functions \( F_\mu, G_\alpha \) and \( H_\alpha \).

Let \( d \) be the saddlepoint associated with the acoustic potentials, \( d_p \) the saddlepoint associated with the pressure potentials, and \( d_0 \) that associated with the shear potentials.
Consider the typical potential, $\phi_2$:

$$F_2 = -2mf(a) + f(r) - f(b) - i\eta(2l\eta + \phi - m\pi),$$

$$F_2'(\eta) = 2m \text{arcsh}(\eta/a) - \text{arcsh}(\eta/r) + \text{arcsh}(\eta/r) + i(2l\pi + \phi - m\pi).$$

(6.1)

The saddlepoint, $d$, is the solution of the equation

$$F_2'(\eta) = 0.$$

In particular, for the case of one internal reflection ($m = 1, l = 0$) let

$$W(\eta) = F_2'(\eta) = 2 \text{arcsh}(\eta/a) - \text{arcsh}(\eta/r) + \text{arcsh}(\eta/b) + i(\phi - \pi),$$

where

$$\text{arcsh}(\eta/r) = \log[\eta/r + (\eta^2/r^2 + 1)^{1/2}].$$

There are two branches from each of the functions $(\eta^2/a^2 + 1)^{1/2}$, $(\eta^2/r^2 + 1)^{1/2}$ and $(\eta^2/b^2 + 1)^{1/2}$, thus contributing to eight Riemann sheets. Let us consider the top sheet (the physical sheet) and make branch cuts from $-ib$ to $-i\infty$ and from $+ib$ to $+i\infty$. Then $W(\eta)$ becomes single-valued in the region contained by the contour $R$.

Fig. 4. Evaluation of the zeros of $w(\eta)$ by the principle of argument.
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It follows from analytic continuation from the imaginary axis that for large $|\eta|$,\[
(\eta^2/r^2+1)^{1/2} \sim \eta/r \quad \text{on the right half plane (r.h.p.)}
\]
\[
\sim -\eta/r \quad \text{on the l.h.p.}
\]

Similarly,\[
\operatorname{arcsh}(\eta/r) \sim \log(-r/2\eta) \quad \text{on the l.h.p.}
\]
\[
\sim \log(2\eta/r) \quad \text{on the r.h.p.}
\]

Whence\[
W(\eta) \sim \zeta + i\phi \quad \text{on the r.h.p.}
\]
\[
\sim -\zeta + i\phi \quad \text{on the l.h.p.}
\]

where $\zeta = \log(4u^2 r/a^2 b)$, $\eta = iu$.

Fig. 5a. $D^-A^-$ and $D^-A^+$ rays for $m = 1$. 

-2 arc sin $\frac{d}{a} + \text{arc sin } \frac{d}{b} = \theta$

-2 arc sin $\frac{d}{a} + \text{arc sin } \frac{d}{b} = 2\pi - \theta$

-2 arc sin $\frac{d}{a} + \text{arc sin } \frac{d}{b} = \theta$

-2 arc sin $\frac{d}{a} + \text{arc sin } \frac{d}{b} = 2\pi - \theta$
We observe that
\[ W(iu) = i[P(u) - (\pi - \phi)], \]
and
\[ W(-iu) = -i[P(u) + (\pi - \phi)], \]
where
\[ P(u) = 2 \arcsin u/a - \arcsin u/r + \arcsin u/b. \]
For \(0 < u \leq b\), \(b < r \leq a\), and \(0 < \arcsin u/r \leq \pi/2\),
\[ P > 0. \]
Therefore \(W(-iu) < 0\) if \(\pi > \phi\).

We may now determine the number of roots (if any) of the equation \(W(\eta) = 0\)
on the top Riemann sheet by the use of the Principle of the Argument. We find that
there is one encirclement of the origin in the \(W\)-plane provided
\[ P(b) - (\pi - \phi) > 0 \text{ and } \pi - \phi > 0. \tag{6.5} \]
This root is positive pure imaginary since, subject to (6.5),
\[ W(ib) = i[P(b) - (\pi - \phi)] > 0, \]
\[ W(i0) = -i(\pi - \phi) < 0, \]

\[ \text{FIG. 5b. } D^- A^+ \text{ and } D^+ A^+ \text{ rays for } m = 1. \]
and for \( u > 0 \), \( W(=iu) < 0 \).

The condition (6.5) implies that the observer is in the light zone (3). A similar analysis holds for the \( P \) and \( S \) wave potentials.

7. Evaluation of wave potentials by Laplace’s method

We can now apply Laplace’s method to the potentials (5.4) to obtain

\[
\Phi_j \sim -A_j (id) [\pi/2 |F_j''(id)|]^\frac{1}{2} \exp(-lg_j), \quad j = 1, \ldots, 4, \tag{7.1}
\]

where

\[
\begin{align*}
g_1 &= 2m(a^2 - d^2)^{\frac{1}{4}} - (r^2 - d^2)^{\frac{1}{4}} - (b^2 - d^2)^{\frac{1}{4}}, \\
g_2 &= 2m(a^2 - d^2)^{\frac{1}{4}} - (r^2 - d^2)^{\frac{1}{4}} + (b^2 - d^2)^{\frac{1}{4}}, \\
g_3 &= 2m(a^2 - d^2)^{\frac{1}{4}} + (r^2 - d^2)^{\frac{1}{4}} - (b^2 - d^2)^{\frac{1}{4}}, \\
g_4 &= 2m(a^2 - d^2)^{\frac{1}{4}} + (r^2 - d^2)^{\frac{1}{4}} + (b^2 - d^2)^{\frac{1}{4}}, \\
F_1'' &= 2m(a^2 - d^2)^{-\frac{1}{4}} - (r^2 - d^2)^{-\frac{1}{4}} - (b^2 - d^2)^{-\frac{1}{4}}, \\
F_2'' &= 2m(a^2 - d^2)^{-\frac{1}{4}} - (r^2 - d^2)^{-\frac{1}{4}} + (b^2 - d^2)^{-\frac{1}{4}}, \\
F_3'' &= 2m(a^2 - d^2)^{-\frac{1}{4}} + (r^2 - d^2)^{-\frac{1}{4}} - (b^2 - d^2)^{-\frac{1}{4}}, \\
F_4'' &= 2m(a^2 - d^2)^{-\frac{1}{4}} + (r^2 - d^2)^{-\frac{1}{4}} + (b^2 - d^2)^{-\frac{1}{4}}.
\end{align*}
\]

The saddlepoints ‘\( d \)’ are determined from the equations:

for \( \Phi_1 \):

\[
2m \arcsin (d/a) - \arcsin (d/r) - \arcsin (d/b) + [2l\pi + \phi - (m - 1)\pi] = 0
\]
\[
(l = 0, 1, \ldots, N_{m,1}),
\]

for \( \Phi_2 \):

\[
2m \arcsin (d/a) - \arcsin (d/r) + \arcsin (d/b) + [2l\pi + \phi - m\pi] = 0
\]
\[
(l = 0, 1, \ldots, N_{m,2}),
\]

for \( \Phi_3 \):

\[
2m \arcsin (d/a) + \arcsin (d/r) - \arcsin (d/b) + [2l\pi + \phi - m\pi] = 0
\]
\[
(l = 0, 1, \ldots, N_{m,3})
\]

and for \( \Phi_4 \):

\[
2m \arcsin (d/a) + \arcsin (d/r) + \arcsin (d/b) + [2l\pi + \phi - (m + 1)\pi] = 0
\]
\[
(l = 0, 1, \ldots, N_{m,4}).
\]

A. Geometrical optical interpretation of the wavefronts

From the laws of geometrical optics, any ray emanating from the source remains tangent to the ‘circle’ radius \( d \), before and after incidence on the boundary \( r = a \). Such a ray may depart before or after tangency to the same circle depending on the position of the observer. Following J. Etienne (3), we shall say that a ray which emanates from the source before tangency departs positively, \( (D^+) \). Conversely, a ray emanating after tangency departs negatively, \( (D^-) \).

In a similar manner, a ray which arrives at the observer before tangency arrives negatively, \( (A^-) \), and one which arrives after tangency arrives positively, \( (A^+) \). This
gives rise to four fundamental possibilities, \( D^-A^- \), \( D^+A^- \), \( D^-A^+ \) and \( D^+A^+ \), which we associate with \( \Phi_t, \Phi_2, \Phi_3 \) and \( \Phi_4 \), respectively. The geometry of clockwise and counterclockwise propagating waves (for the case of one internal reflection) is depicted in Figs. 5a–c. The totality of clockwise and counterclockwise propagating waves in each of the four fundamental forms defines a unique light zone. For example, those for the case \( m = 1 \) (\( l = 0 \)) are given in (3).

B. Evaluation of the transmitted waves

Just as in the case of the reflected function, we may evaluate the transmitted waves by the method of Laplace.

\[
\begin{align*}
\Psi_3 & \sim B(id_p) \left[ \frac{\pi}{2 |G'''(id_p)|} \right]^i \exp -kf_3, \\
\Psi_4 & \sim -iB(id_p) \left[ \frac{\pi}{2 |G''(id_p)|} \right]^i \exp -kf_4,
\end{align*}
\]

(7.3)

where

\[
f_3 = 2(m-1)(a^2 - d^2)^i + (a^2 - d^2)^i - (a^2 - d^2)^i - (b^2 - d^2)^i + (r^2 - d^2)^i,
\]

\[
f_4 = \frac{2(m-1)}{2 |G''(id_p)|} \left[ \frac{\pi}{2 |G'''(id_p)|} \right]^i \exp -kf_4,
\]

\[
\Psi_3 = B(id_p) \left[ \frac{\pi}{2 |G'''(id_p)|} \right]^i \exp -kf_3,
\]

\[
\Psi_4 = -iB(id_p) \left[ \frac{\pi}{2 |G''(id_p)|} \right]^i \exp -kf_4,
\]

Fig. 5c. Light zones for \( m = 1 \).
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\( f_4 = 2(m-1)(a_1^2 - d_p^2)^\frac{1}{4} + (a_1^2 - d_p^2)^\frac{1}{4} - (a^2 - d_p^2)^\frac{1}{4} + (b_1^2 - d_p^2)^\frac{1}{4} + (r^2 - d_p^2)^\frac{1}{4}, \)

\( G_3'' = 2(m-1)(a_1^2 - d_p^2)^{-\frac{1}{4}} + (a_1^2 - d_p^2)^{-\frac{1}{4}} - (a^2 - d_p^2)^{-\frac{1}{4}} - (b_1^2 - d_p^2)^{-\frac{1}{4}} + (r^2 - d_p^2)^{-\frac{1}{4}}, \)

and

\( G_4'' = 2(m-1)(a_1^2 - d_p^2)^{-\frac{1}{4}} + (a_1^2 - d_p^2)^{-\frac{1}{4}} - (a^2 - d_p^2)^{-\frac{1}{4}} + (b_1^2 - d_p^2)^{-\frac{1}{4}} + (r^2 - d_p^2)^{-\frac{1}{4}}. \)

The saddlepoints are determined from

\[
\begin{align*}
\Psi_3: & \quad 2(m-1) \arcsin \left( \frac{d_p}{a_1} \right) + \arcsin \left( \frac{d_p}{a_1} \right) - \arcsin \left( \frac{d_p}{a_1} \right) - \arcsin \left( \frac{d_p}{a_1} \right) \\
& \quad + \arcsin \left( \frac{d_p}{a} \right) + [2\pi + \phi - (m-1)\pi] = 0, \quad (l = 0, 1, \ldots, N_{m,3}) \\
\Psi_4: & \quad 2(m-1) \arcsin \left( \frac{d_p}{a_1} \right) + \arcsin \left( \frac{d_p}{a_1} \right) - \arcsin \left( \frac{d_p}{a_1} \right) + \arcsin \left( \frac{d_p}{a_1} \right) \\
& \quad + \arcsin \left( \frac{d_p}{a} \right) + (2\pi + \phi - m\pi) = 0. \quad (l = 0, 1, \ldots, N_{m,4}).
\end{align*}
\]

For the transmitted shear wave,

\[
\begin{align*}
\Pi_3 & \sim -i\gamma^{-1} C(id_0) \left[ \frac{\pi}{2 |H_3''(id_0)|} \right] \exp - \gamma h_3, \\
\Pi_4 & \sim -\gamma^{-1} C(id_0) \left[ \frac{\pi}{2 |H_4''(id_0)|} \right] \exp - \gamma h_4.
\end{align*}
\]

Fig. 6a. Transmitted \( D^- \cdot P \cdot \) ray for \( m = 1. \)
For $I = 1$, $2(m-1)$ \[\frac{\sin d_\rho}{b_1} - \frac{\sin d_\rho}{a_1} + \frac{\sin d_\rho}{a} - \frac{\sin d_\rho}{r} = 2\pi - \theta\]

Fig. 6b. Transmitted $D^-' P'$ ray for $m = 1$.

where

\[h_3 = 2(m-1)(a_s^2 - d_0^2) + (a_s^2 - d_0^2) - (a^2 - d_0^2) + (r^2 - d_0^2),\]

\[h_4 = 2(m-1)(a_s^2 - d_0^2) + (a_s^2 - d_0^2) - (a^2 - d_0^2) + (b_s^2 - d_0^2) + (r^2 - d_0^2),\]

\[H_3'' = 2(m-1)(a_s^2 - d_0^2)^4 + (a_s^2 - d_0^2)^4 - (a^2 - d_0^2)^4 + (b_s^2 - d_0^2)^4 + (r^2 - d_0^2)^4,\]

and

\[H_4'' = 2(m-1)(a_s^2 - d_0^2)^4 + (a_s^2 - d_0^2)^4 - (a^2 - d_0^2)^4 + (b_s^2 - d_0^2)^4 + (r^2 - d_0^2)^4.\]

The saddle points corresponding to the transmitted shear potentials are determined from the following equations.

For $\Pi_4$:

\[
2(m-1) \sin (d_\rho/a_s) + \sin (d_\rho/a) - \sin (d_\rho/a) - \sin (d_\rho/b_s) + \sin (d_\rho/r) + [2\pi + \phi - (m-1)\pi] = 0, \quad (l = 0, 1, \ldots, N_{m, 3})
\]

and for $\Pi_4$,

\[
2(m-1) \sin (d_\rho/a_s) + \sin (d_\rho/a) - \sin (d_\rho/a) + \sin (d_\rho/b_s) + \sin (d_\rho/r) + [2\pi + \phi - m\pi] = 0, \quad (l = 0, 1, \ldots, N_{m, 4}).
\]

Just as we have demonstrated in the case of internal reflection in Section 7 (see Fig. 5), the saddlepoint equations for the transmitted waves are consistent with the laws of reflection and refraction. We shall consider a typical refracted $P$ ray.
Let $\iota =$ angle of incidence, $\nu =$ angle of refraction. Then, by Snell’s law,

$$\frac{\sin \nu}{\sin \iota} = \frac{c_1}{c} = \alpha,$$

since from equation (2.8), $c_1 = \alpha c$. We find that the parameter $\alpha$ is the refraction index for the transmitted $P$ ray. The geometry of refracted $P$ rays is shown in Figs. 6a–d. A ray emitted from the source $P(b, 0)$ remains tangent to the 'circle', radius $d = d_p/\alpha$ before refraction. The refracted ray is tangent to the 'circle' of radius, $d_p$. The saddle-point equations for clockwise ($\theta$) and counterclockwise ($2\pi - \theta$) propagation are precisely the mathematical expression of these conditions. Unlike the phenomenon of reflection, we can no longer talk of positive or negative arrival, but rays may depart 'positively' or 'negatively' from $P(b, 0)$.

The ray path $P IQ$ gives automatically the time of arrival at the observation station, $Q$, measured from the instant the disturbance is generated at $P$. For example, let $\tau$ denote such a period of time for the potential $\Psi_1$ for $m = 1$.

$$\tau = f_8/c_1 \quad (m = 1, l = 0)$$

$$= \frac{1}{c_1} [\sqrt{(a_1^2 - d_p^2) - \sqrt{(b_1^2 - d_p^2)}}] + \frac{1}{c_1} [\sqrt{(r^2 - d_p^2) - \sqrt{(a^2 - d_p^2)}}]$$

$$= \frac{\alpha}{c_1} [\sqrt{(a^2 - d^2) - \sqrt{(b^2 - d^2)}}] + \frac{\alpha}{c_1} [\sqrt{(r^2 - d_p^2) - \sqrt{(a^2 - d_p^2)}}]. \quad (7.7)$$

**Fig. 6c.** Transmitted $D^+ ' P'$ ray for $m = 1.$
In arriving at the last expression, we have used the facts that \( d_p = ad \) and \( a_1 = a a_1 \).

Now, since \( c_1 = ac \),

\[ \tau = \tau_1 + \tau_2, \]

where

\[ \tau_1 = \frac{1}{c} \left[ \sqrt{(a^2 - d^2)} - \sqrt{(b^2 - d^2)} \right] \]

and

\[ \tau_2 = \frac{1}{c} \left[ \sqrt{(r^2 - d_p^2)} - \sqrt{(a^2 - d_p^2)} \right]. \]

\( \tau_1 \) represents the time taken for the disturbance to travel through the fluid medium from \( P \) to \( I \) and \( \tau_2 \), the time from \( I \) to \( Q \) in the elastic medium. (See Fig. 6a.) For typical numerical values of the saddlepoints, we refer the reader to the appendix (A3).

So far, the analysis has been based on the requirement that the imaginary parts of the complex variables \( \eta, z \) and \( \xi \) be positive. It can be shown that there are no saddlepoints for negative imaginary parts of these variables. Furthermore, if we replaced \( z \) by \( q_\eta \) as opposed to equation (5.1), the position of the saddlepoints is reversed and the same results hold true.

It is important to emphasize that of all the various regions of uniformity of the Debye asymptotic representations of the modified Bessel and Hankel functions, our analysis for \( P \) and \( S \) waves is based on only one. Considering the function \( B_\eta(qa) = B_{qa}(qa) \), this is the region \( |q_\eta| < qa \), \( \eta \) being pure imaginary. Thus this region excludes the poles of \( B_{qa}(qa) \) which lie on the imaginary \( \eta \)-axis for \( |q_\eta| \sim qa \).
The latter portion has been the subject of a detailed analysis by Jeffreys & Lapwood (4). Moreover, unlike (4), the direct geometrical optical solution obtained here is valid for Im(q) = 0. Thus, elementary sinusoidal waves are not considered.

An examination of equations (4.5) and (5.4) reveals that $\Phi_3$ and $\Phi_4$ have a different behaviour from $\Phi_1$ and $\Phi_2$. In particular for the case of one internal reflection ($m = 1$), $\Phi_1$ and $\Phi_2$ are real whereas $\Phi_3$ and $\Phi_4$ have a multiplying factor of $i$, indicating a phase change of $\pi/2$ if elementary sinusoidal waves were considered (4). For this case $\Phi_2$ and $\Phi_3$ would yield PP waves (or ‘Allied Functions’ associated with the $\delta$-function in time). On the other hand, for $\Phi_1$ (pP wave) and $\Phi_4$ (PP wave) the time function is the same as for the incident pulse. With this limitation in the analysis and keeping in mind our concern with pP and PP waves, we now give the time forms of the reflected pulses.

8. Space-time representation of P and S waves

We recall from equation (2.3) that the ‘cause’ is the retarded potential,

$$(4\pi R)^{-1} \delta(t - R/c) ; \quad R = (r^2 + b^2 - 2br \cos \theta)^{1/2}.$$ 

Subject to this cause, we will now examine the effects in space-time.

A. For the reflected potential

$$\Phi_j(r, \theta, t) \sim - A_j(id) \left[ \frac{\pi}{2|F_j''(id)|} \right] \delta(t - g_j/c). \quad (8.1)$$

This is a pulse with spherical wavefronts, $g_j = ct$.

B. Transmitted potentials

Unlike the reflected potential which is a delta function, the transmitted P and S potentials are Heaviside functions.

$$\Psi_3 \sim B(id) \left[ \frac{\pi}{2|G_3''(id)|} \right] \delta(t - f_3/c),$$

$$\Psi_4 \sim -iB(id) \left[ \frac{\pi}{2|G_4''(id)|} \right] \delta(t - f_4/c),$$

$$\Pi_3 \sim -ic_2 C(id) \left[ \frac{\pi}{2|H_3''(id)|} \right] \delta(t - h_3/c),$$

$$\Pi_4 \sim -c_2 C(id) \left[ \frac{\pi}{2|H_4''(id)|} \right] \delta(t - h_4/c). \quad (8.2)$$

We observe that the P and S waves transmitted into the elastic region are very severely ‘damped’ in comparison with the incident pulse in the acoustic region (equation 2.3). This phenomenon will be investigated further in the next section.

9. Transmission coefficients

Because of the ‘damping’ of the incident pulse upon transmission into the elastic medium, we can conveniently introduce the concept of ‘transmission factors’ across the two media. Let us denote such a factor by $\delta$ and define $\delta$ as ‘the ratio of the modulus of the amplitude of the incident P pulse at the interface, $r = a$ to the modulus of the amplitude of the transmitted P wave for all $r$’.
Then with \( \Psi_s \) and \( \Psi_4 \) we associate the two factors \( \delta_s \) and \( \delta_4 \) respectively. Define

\[
\Delta = \left| \frac{Q_p(i\eta_p)}{a_1^2(\omega f)dp(a_0^2 - 2d_0^2)} \right| \left\{ \frac{(r^2 - d_p^2)(b_t^2 - d_p^2)}{(a_1^2 - d_p^2)(a_2^2 - d_p^2)} \right\} \left( d_p \sin[\pi - \theta] \right)^{\frac{1}{2}},
\]

\[
|\Psi_s| = \frac{(br)^{-\frac{1}{2}}}{4\pi\Delta} \left| G_s''(i\eta_p) \right|^{-\frac{1}{4}} = \frac{(br)^{-\frac{1}{4}}}{4\pi} |\Psi_s|, \tag{9.1}
\]

\[
|\Psi_4| = \frac{(br)^{-\frac{1}{4}}}{4\pi\Delta} \left| G_4''(i\eta_p) \right|^{-\frac{1}{4}} = \frac{(br)^{-\frac{1}{4}}}{4\pi} |\Psi_4|.\]

Then, following our previous definition,

\[
\delta_s = (4\pi R_a |\Psi_s|)^{-1} = \Delta \left| G_s''(i\eta_p) \right|^{-\frac{1}{4}} (br)^{\frac{1}{4}} R_a^{-1},
\]

\[
\delta_4 = (4\pi R_a |\Psi_4|)^{-1} = \Delta \left| G_4''(i\eta_p) \right|^{-\frac{1}{4}} (br)^{\frac{1}{4}} R_a^{-1}, \tag{9.2}
\]

where \( R_a = (a^2 + b^2 - 2ab \cos \theta)^{1/4} \).

Thus, for each observer in the light zone (in the elastic region) we can calculate a transmission factor which is a measure of how damped the observed amplitude of the transmitted \( P \)-wave is in comparison with that of the original incident pulse.

In Figs. 7a and b, the scaled transmission factor, \( 2a \delta_s \), is plotted against \( r/a \) for a Poisson material and for \( a^2 = 3 \); \( r \) is the radial coordinate of an observer in the elastic zone, and the angular coordinate \( \theta \) is kept fixed at \( \pi/12 \). The ratio of the density of the acoustic medium to the elastic medium, \( \rho_1/\rho_2 \), is denoted by \( \rho \). Since the reciprocal, \( 1/2a \delta_s \), is a measure of the damping of the first transmitted pressure potential, we find that damping increases with decreasing \( \rho \). Furthermore, for a fixed value of \( \rho \), damping also increases with increasing \( r/a \) ratio. This is as it should be expected, since the transmitted potentials tend to zero as \( r \) tends to infinity. The factor \( 2a \delta_s \) behaves in a similar manner.
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\[ \frac{a}{b} = 50 \]

\[ \rho = \frac{\rho_{+}}{\rho_{\infty}} = \rho \]

Thus, the decaying of the amplitudes of transmitted pressure potentials in the elastic region is governed by the density ratio \( \rho \), and by the radial location of the observer with respect to the core radius, \( r/a \).

10. General remarks

In problems of wave propagation in closed regions, additional complications arise because of the curvature of the bounding surfaces. This is evident in this investigation. However, the results, though by no means complete, are general and may be applied to the transient behaviour of seismic waves generated by earthquakes or underground explosions. It is interesting to observe that fundamental as the approach is, the results are in agreement with the predictions of geometrical optics.

Appendices

(A1) Let \( \eta = iu, u > 0 \). [See (1), (3), (7).]

Then for \( k \) real, large and positive, and \( u < r \)

\[ I_{\kappa}(kr) \sim (2\pi k)^{-1/4}(\eta^2 + r^2)^{-1/4} \exp kf(r), \]

\[ B_{\kappa}(kr) \sim i\eta I_{\kappa}(kr), \]

\[ K_{\kappa}(ka) \sim (\pi/2)^1(\eta^2 + r^2)^{-1/4} \exp -kf(r), \]

\[ krI_{\kappa}(kr) \sim k(\eta^2 + r^2)^{1/4} I_{\kappa}(kr), \]

\[ krB_{\kappa}(kr) \sim k(\eta^2 + r^2)^{1/4} B_{\kappa}(kr), \]
Saddlepoints for the generalized problem of Lamb

A programme for the computation of all saddlepoints arising in this investigation has been written by the Geophysics Division of the Illinois Institute of Technology Research Institute, and could be made available upon request. All requests should be addressed to Mr D. A. Anderson.*

The following is a short table of saddlepoints for some positions of observer and source. All results are based on one internal reflection \((m = 1)\).

I. \textit{Transmitted P and S waves}

Let \(d_0 = a y\), \(m = 1\).

Then the saddlepoint equations (7.4) become:

for \(\Psi_3\):
\[
\begin{align*}
\arcsin (y/a) - \arcsin y - \arcsin (ya/\alpha b) & \arcsin (ya/r') + \theta_1 = 0; \\
\end{align*}
\]

for \(\Psi_4\):
\[
\begin{align*}
\arcsin (y/a) - \arcsin y + \arcsin (ya/\alpha b) + \arcsin (ya/r') - (\pi - \theta_2) = 0,
\end{align*}
\]

where \((r', \theta_1)\) and \((r', \theta_2)\) are observation stations in the elastic region.

Each solution in \(y\) determines the saddlepoint corresponding to the appropriate potential.

Similarly, from equation (7.6) we find that

for \(\Pi_3\):
\[
\begin{align*}
\arcsin \frac{\sqrt{2\omega}}{\alpha} - \arcsin y - \arcsin \frac{\sqrt{2\omega}}{\alpha} \frac{a}{b} + \arcsin (ya/r') + \theta_1 = 0,
\end{align*}
\]

and for \(\Pi_4\):
\[
\begin{align*}
\arcsin \frac{\sqrt{2\omega}}{\alpha} - \arcsin y + \arcsin \frac{\sqrt{2\omega}}{\alpha} \frac{a}{b} + \arcsin (ya/r') - (\pi - \theta_2) = 0,
\end{align*}
\]

where \(d_0 = a y\).

Tables 1–5 are based on the following set of fixed data:

\[
\begin{align*}
\alpha^2 = c_1^2/c_2^2 = 2\omega = 3, \quad \theta_1 = \pi/12, \quad \theta_2 = 11\pi/12, \quad \theta_3 = 3\pi/4, \quad \theta_4 = 5\pi/4.
\end{align*}
\]

* Mr D. A. Anderson, c/o Geophysics Division, Iitri, Chicago, Illinois 60616.
The ratio of the core radius to the source location, \( a/b \), is varied from 30 in Table I to 50 in Table 5; \( y(\Psi_3) \) indicates the value of \( y \) obtained from \( \Psi_3 \), etc.

### Table 1

\( (a/b = 30) \)

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<th>( r'/a )</th>
<th>( y(\Psi_3) \times 10^3 )</th>
<th>( y(\Psi_3') \times 10^3 )</th>
<th>( y(\Pi_3) \times 10^3 )</th>
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### Table 2

\( (a/b = 35) \)

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### Table 3

\( (a/b = 40) \)

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II. Reflected P waves

For $d = ay$, $m = 1$, the equations (7.2) become

$$\Phi_1: 2 \arcsin y - \arcsin (ay/r) - \arcsin (ay/b) + \theta_1 = 0,$$

$$\Phi_2: 2 \arcsin y - \arcsin (ay/r) + \arcsin (ay/b) - (\pi - \theta_4) = 0,$$

$$\Phi_3: 2 \arcsin y + \arcsin (ay/r) - \arcsin (ay/b) - (\pi - \theta_3) = 0,$$

$$\Phi_4: 2 \arcsin y + \arcsin (ay/r) + \arcsin (ay/b) - (2\pi - \theta_4) = 0,$$

where $(r, \theta_1) \ldots (r, \theta_4)$ are observation stations in the acoustic region.

Tables 6-10 are based on the following set of fixed data:

$$\alpha^2 = c_1^2/c_2^2 = 2\omega = 3, \quad \theta_1 = \pi/4, \quad \theta_2 = \theta_3 = 3\pi/4, \text{ and } \theta_4 = 5\pi/4.$$

Each table is based on a different value of $a/b$ ratio. No entry implies no saddle-point for that particular observation station.
The generalized problem of Lamb, I

Table 6
(a/b = 5)

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<th>r/a</th>
<th>y(\Phi_1) \times 10^3</th>
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<th>y(\Phi_3) \times 10^3</th>
<th>y(\Phi_4) \times 10^3</th>
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(a/b = 30)

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Table 10

\( (a/b = 50) \)

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Acknowledgments

The author is grateful to Professor R. A. Ross of the University of Toronto for his encouragement and to the U.S. Defense Atomic Support Agency (Blast and Shock Division) and the Illinois Institute of Technology Research Institute (Geophysics Division) for co-sponsoring the investigation.

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Carleton University,
Ottawa 1, Canada.
1966 April.

References